Gravitational Waves
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Lecture 1 – Elementary Theory of Gravitational Waves and their Detection

Special and General Relativity

Lectures assume familiarity with relativistic electromagnetism and with Minkowski geometry. The metric (interval) is

$$ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta,$$

where the symbol η denotes the matrix $\mathrm{diag}(-1, 1, 1, 1)$. I will take $c = 1$ and use the usual summation convention on repeated indices. Greek indices sum over all four coordinates (0..3), Latin over the three spatial coordinates (1..3).

General relativity describes gravitation as geometry. Inspiration: the *principle of equivalence*, roots back to Galileo.

- Any smooth geometry is locally flat, and in GR this means that it is locally Minkowskian. *Local* means in space and time: the local Minkowski frame is a freely-falling observer.

- In a general coordinate system the Minkowski equation is replaced by

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta,$$

where g is a position-dependent symmetric 4×4 matrix. As in special relativity, *the metric measures proper time and proper distance*. Coordinates are arbitrary in GR, but most situations are easier to analyse in appropriately chosen coordinates. A free particle follows a *geodesic* of this metric, defined as a locally straight world-line.

- There are 4 degrees of freedom to choose coordinates, and 10 components of g: 6 "true" functions left for the geometry. The tensorial description of the geometry is through the *Riemann curvature tensor*, which contains second derivatives of g. We will explore its meaning later.

- Derived from the Riemann tensor is the Einstein tensor G, which is basis of the field equations

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta},$$

where T is the stress-energy tensor, whose components contain the energy density, momentum density, and stresses inside the source of the field. (We also take Newton's constant $G = 1$.) In GR momentum and stress as well as energy density create gravity.

- In non-relativistic situations, the energy density dominates, which is the 0-0 field equation. The most important curvature for nearly-Newtonian systems is the *curvature of time*, which is measurable by the gravitational redshift of clocks. *All* of Newtonian orbital motion can be derived from time-curvature: if you know the gravitational redshift everywhere, you know the Newtonian gravitational field everywhere. Relativistic particles, such as photons, are not described by Newtonian physics; their motion is sensitive also to the spatial curvature.

- The coordinate invariance of the theory implies that not all field equations are independent; mathematically the Einstein tensor is divergence-free for any metric:

$$\nabla_\alpha G^{\alpha\beta} = 0.$$

This is called the *Bianchi identity.* It implies, from the field equation,

$$\nabla_\alpha T^{\alpha\beta} = 0.$$

This is the *equation of conservation of energy and momentum* in the matter sources. In field theory language, coordinate invariance is a gauge group, the conservation laws of the Bianchi identities arise as Noether identities.

- Derivatives like ∇_α are defined so that in a freely-falling frame they are the derivatives of special relativity. Called *covariant derivatives.* In a general coordinate system they involve derivatives of tensor components and of basis vectors $\mathbf{e}_\alpha$, so expressions are complicated. They involve *Christoffel symbols* $\Gamma^\alpha{}_{\beta\delta}$:

$$\nabla_\delta \mathbf{e}_\beta = \Gamma^\alpha{}_{\beta\delta}\mathbf{e}_\alpha.$$

The Christoffel symbols involve the first derivatives of the metric tensor. They vanish in a local freely falling frame, but only at the single event where the frame is perfectly freely falling. The second derivatives of the metric cannot in general be made to vanish by going to any special coordinate system. You first meet the Christoffel symbols (but nobody introduces you to them by name!) in elementary vector calculus in Euclidean space, when you compute divergences in polar coordinates and find that you need more than just the derivatives of vector components.

- It is important to understand what the conservation law of the stress-energy tensor means. It is conservation in a freely-falling frame. This is the equivalence principle: in a freely falling frame, the *local* matter energies are conserved. It does *not* represent a conservation law with gravitational potential energy, for example. Such *global* energy conservation laws are valid only if the metric is independent of time. (We will return to this important point below.) The local conservation law is an equation of motion. It implies, for example, that isolated small particles fall on geodesics of the metric.

- If a tensor has zero covariant derivative in a given direction, it is said to be *parallel-transported*. Thus, a vector V is parallel-transported in the direction W if

$$W^\alpha \nabla_\alpha V^\beta = 0$$

for all β. Parallel-transport means that the field is held constant in a freely-falling frame.

- A special case of parallel transport is the geodesic equation, which is the statement that the four-velocity U — whose components are $U^\alpha = dx^\alpha/d\tau$ (where τ is proper time) — is parallel-transported along itself:

$$\frac{dU^\alpha}{d\tau} + \Gamma^\alpha{}_{\mu\nu} U^\mu U^\nu = 0.$$

- The second derivatives of the metric contain coordinate-invariant information that is collected in the Riemann curvature tensor R. An interesting definition of R involves the commutator of covariant derivatives of a vector field V:

$$[\nabla_\alpha, \nabla_\beta]V^\mu = R^\mu{}_{\nu\alpha\beta}V^\nu.$$

  In Minkowski spacetime, derivatives commute and the curvature is zero. In a curved space(time), covariant derivatives in different directions do not commute. This is most easily seen in terms of parallel transport: vector fields that are parallel transported tangent to a sphere along different great circles, for example, do not coincide with one another when the great circles intersect again.

- Another interpretation of the Riemann tensor is in terms of the failure of parallelism in a curved space. If we start two nearby geodesics off in the same direction, with a tangent vector U, and if we place a *connecting vector* ξ between the two geodesics and carry it along so it always links them at the same elapsed proper time, then the connecting vector will not remain constant if the space is not flat:

$$\frac{d^2\xi^\alpha}{d\tau^2} = -R^\alpha_{\mu\beta\nu}U^\mu\xi^\beta U^\nu.$$

  This is called the equation of geodesic deviation.

- The mathematics of tensor calculus can get very complicated. The expressions for the Riemann tensor in terms of the components of the metric tensor are long and not very informative. We will not go into such things in these lectures. They are treated in the textbooks.

- Approximation methods are crucial in general relativity.
  - We will deal mostly with *linearized theory* in these lectures, where the curvature is small and spacetime is nearly Minkowskian. Only terms of first order in the difference between the true metric and the Minkowski metric are considered.
  - Another important approximation is the *post-Newtonian approximation*. GR contains Newtonian gravity as a (degenerate) limit, where the field equations lose their time-derivatives. One can develop an asymptotic approximation to GR away from this limit. The limit links two parameters: the gravitational field is weak and the velocities of the sources are small. In linearized theory the field is weak but the sources do not have to be non-relativistic.
  - *Perturbation theory* is the study of solutions near a known solution. This is a generalization of linearized theory. It can study stellar stability, the orbits (with radiation reaction) of particles near black holes, and so on.
  - A final approximation method is *numerical relativity*: simulations of black holes and other situations are increasingly useful in understanding GR. It can attack any problem in principle, no matter how complicated, but of course one has to be careful to be sure the result is a close enough approximation to the true solution.

Elements of gravitational waves

GR is nonlinear, fully dynamical $\Rightarrow$ in general no clear distinction between waves and the rest of the metric. The notion of a *wave* is OK in certain limits:

- in linearized theory;
- as small perturbations of a smooth background metric (e.g. waves propagating in cosmology or waves being gravitationally lensed by the metric of a star, galaxy, or cluster);
- in post-Newtonian theory (far zone, i.e. more than one wavelength distant from the source).

We will concentrate on linearized theory, but much of this work carries over to the other cases in a straightforward way.

Mathematics of linearized theory

- In **linearized theory** metric is nearly that of flat spacetime:

$$ds^2 = (\eta_{\alpha\beta} + h_{\alpha\beta})\, dx^\alpha dx^\beta, \quad |h_{\alpha\beta}| \ll 1.$$

- Define trace-reversed metric perturbation $\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}\eta^{\mu\nu}h_{\mu\nu}$ and adopt *Lorentz gauge*:

$$\bar{h}^{\alpha\beta}{}_{,\beta} = 0,$$

where a subscripted comma denotes the partial derivative with respect to the coordinate associated with the index that follows the comma. Lorentz gauge is just a gauge (coordinate) choice: four equations use up 4 degrees of freedom to specify spacetime coordinates. Initial data for these equations is still free.

- In Lorentz gauge, the Einstein field equations are just a set of decoupled wave equations

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right) \bar{h}^{\alpha\beta} = -16\pi T^{\alpha\beta}.$$

- To understand propagation, it is easiest to look at *plane waves*:

$$\bar{h}^{\alpha\beta} = A^{\alpha\beta} \exp(2\pi\imath k_\mu x^\mu),$$

for constant amplitudes $A^{\alpha\beta}$ and wave vector k_μ. Then the Einstein equations imply that the wave vector is null $k_\alpha k^\alpha = 0$ (propagation at the speed of light), and the gauge condition implies that the amplitude and wave vector are orthogonal, $A^{\alpha\beta}k_\beta = 0$.

- Further gauge conditions (adjustments of the initial data for the Lorentz gauge equations) are possible. Just state them here: we will explicitly construct them in Chapter 4. We can demand that

1. $A^{0\beta} = 0 \quad \Rightarrow \quad A^{ij}k_j = 0$: *Transverse* wave; and
2. $A^j{}_j = 0$: *Traceless* wave amplitude.

These conditions can only be applied outside a sphere surrounding the source. Together, they put the metric into the *transverse-traceless (TT) gauge*. In TT gauge, $\bar{h}_{\alpha\beta} = h_{\alpha\beta}$.

Using the TT gauge to understand gravitational waves

- **Only two independent polarizations** The TT gauge leaves only two independent wave amplitudes out of the original 10. Take the wave to move in the z-direction, so that $k_z = k$, $k_x = k_y = 0$. Then gauge + transversality $\Rightarrow A^{0\alpha} = A^{z\alpha} = 0$, leaving only A^{xx}, $A^{xy} = A^{yx}$, and A^{yy} nonzero. Tracelessness $\Rightarrow A^{yy} = -A^{xx}$. So there are only 2 independent amplitudes, 2 independent degrees of freedom for polarization.

- A wave for which $A^{xy} = 0$ produces a metric of the form

$$ds^2 = -dt^2 + (1 + h_+)dx^2 + (1 - h_+)dy^2 + dz^2,$$

  where $h_+ = A^{xx} \exp[ik(z-t)]$. This produces opposite effects on proper distance on the two axes, contracting one while expanding the other.

- If $A^{xx} = 0$ then only the off-diagonal term $h_{xy} = h_\times$ is non-trivial, and these can be obtained from the previous case by a $45°$ rotation.

- A general wave is a linear combination of these two. If one lags the other in phase, the polarization is circular or elliptical. The existence of only two polarizations is a property of any non-zero spin field that propagates at the speed of light.

- The effect of a wave in TT gauge on a particle at rest can be computed from the Christoffel symbols. Its initial acceleration is

$$\frac{d^2}{d\tau^2}x^i = -\Gamma^i{}_{00} = -\frac{1}{2}\left(2h_{i0,0} - h_{00,i}\right) = 0.$$

So the particle does not "move", does not change coordinate location. *The TT gauge represents a coordinate system that is comoving with freely-falling particles.* Because $h_{0\alpha} = 0$, TT-time is proper time on the clock of a freely falling particle at rest.

- **Tidal forces** show the action of the wave independently of coordinates. For example, the geodesic deviation equation for the separation ξ of two freely falling particles initially at rest is

$$\frac{d^2}{d\tau^2}\xi^i = -R^i{}_{0j0}\xi^j = \frac{1}{2}h_{ij,00}\xi^j.$$

  This contains the same information as we saw in the metric above. The Riemann tensor is gauge-invariant in linearized theory.

Measuring gravity using light propagation

How does one measure a distorted spacetime geometry? Measuring rods are not good enough: can we manufacture an "ideal" standard-length rod that is not affected by gravity, not compressed in some way? In GR there are two things that are not affected by gravity: the geometry of a *local* patch, whose size is chosen small enough that tidal effects are negligible, and the speed of light. Light always propagates at speed $c = 1$ in every local inertial frame.

We can use these to make a geometry-measuring tool: send light out to a distant point, reflect it back, and measure the return time on a clock that has not moved. The clock is in a local inertial frame so it measures proper time. The propagation time of light is a measure of the proper distance to the distant point. With a lot of such measurements from clocks located all over space and in various states of motion, one can build up a database from which the overall geometry can be reconstructed.

Although this sounds like a thought-experiment, we have been using light-clocks as distance measuring tools for decades. We call it radar. Today it is even more practical to use such a technique because lasers give such precise control over the generation of the light that ultra-precise measurements of distance are possible. Photon-based distance measurements are the basis of all methods being used today to directly measure time-dependent spacetime geometry, i.e. to detect gravitational waves. Such detectors are called *beam detectors.* To understand how beam detectors work we need to look at the propagation of light in a gravitational wave geometry.

Let us compute the proper distance between two freely-falling bodies (no forces on them other than gravity), which start off at rest with respect to one another before the gravitational wave arrives. We shall do the computation in TT gauge because in these coordinates the bodies remain at fixed coordinate locations, which simplifies the computation. We need to calculate the effect of the waves on the coordinate speed of light. In the "+" metric earlier, where the wave is traveling in the z-direction, a null geodesic moving in the x-direction has effective speed

$$\left(\frac{dx}{dt}\right)^2 = \frac{1}{1+h_+}.$$

This is a *coordinate speed*, no contradiction to special relativity.

- Suppose light is directed along the x-direction and the gravitational wave is moving in the z-direction with a $+$-polarization of *any* waveform $h_+(t)$ along this axis. (It is a plane wave, so its waveform does not depend on x.) Then a photon emitted at time t from the origin reaches the distant object, which at a fixed coordinate position $x = L$, at the coordinate time

$$t_{far} = t + \int_0^L \left[1 + h_+(t(x))\right]^{1/2} dx,$$

where the argument $t(x)$ denotes the fact that one must know the time to reach position x in order to calculate the wave field. This implicit equation can be solved in linearized theory by using the fact that h_+ is small. Then we can set $t(x) = t + x$ and expand the square root. The result is

$$t_{far} = t + L + \frac{1}{2}\int_0^L h_+(t + x)dx.$$

In our distance meter, the light is reflected back, so the whole trip takes

$$t_{return} = t + 2L + \frac{1}{2}\left[\int_0^L h_+(t + x)dx + \int_0^L h_+(t + L + x)dx\right].$$

- In practice, to see if a gravitational wave has arrived, one monitors *changes* in the time for the return trip as a function of time at the origin. The rate of change of the return time as a function of the start time t:

$$\frac{dt_{return}}{dt} = 1 + \frac{1}{2}\left[h_+(t + 2L) - h_+(t)\right].$$

This depends only on the wave amplitude when the beam leaves and when it returns. Interestingly, for this special geometry, it does not involve the wave amplitude at the other end.

- The wave amplitude at the other end does get involved if the wave travels at an angle θ to the z-axis in the $x - z$ plane. If we re-do this calculation, allowing the phase of the wave to depend on x in the appropriate way, and taking into account the fact that h_{xx} is reduced if the wave is not moving in a direction perpendicular to x, we can find

$$\begin{aligned}\frac{dt_{return}}{dt} &= 1 + \frac{1}{2}\{(1 - \sin\theta)h_+(t + 2L) - (1 + \sin\theta)h_+(t) \\ &\quad +2\sin\theta h_+[t + L(1 - \sin\theta)]\} .\end{aligned}$$

This three-term relation is the starting point for analyzing the response of all beam detectors.

Beam detectors

If a detector is small compared to the wavelength λ of a gravitational wave, then once can do a Taylor expansion on L. For the simple case of light moving along the x-axis, we get from above

$$\frac{dt_{return}}{dt} = 1 + \dot{h}_+(t)L + O(L^2).$$

(Since $\dot{h}_+ \sim 2\pi h_+/\lambda$, this really is a Taylor expansion in the small dimensionless parameter L/λ.) If we take another time-derivative to get

$$\frac{d^2t_{return}}{dt^2} = \ddot{h}_+(t)L + O(L^2),$$

we can link this with the equation of geodesic deviation that we wrote down earlier:

$$\frac{d^2}{d\tau^2}\xi^i = \frac{1}{2}h_{ij,00}\xi^j.$$

From this point of view, the central body can consistently assume that the distant body is subject to a simple force, proportional to the second time-derivative of the metric. We call this the tidal force, and it is a consistent approximation provided the experimental region is small. This is true for laser interferometric detectors on the Earth, and indeed for the question of how tides are raised on the Earth by the Moon and Sun. But a gravitational wave detector in space would be too large for this approximation to hold, and for pulsar timing the approximation makes no sense at all.

The experiment as described assumes that we can measure time accurately enough on the central clock to detect the gravitational wave. Unfortunately, wave amplitudes are so small that the time variations are below the accuracies of our best clocks. The only way to make such measurements is to *compare* two different directions. An interferometer can be thought of as a light-time-return comparison machine, sending photons off in two perpendicular directions and sensing variations in their relative return times by watching for shifts in their interference patterns when they return. Since the action of a gravitational wave is anisotropic (different in x, y, and z directions), this comparison can detect the wave in almost all cases.

There are several kinds of beam detectors:

- **Ranging to spacecraft.** Both NASA and ESA perform experiments in which they monitor the return time of communication signals with interplanetary spacecraft for the characteristic effect of gravitational waves. For missions to Jupiter and Saturn, for example, the return times are of order $2 - 4 \times 10^3$ s. Any gravitational wave event shorter than this will appear 3 times in the time-delay: once when the wave passes the Earth-based transmitter, once when it passes the spacecraft, and once when it passes the Earth-based receiver. Searches use a form of data analysis using pattern matching. Using two transmission frequencies and very stable atomic clocks, it is possible to achieve sensitivities for h of order 10^{-13}, and even 10^{-15} may soon be reached. This technique really does use only one arm and a clock, so it is limited by clock accuracies.

- **Pulsar timing.** Many pulsars, particularly the old millisecond pulsars, are extraordinarily regular clocks, with random timing irregularities too small for the best atomic clocks to measure. If one assumes that they emit pulses perfectly regularly, then one can use observations of timing irregularities of single pulsars to set upper limits on the background gravitational wave field. Here the 3-term formula is replaced by a simpler two-term expression, because we only have a one-way transmission. Moreover, the transit time of a signal to the Earth from the pulsar may be thousands of years, so we cannot look for correlations between the two terms in a given signal. Instead, the delay is a combination of the effects of waves at the pulsar when the signal was emitted and waves at the Earth when it is received.

  If one simultaneously observes two or more pulsars, the Earth-based part of the delay is correlated, and this offers a means of actually detecting long-period gravitational waves. Observations require timescales of several years in order to achieve the long-period stability of pulse arrival times, so this method is suited to looking for strong gravitational waves with periods of several years. Observations are currently underway at a number of observatories. These include Parkes Pulsar Timing Array, the European Pulsar Timing Array, and the American Nanograv collaboration. Detections of random backgrounds of gravitational waves from the mergers of supermassive black holes may well take place before 2020. Longer-range plans include SKA (the Square Kilometer Array) and its pathfinders.

- **Interferometry.** As mentioned above, an interferometer essentially measures the difference in the return times along two different arms. The response of each arm will follow the three-term formula, but with different values of θ, depending in a complicated way on the orientation of the arms relative to the direction of travel and polarization of the wave. Ground-based interferometers are small enough to use the small-L formulas we derived above. These include the two LIGO sites in the USA (Hanford, Washington, and Livingston, Louisiana), the VIRGO detector near Pisa, Italy, the GEO600 detector near Hannover, Germany, and the LCGT detector under construction in Japan in the Kamiokande underground facility.

But LISA, the proposed space-based interferometer, would be larger than a wavelength of gravitational waves for frequencies above 10 mHz, so a detailed analysis based on the 3-term formula is needed.

Energy and Gravitational Radiation

- **How waves carry energy.** Energy has been one of the most confusing aspects of gravitational wave theory and hence of general relativity. It caused much controversy, and even Einstein himself took different sides of the controversy at different times in his life. Physicists today have reached a wide consensus. The problem is difficult because of the equivalence principle: in a local frame there are no waves and hence no local definition of energy that can be coordinate-invariant. Moreover, a wave is a time-dependent metric, and in such spacetimes there is no global energy conservation law. (Recall that conservation laws are associated with symmetry. Angular momentum is conserved only in axisymmetric systems, or in systems governed by axisymmetric forces or fields. Likewise, energy is conserved on in time-invariant systems.) Energy is only well-defined in certain regimes, which coincide with those for which waves can be cleanly separated from "background" metrics.

- **Asymptotically flat spacetimes.** Relativists have introduced the concept of *asymptotic flatness*, which idealises an isolated body, one whose geometry becomes flat far away. If the body is stationary (time-independent) then test fields (falling particles) will have conserved energy. If there is a small perturbation that can be treated as a wave, then its energy will be conserved, in the sense that the total mass-energy of the system (as measured by planetary orbits far away) decreases as the waves leave.

- **Wave flux.** The energy carried by a wave as it leaves a body or as it moves through a nearly-flat spacetime can be written as an effective stress-energy tensor for the wave. It is known as the Isaacson tensor, and in linearized theory it has the following expression:

$$T^{(GW)}_{\alpha\beta} = \frac{1}{32\pi} h_{\mu\nu,\alpha} h^{\mu\nu}{}_{,\beta}.$$

It represents the localisation of energy to regions whose size is of order a wavelength, but not smaller. It can be defined in the same way if h represents a perturbation away from a background curved spacetime, in which case the derivatives are replaced by covariant derivatives.

- **The status of energy in GR.** It is important to understand that energy is a useful but not fundamental concept in gravitational wave studies. The fundamental quantity is the metric perturbation itself. We don't need energy concepts to calculate the radiated wave amplitude (below) nor to compute the effect of the wave on a detector (above). Energy is a useful concept only when it is conserved, where it makes source calculations easier and allows one to understand sources better. But we could get along without it!

- **Energy in cosmology.** Cosmology particularly illustrates the differences between GR and special-relativistic physics concerning energy. In cosmology, the metric is time-dependent, so the energy of things in the spacetime is not conserved. The cosmological redshift of photons arises from this. The energy does not go anywhere, it is not transformed into something else as we might habitually expect in physics: it simply disappears because it is not conserved. Energy as measured in a local inertial frame is still conserved, so that particle interactions behave as in special relativity. But when a particle is followed over a long time, so that the local inertial frame is not a good approximation, then the energy changes.

- **Gravitational waves in cosmology.** As remarked before, it is not possible to talk sensibly about gravitational waves unless the wavelength is small compared to the curvature scale. In cosmology that is the "radius" of the universe, or the Hubble scale/horizon size. So any perturbations smaller than the horizon size will propagate as gravitational waves following the same laws of geometrical optics that photons follow. In particular the energy of the waves gets redshifted in the same way as for photons.

Practical applications of the energy formula

- **Relation between typical wave amplitude and the energy radiated by a source.** If we are far from a source of gravitational waves, we can treat the waves by linearized theory. Then if we adopt TT gauge and specialize the stress-energy tensor of the radiation to a flat background, we get

$$T^{(GW)}_{\alpha\beta} = \frac{1}{32\pi} h^{TT}_{\mu\nu\ ,\alpha} h^{TT\mu\nu}{}_{,\beta}.$$

Since there are only two components, a wave traveling with frequency f (wave number $k = 2\pi f$) and with a typical amplitude h in both polarizations carries an energy flux F_{gw} equal to (see Exercise 6)

$$F_{gw} = \frac{\pi}{4} f^2 h^2.$$

Putting in the factors of c and G and scaling to reasonable values gives

$$F_{\rm gw} = 3\ \mathrm{mW\ m^{-2}} \left[\frac{\mathrm{h}}{1\times 10^{-22}}\right]^2 \left[\frac{\mathrm{f}}{1\ \mathrm{kHz}}\right]^2,$$

which is a very large energy flux even for this weak a wave. It is twice the energy flux of a full moon! Integrating over a sphere of radius r, assuming a total duration of the event τ, and solving for h, again with appropriate normalisations, gives

$$h = 10^{-21} \left[\frac{E_{gw}}{0.01 M_\odot c^2}\right]^{1/2} \left[\frac{r}{20\ \mathrm{Mpc}}\right]^{-1} \left[\frac{f}{1\ \mathrm{kHz}}\right]^{-1} \left[\frac{\tau}{1\ \mathrm{ms}}\right]^{-1/2}.$$

This is the formula for the "burst energy", normalized to numbers appropriate to a gravitational collapse occurring in the Virgo cluster. It explains why physicists and astronomers regard the 10^{-21} threshold as so important. But this formula could be applied to binary systems radiating away their orbital gravitational binding energy over long periods of time τ, for example.

- **Curvature produced by waves.** Although the Isaacson flux tensor is an approximation, it is a very robust and satisfying approximation. Isaacson showed that the background spacetime will actually exhibit a small average curvature when the waves are contained on it, and that this curvature has an Einstein tensor given by.
$$G_{\alpha\beta} = 8\pi T^{(GW)}_{\alpha\beta}.$$
This is the Einstein field equation for a spacetime with the wave-energy as its source. It only holds to lowest order in the (small) wave amplitude, but we would not expect the Isaacson flux expression to be meaningful at higher orders of approximation anyway.

- **Cosmological background of radiation.** This self-consistent picture allows us to talk about, for example, a cosmological gravitational wave background that contributes to the curvature of the Universe. Since the energy density is the same as the flux (when $c = 1$), we have
$$\rho_{gw} = \frac{\pi}{4} f^2 h^2,$$
but now we must interpret h in a statistical way. Basically it is done by replacing h^2 by a statistical mean square amplitude per unit frequency (Fourier transform power per unit frequency) called $S_h(f)$, so that the energy density *per unit frequency* is proportional to $f^2 S_h(f)$. It is then conventional to talk about the energy density per unit logarithm of the frequency, which means multiplying by f. The result, after being careful about

averaging over all directions of the waves and all independent polarization components, is

$$\frac{d\rho_{gw}}{d\ln f} = 4\pi^2 f^3 S_h(f).$$

Finally, what is of most interest is the energy density as a fraction of the closure or critical cosmological density, given by the Hubble constant H_0 as $\rho_c = 3H_0^2/8\pi$. The resulting ratio is called $\Omega_{gw}(f)$:

$$\Omega_{gw}(f) = \frac{32\pi^3}{3H_0^2} f^3 S_h(f).$$

Status of Interferometric detectors

The interferometric gravitational wave detectors operating today are the most sensitive measuring instruments ever built. Capable of measuring small changes in the proper separation of their mirrors of order 10^{-16} cm, which is significantly smaller than the diameter of a proton. It has taken decades of development to reach this sensitivity and to be able to control the instruments well enough that they remain on-line for most of the time during an observing run (over 90% of the time in the case of GEO600). The observing frequency band of these detectors is from about 40 Hz up to a few kHz.

Yet even with this sensitivity the detectors would have to be very lucky, on our present understanding of potential sources, to have detected anything yet. LIGO, GEO600, and VIRGO have done two observing runs, called S5 (2005-7) and S6 (2009-10) with a small sensitivity upgrade in between, and from these runs have come a large number of papers setting upper limits on possible sources, including on the stochastic background from the Big Bang. We will consider that limit in the third lecture. Other limits have constrained the smoothness of some neutron stars (some are smoother than parts in 10^7, much smoother than the Earth) and the fraction of the energy-loss of the Crab pulsar that is going into gravitational waves (less than 1%). But a further upgrade is needed to make direct detections happen within a reasonable amount of time.

LIGO at present is upgrading to what is called Advanced LIGO, which will be 10 times more sensitive than in S5 when it begins operating again, probably around 2015. VIRGO is currently in an observing run (S6e) with GEO600 but will soon begin a similar upgrade. GEO600 is upgrading in a different manner, aimed at higher frequencies (several kHz), but is also mainly providing cover ("astrowatch") while the bigger detectors are upgrading. Regular detections are confidently expected starting in the time-frame 2015-16 because a factor of 10 increase in sensitivity translates in to a factor 10 improvement in the detection range, which increases the detection volume by 1000. Even pessimistic source rate estimates suggest a few events per year in such a volume, and optimistic scenarios suggest hundreds of events per year.

Recently the Japanese have approved and funded the Large Cryogenic Gravitational-wave Telescope (LCGT), which will hopefully be able to operate soon after 2016 (depending on the speed with which funds can be released). It will compete in sensitivity with Advanced LIGO, but will at the same time introduce two new technologies that are needed for further improvements in sensitivity: cryogenic cooling of the mirrors and going underground.

The Europeans have looked ahead and, after a multi-year study funded by the EU, have proposed a design for a so-called third-generation detector called the Einstein Telescope (ET). This would advance the sensitivity a further factor of 10 beyond Advanced LIGO and push the lower frequency limit down to a few Hz. This upgrade cannot be done in the existing instruments, and would require a new instrument, cryogenically cooled, 10 km long, and a few hundred metres beneath the ground. The sensitivity, however, would be enough to survey the entire universe for binary mergers of neutron stars and/or black holes, a source that we will treat in more detail in the second lecture.

Pulsar timing arrays were started about 5 years ago but are now being very actively built up, and observatories are giving them more and more time. Because pulsars are good clocks only when averaged over a long time, these arrays are sensitivity to gravitational waves of periods of a year or more. They are expecting that the largest signal will be a random background from countless orbiting systems of supermassive black holes, and we will return to this in the third lecture.

The frequency band around 1 mHz is very rich in interesting sources, including binaries in our Galaxy and binaries of massive black holes in external galaxies. To observe in this band requires going into space, and ESA adopted the proposed LISA mission as long ago as 1995. The new technology required for LISA will be launched in LISA Pathfinder sometime in 2013-14. The recent funding problems in NASA have caused it to withdraw from its partnership in LISA, so ESA is again considering whether it can proceed with a somewhat de-scoped mission by itself.

Detector Networks

The current suite of ground-based interferometers all work closely together. LIGO and GEO are part of the same collaboration, the LIGO Scientific Collaboration (LSC), where technologies pioneered in GEO600 are transferred to LIGO for its upgrade, and where scientists work in teams on the experiment and data analysis. Data from the LSC detectors is exchanged with data from VIRGO, and jointly analysed. When LCGT comes online it is expected that this model will continue to hold.

This cooperation is driven by the needs of the science. No single detector could have enough confidence to claim a detection of such a weak signal by itself: one must see the signal in more than one detector for confidence. But also recall that interferometers are not highly directional; they are not pointed telescopes. So to get directional information one uses time-delays among the arrival time of the signals at different detectors. Having multiple detectors also is needed in order to measure both polarizations of the gravitational wave. In these respects, detectors behave in the same way as sound microphones: one needs stereo (or better) for directionality. Key scientific results, like measuring the distance to binary systems (see the next lecture), depend on having the information a network can measure: polarization and sky location.

LIGO is currently exploring the possibility of moving one of the two large Advanced detectors it has been planning for its Hanford location, and putting it in Australia. This would greatly improve the information available from multiple locations, especially the localization of directions. it would also improve the ability of the network to distinguish real signals from random background noise events. If it is not possible to find the funding in Australia, another potential good location for the detector would be India.

LISA's design, with three spacecraft and three active arms measuring the changes in the separations, allows the signals to be combined into three different gravitational wave measurements, essentially treating each spacecraft as the central station of an interferometer based on the two arms that converge there. With these signals, LISA measures the polarization of an incoming wave automatically, and the configuration also gives some directional capabilities. But LISA's direction finding comes mainly from its orbital motion around the Sun and the Doppler shift in the signals that this produces. For a short-lived signal this does not help, but for a signal that lasts a good fraction of a year, which is the case for most of LISA's sources, then the Doppler effect is measurable and points to the location of the source on the sky.