Gauge/Gravity Correspondence and Black Hole Attractors in Various Dimensions

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Gauge/Gravity Correspondence and Black Hole Attractors in Various Dimensions

Abstract

This thesis investigates several topics on Gauge/Gravity correspondence and black hole attractors in various dimensions.

The first chapter contains a brief review and summary of main results. Chapters 2 and 3 aim at a microscopic description of black objects in five dimensions. Chapter 2 studies higher-derivative corrections for 5D black rings and spinning black holes. It shows that certain $R^2$ terms found in Calabi-Yau compactifications of M-theory yield macroscopic corrections to the entropies that match the microscopic corrections. Chapter 3 constructs probe brane configurations that preserve half of the enhanced near-horizon supersymmetry of 5D spinning black holes, whose near-horizon geometry is squashed $AdS_2 \times S^3$. There are supersymmetric zero-brane probes stabilized by orbital angular momentum on $S^3$ and one-brane probes with momentum and winding around a $U(1)_L \times U(1)_R$ torus in $S^3$.

Chapter 4 constructs and analyzes generic single-centered and multi-centered black hole attractor solutions in various four-dimensional models which, after Kaluza-Klein reduction, admit a description in terms of 3D gravity coupled to a sigma model whose target space is symmetric coset space. The solutions correspond to certain
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Nilpotent generators of the coset algebra. The non-BPS black hole attractors are found to be drastically different from their BPS counterparts.

Chapter 5 examines three-dimensional topologically massive gravity with negative cosmological constant in asymptotically $AdS_3$ spacetimes. It proves that the theory is unitary and stable only at a special value of Chern-Simons coupling, where the theory becomes chiral. This suggests the existence of a stable, consistent quantum gravity theory at the chiral point which is dual to a holomorphic boundary $CFT_2$.

Finally, Chapter 6 studies the two-dimensional $\mathcal{N} = 1$ critical string theory with a linear dilaton background. It constructs time-dependent boundary state solutions that correspond to D0-branes falling toward the Liouville wall. It also shows that there exist four types of stable, falling D0-branes (two branes and two anti-branes) in Type 0A projection and two unstable ones in Type 0B projection.
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Chapter 1

Introduction and Summary

1.1 Main theme

1.1.1 Gauge/Gravity Correspondence and its Various Manifestations

The conjectured Gauge/Gravity correspondence states that the IR limit of the world-volume theory on a given D-brane (or M-brane) configuration is equivalent to the string theory (or M-theory) living in the near-horizon geometry sourced by the corresponding brane configuration.

The Gauge/Gravity correspondence arises from the two dual descriptions of D-branes. From the open-string point of view, D-branes are hypersurfaces where open strings end. Their dynamics is described by the world-volume theory living on the D-branes. The string system with a given D-brane configuration can be schematically
decomposed into

\[ S = S_{\text{brane}} + S_{\text{bulk}} + S_{\text{interaction}}, \]  

(1.1)

where \( S_{\text{brane}} \) describes the world-volume theory on the D-branes and \( S_{\text{bulk}} \) the bulk gravity theory; and \( S_{\text{interaction}} \) encodes the interaction between the two.

With certain D-brane configurations (which will be discussed later), \( S_{\text{interaction}} \rightarrow 0 \) as we take an appropriate low energy limit of the theory. It is called “decoupling limit” since during which the world-volume theory living on the D-branes is decoupled from the gravity theory living in the bulk.

From the closed-string point of view, D-branes are non-perturbative states in the closed-string spectrum. They generate background p-brane geometry with RR-fluxes. In this context, the above decoupling limit results in two decoupled gravity systems: one in the near-horizon region and one in the asymptotic region.

Since in both open-string and closed-string formulations, the decoupling limit gives rise to a pair of two decoupled subsystems with one of them being the gravity theory living in the asymptotic region, we can identify the remaining subsystems in these two different formulations. Namely, the low energy limit of the world-volume theory living on the D-branes is equivalent to the gravity theory living in the near-horizon geometry sourced by the same D-branes configuration.

Different brane configurations give rise to different manifestations of Gauge/Gravity correspondence. Among those most studied are following:

**AdS/CFT Correspondence.** The most prominent example of Gauge/Gravity correspondence is AdS/CFT correspondence. In this duality, the superconformal gauge theory living on the \( d \)-dimensional boundary of a \((d + 1)\)-dimensional anti-de Sitter
space $AdS_{d+1}$ is conjectured to be equivalent to the string theory living in $AdS_{d+1} \times M^{9-d}$ (or M-theory living in $AdS_{d+1} \times M^{10-d}$), where $M$ is a compact manifold [102].

Consider a D-brane configuration whose near-horizon geometry is $AdS_{d+1} \times M$. In the open-string sector, the appropriate decoupling limit is simply the low energy limit in the world-volume theory on the D-branes. It is equivalent to the limit of

$$\ell_s \to 0$$

with all dimensionless parameters and the Yang-Mills gauge coupling

$$g_{YM}^2 = 2(2\pi)^{d-3} g_s \ell_s^{d-4}$$

fixed.

The ten-dimensional and eleven-dimensional gravitational couplings are:

$$\kappa_{10} \sim g_s \ell_s^4 \sim g_{YM}^2 \ell_s^{8-d} \sim \ell_s^4, \quad \kappa_{11} \sim (g_s \ell_s^3)^{3/2} \sim (g_{YM}^2 \ell_s^{7-d})^{3/2} \sim \ell_s^{9/2}.$$ (1.4)

For systems with $d \leq 6$, $\kappa_{10}, \kappa_{11} \to 0$ in the above decoupling limit. Therefore, in the whole coupling region, the gravity theory in the closed string sector is decoupled from the world-volume theory on the open string sector in this limit.

This limit is effectively a low-energy limit in both open string and closed string sectors. In the closed string sector, for systems with $d \leq 6$, it is translated into the near-horizon limit: the closed string sector physics reduces into the gravity theory living in $AdS_{d+1} \times M$ (with background fluxes given by the D-brane configuration).

---

1 See [1] for a nice review on AdS/CFT correspondence.

2 We need to consider $\kappa_{11}$ as well since the ten-dimensional string theory transforms into eleven-dimensional M-theory in the strong coupling regime.
In the open string sector, the low-energy limit of the world-volume theory on the D-branes yields a superconformal gauge theory living on the boundary of $AdS_{d+1}$. The conjectured $AdS/CFT$ correspondence then states that the superconformal gauge theory living on the $d$-dimensional boundary of $AdS_{d+1}$ is equivalent to the string theory living in $AdS_{d+1} \times M$.

At energy scale $E$, the dimensionless effective Yang-Mills coupling is

$$g_{\text{eff}}^2 \sim g_{YM}^2 N \cdot E^{d-4}.$$  \hspace{1cm} (1.5)

The effective string coupling $e^\phi$ and the curvature (in string unit) of the $AdS_{d+1}$ geometry are related to $g_{\text{eff}}$ as

$$e^\phi \sim \frac{(g_{\text{eff}})^{(8-d)/2}}{N}, \quad R\ell_s^2 \sim \frac{1}{g_{\text{eff}}}.$$  \hspace{1cm} (1.6)

The perturbative Yang-Mills theory is a good approximation when $g_{\text{eff}} \ll 1$. The supergravity approximation is valid when both $e^\phi \ll 1$ and $R\ell_s^2 \ll 1$, which requires $g_{\text{eff}} \gg 1$ and $N \gg 1$. Therefore, the UV behavior in the open string sector is dual to the IR behavior of the closed string sector and vice versa.

The $AdS/CFT$ correspondence was initially discovered in the case of $N$ coincident D3-branes. Its near-horizon geometry is $AdS_5 \times S^5$, and the low-energy limit of the world-volume theory is $\mathcal{N} = 4$ super Yang-Mills theory. Thus it produces the duality between Type IIB string theory in $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super Yang-Mills theory on the four-dimensional boundary of $AdS_5$ [102]. Similarly, a stack of $N$ M2-branes generates a $AdS_4/CFT_3$ correspondence, a stack of $N$ M5-branes generates a $AdS_7/CFT_6$ correspondence, and so forth. Gauge/Gravity correspondence can also be generated by D-brane bound states: for example, the two-dimensional $(0,4)$ CFT
on the long string created by a triple intersection of three stacks of M5-brane is dual to M-theory living in $AdS_3 \times S^3 \times M$ created by this M5-brane configuration [104].

As will be shown later, $AdS/CFT$ correspondence is crucial in microscopic descriptions of black holes. In fact, the first microscopic counting of black hole entropy is in the case of five-dimensional $\mathcal{N} = 4$ supersymmetric black holes created by D1-D5-P bound states in Type IIB compactification on $K3 \times S^1$. Taking $S^1$ to be large, the near-horizon geometry is $AdS_3 \times S^3 \times K3$. The black hole entropy is then computed by the growth of states in the dual two-dimensional conformal field theory via Cardy’s formula [140].

**MQM/2D string duality**  This is the earliest example of Gauge/Gravity correspondence. In this duality, the matrix quantum mechanics (MQM) is conjectured to be equivalent to two-dimensional critical string theory in the linear dilaton background [107, 143, 57].

The two-dimensional critical string has a linear dilaton field $\Phi = \frac{Q}{2} X^1$, where $X^1$ is the spatial dimension and $Q$ is tuned such that $c = 26$ for bosonic string and $\hat{c} = 10$ for superstring.

The closed string sector is the two-dimensional world-sheet gravity described by the Liouville field theory (LFT) coupled to a $c = 1$ matter field $X$, which is identified as the time dimension of the critical string. The space dimension is provided by the Liouville mode of LFT. To quantize the 2D worldsheet gravity, one first discretize the 2D Riemann surface via a random triangulation:

$$Z \longrightarrow Z_{\text{discrete}} : \sum_h \int \mathcal{D}g \longrightarrow \sum_h \sum_{\text{triangulations}} . \quad (1.7)$$
The discretized path integral is
\[
Z_{\text{discrete}} = \sum_h \sum_{\text{triangulations}} g_s^{2h-2} e^{-\lambda_0 A} \int \prod_{i=1}^A dX_i e^{-\sum_{<ij>} \frac{(X_i-X_j)^2}{2\ell^2}} ,
\]  
where \(\lambda_0\) is the bare world-sheet cosmological constant, and \(<ij>\) denotes nearest neighbors. The area of each triangle is normalized to be 1, so the number of triangles \(A\) gives the total area of the 2D Riemann surface. Exponential of \(Z_{\text{discrete}}\) can be evaluated as the path integral of the quantum mechanics system of a Hermitian \(N \times N\) matrix \(M(t)\):
\[
e^{Z_{\text{discrete}}} = \int D M(t) e^{-\beta S_m} ,
\]  
where the Euclidean action of MQM is
\[
S_m = \int_{-T/2}^{T/2} dt \text{Tr}[\frac{1}{2} (\partial_t M)^2 + V(M)] ,
\]  
with \(T\) being the length of the Euclidean time-direction. The coupling constant of MQM is
\[
k = e^{-\lambda_0} = \sqrt{\frac{N}{\beta}} .
\]  
One then only needs to find the appropriate continuum limit to bring \(Z_{\text{discrete}}\) back to the original path integral \(Z\).

On the other hand, in the open string sector of the 2D critical string, there are D0-branes, defined by the Dirichlet boundary condition in the spatial direction (i.e. the Liouville direction). The fields living on the one-dimensional world-volume of \(N\) coincident D0-branes include a Hermitian \(N \times N\) matrix field \(M(t)\) and a \(U(N)\) gauge field \(A(t)\). The D0-brane world-volume theory has the DBI action:
\[
S_{\text{open}} = -\int dt \text{Tr} V(M(t)) \sqrt{1 - (D_t M(t))^2} ,
\]
where $D_tM = \partial_tM + [A_t, M]$. $A(t)$ can be gauged away and acts merely as a lagrangian multiplier that projects $M(t)$ onto $SU(N)$ singlets. The potential $V(M) = 1/(g_0\ell_s \cosh (M/2))$. At low energy, the action of world-volume theory $S_{\text{open}}$ reduces to that of MQM in (1.10). The open string coupling $g_o$ is related to the inverse temperature $\beta$ of the MQM system by

$$g_o = \frac{1}{\beta}.$$  

(1.13)

The decoupling limit then is simply the appropriate continuum limit that connects the matrix quantum mechanics with the continuum 2D gravity. Since the physical states are $SU(N)$ singlets, the system is reduced into a collection of non-interacting fermions, with Hamiltonian

$$H = \sum_{i=1}^{N} \left[ -\frac{1}{2\beta^2} \frac{d^2}{dm_i^2} + V(\lambda_i) \right],$$

(1.14)

where $m_i$ is the $i^{th}$ eigenvalue of $M$. We see that in this normalization, $1/\beta (\sim 1/N)$ plays the role of Planck constant $\hbar$.

The ground state of the fermionic MQM system has $N$ fermions filling the Fermi sea up to Fermi level $\mu_F$, which is determined by

$$\int dm \sqrt{2(\mu_F - V)} = \frac{\pi N}{\beta} = \pi k^2.$$ 

(1.15)

We define $\mu_c$ as the top of potential $V$: $\mu_c \equiv V(0) = \frac{1}{g_0\ell_s}$. The renormalized world-sheet cosmological constant is then $\lambda = \pi(k_c^2 - k^2)$.

It might appear that the continuum limit is simply $N \to \infty$. However, this would only select the lowest genus surface since the genus-$h$ partition function is weighted

$^3$The degrees-of-freedom of the matrix quantum mechanics becomes fermionic after taking into account the Vandermonde determinant from the path integral.
by $N^{2-2h}$. Evaluating $Z_{\text{discrete}}$ shows that it has a critical point at $\mu_F = \mu_c$, at which point the length scale diverges. This means that the average area of the Riemann surface diverges, which is equivalent to shrinking individual triangles to zero size while keeping the total area fixed — namely the continuum limit. Thus the correct continuum limit is the one that brings the system to the critical point $[30, 75, 120, 78]$: \footnote{Different parameterizations of MQM might lead to different formulations of the decoupling limit. We follow the conventions of [78].}

$$N \to \infty \quad \text{and} \quad \beta \to \infty \quad \text{with} \quad \frac{N}{\beta} = k^2 \to k_c^2,$$ \hfill (1.16)

with $k_c$ defined by (1.15) with $\mu_F$ replaced by the critical value $\mu_c$, i.e. $\mu_F \to \mu_c$ in this limit. The details of $V(x)$ is lost in this limit since only the immediate neighborhood around the critical point is relevant.

One can show that in this limit, genus-$h$ partition function is weighted by $\bar{\mu}^{2-2h}$, where $\bar{\mu} \equiv \beta(\mu_c - \mu_F)$. To include contributions from all topologies, the decoupling limit should be the double-scaling limit

$$N \to \infty \quad \text{and} \quad \beta \to \infty \quad \text{with} \quad \bar{\mu} \equiv \beta(\mu_c - \mu_F) \text{ fixed.} \quad (1.17)$$

Recall that $1/\beta$ plays the role of $\hbar$, hence gives spacing between energy levels. In the double-scaling limit, both the energy spacings and the distance between Fermi level $\mu_F$ and critical level $\mu_c$ vanish, with their ratio kept constant.

In the double-scaling limit, the effective closed string coupling $g_c$ emerges as

$$g_c = \frac{g_o}{\nu},$$ \hfill (1.18)

where $\nu$ is defined via $\lambda = -\nu \log \nu$. Thus the double-scaling limit can also be written
as
\[ g_0 \to 0 \quad \text{and} \quad \nu \to 0 \quad \text{with} \quad g_c = \frac{g_0}{\nu} \text{ fixed.} \quad (1.19) \]

The Gauge/Gravity duality generated by \( N \) D0-branes in 2D critical string theory states that \( SU(N) \) gauged matrix quantum mechanics in the double-scaling limit (1.19) is equivalent to two-dimensional critical string theory in the linear dilaton background. The matrix quantum mechanics is an integrable model since its eigenmodes are non-interacting. Hence many questions of 2D string theory are exactly solvable once phrased in the language of matrix quantum mechanics.

**Matrix theory formulation of M-theory**  In this duality, the supersymmetric matrix quantum mechanics with \( SU(N) \) gauge symmetry living on \( N \) coincident D0-branes in Type IIA theory in the infinite-momentum frame is conjectured to be dual to the Discrete Light-Cone Quantization (DLCQ) of M-theory [17, 141, 134]. It provides a non-perturbative description of M-theory in this background.

In Type IIA string theory, the world-volume theory on a stack of \( N \) coincident D0-branes is the ten-dimensional super Yang-Mills theory dimensionally reduced to 0+1 dimensions, plus higher-order corrections from DBI action. When lifted to M-theory, the near-horizon geometry sourced by this D0-brane configuration has M-theory circle \( R(r) \to R_s = g_s \ell_s \) [86, 128]. The \( N \) D0-brane bound state corresponds to a graviton in M-theory, with momentum \( p_{10} = N/R_s \). The eleven-dimensional Planck length is \( \ell_{11} = g_s^{1/3} \ell_s \).
We can boost the M-theory compactified on $R_s$-circle to a new frame
\[
\begin{pmatrix}
X^0 \\
X^{10}
\end{pmatrix}
\sim
\begin{pmatrix}
X^0 \\
X^{10} + 2\pi R_s
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
\tilde{X}^0 \\
\tilde{X}^{10}
\end{pmatrix}
\sim
\begin{pmatrix}
\tilde{X}^0 - 2\pi \frac{R_s}{\sqrt{2}} \\
\tilde{X}^{10} + 2\pi \sqrt{\frac{R^2}{2} + R_s^2}
\end{pmatrix}
\]
with boost parameter
\[
\beta = \frac{1}{\sqrt{1 + 2(R_s/R)^2}}.
\tag{1.20}
\]
The energy and momentum in the new frame are related to those in the old frame by
\[
\begin{pmatrix}
\tilde{E} \\
\tilde{p}^{10}
\end{pmatrix}
= \begin{pmatrix}
\frac{1}{\sqrt{1-\beta^2}} & -\frac{\beta}{\sqrt{1-\beta^2}} \\
-\frac{\beta}{\sqrt{1-\beta^2}} & \frac{1}{\sqrt{1-\beta^2}}
\end{pmatrix}
\begin{pmatrix}
E \\
p^{10}
\end{pmatrix}.
\]
Now consider the limit
\[
R_s \to 0 \quad \text{with} \quad \ell_{11} \text{ fixed},
\tag{1.21}
\]
which amounts to
\[
g_s \to 0 \quad \ell_s \to \infty \quad \text{with} \quad \ell_{11} = g_s^{1/3} \ell_s \text{ fixed}.
\tag{1.22}
\]
It brings the system into a “infinite momentum frame” since $p^{10} \to \infty$. The ten-dimensional Planck length $\ell_{10} \sim g_s^{1/4} \ell_s = (\ell_{11}/R_s)^{1/8} \to \infty$ in this limit, so the gravitational back-reaction to D0-branes can be neglected.

In the infinite momentum frame, the energy behaves as
\[
E = \sqrt{p_{10}^2 + E_{10}^2} \approx p_{10} + \frac{E_{10}^2}{2p_{10}} = p_{10} + \Delta E,
\tag{1.23}
\]
where $E_{10}$ is the energy in the ten-dimensional rest frame, and $\Delta E$ is the excitation energy relevant in M-theory and is related to the light-front energy in the new frame $\tilde{P}^{-} (\equiv \frac{E - \tilde{p}_{10}}{\sqrt{2}})$ by
\[
\Delta E = \frac{\sqrt{2}}{1 + \beta} \tilde{P}^{-} \approx \frac{R_s}{R} \tilde{P}^{-} = \sqrt{R_s \ell_{11}^3} \cdot \left( \frac{\tilde{P}^{-}}{R} \right) \cdot \left( \frac{1}{\ell_s} \right).
\tag{1.24}
\]
We can see that in the limit (1.21), although $\ell_s \to \infty$, but since $\Delta E \cdot \ell_s \to 0$, namely $\Delta E$ approaches zero faster than $1/\ell_s$ does, the limit (1.21) is actually a low-energy limit. Moreover, it is easy to show that interactions between open-string and closed-string sectors scale as $1/p_{10}$, hence are also suppressed in this limit.

In summary, the limit (1.21) is the decoupling limit in which the world-volume theory on D0-branes decouples from the closed-string sector and reduces into a supersymmetric matrix quantum mechanics with $SU(N)$ gauge symmetry.

In the M-theory side, since $\beta \to 1$, the boosting is at the speed of light and the resulting M-theory is compactified on a null circle with radius $R$, with compact momentum $\tilde{p}_{10} = \frac{N}{R}$. That is, the closed string sector in the decoupling limit (1.21) gives the sector of M-theory in the Discrete Light-Cone Quantization (DLCQ).

The Gauge/Gravity correspondence then states that the supersymmetric matrix quantum mechanics with $SU(N)$ gauge symmetry in the infinite-momentum frame is dual to DLCQ M-theory.

Depending on which of its various facets is emphasized, Gauge/Gravity is also termed “Boundary/Bulk correspondence” (gauge theory living on the boundary is dual to Gravity theory living in the bulk), “Open/Closed duality” (the D-brane excitation in the open string sector is dual to the gravity dynamics in the closed string sector), or “UV-IR correspondence” (the UV behavior in the open string sector is dual to the IR behavior of the closed string sector and vice versa).

A complete proof of the conjecture is hampered by the lack of a complete definition of string theory (in fact, it has been proposed that the non-perturbative completion of string theory should simply be defined by the Gauge/Gravity correspondence).
However, abundant evidence points to its validity, and no falsification of the conjecture has been found so far. In the present thesis, we will assume the validity of the conjecture, and examine its various implications and apply it to study the gravity side of the conjecture.

The present thesis will only consider the first two manifestations of Gauge/Gravity correspondence, namely, $AdS/CFT$ correspondence and MQM/2D-string duality. In particular, the first half of Chapter 2 and Chapter 5 apply $AdS_3/CFT_2$ correspondence to study black hole solutions with a $AdS_3$ factor in their horizons and gravity theory living in $AdS_3$; the second half of Chapter 2, Chapter 3 and Chapter 4 are related to the conjectured $AdS_2/CFT_1$ duality; finally, Chapter 6 are connected to MQM/2D string theory duality.

1.1.2 Black Hole Attractors

Apart from the big bang singularity, the black hole is the object where the tension between gravity and quantum physics is strongest, hence provides the best stage to test and unlock the full power of string theory.

Black holes were originally discovered as singular solutions of classical general relativity. Then it is realized that they obey thermodynamics laws. In particular, the temperature of a black hole is determined by the surface gravity of its horizon and its entropy is given by the area law: \({}^5\)

$$S = \frac{A}{4G_d \hbar}, \quad (1.25)$$

\(^5\)The area law receives higher-order correction in the presence of higher derivative terms in the action.
where $A$ is the area of its event horizon. The finite entropy signifies that a black hole has quantum states. How to count these micro-states to correctly reproduce the entropy computed macroscopically is a question all candidates for a theory of quantum gravity need to answer.

The area law (1.25) asserts that the number of degrees-of-freedom of a black hole is proportional to its horizon area, instead of its volume as required by all non-gravitational quantum field theories. This can be generalized to the “holographic principle” which states that the entropy of any gravitational system is bounded above as

$$S_{\text{max}} \leq \frac{A}{4G_d \hbar},$$

(1.26)

where $A$ is the horizon area of a black hole with mass $M$ equal to the total mass contained in the system. Hence a quantum field theory coupled to gravity has drastically fewer degrees-of-freedom than its non-gravitational version.

Another puzzle is the long-standing Information Paradox. The non-zero temperature of a black hole means that it will radiate off its entire mass via the so-called Hawking radiation. The semi-classical computation of Hawking radiation shows that it is a thermal radiation [84]. This raised the alarm that the information stored in the black hole might be lost during its evaporation — thus violating unitarity of quantum mechanics (barring the very unlikely possibility that the end-point of the evaporation is a planck-mass remnant that contains all the original information, or the even more radical one that unitarity of quantum mechanics simply breaks down in a gravitational system).

The successful microscopic description of black holes by string theory solved all
these puzzles. The crucial element is the AdS/CFT correspondence. For a supersymmetric black hole with an $AdS$ factor in its horizon geometry, there exists a dual description via a superconformal gauge theory living on the boundary of the $AdS$ space. One can count the micro-states in the weak-coupling region of the gauge theory. Supersymmetry then allows us to extrapolate the result to the strong-coupling region of the gauge theory, which is dual via AdS/CFT correspondence to the weak-coupling region of the string theory side. This is the region where the supergravity description is valid, thus one can use the above state-counting result to correctly reproduce the black hole entropy.

As mentioned earlier, the first microscopic counting of black hole entropy was performed by A. Strominger and C. Vafa in 1996 in the case of five-dimensional $\mathcal{N} = 4$ supersymmetric black hole created by D1-D5-P bound states in the Type IIB compactification on $K3 \times S^1$. Taking $S^1$ to be large, its near-horizon geometry is $AdS_3 \times S^3 \times K3$. Via AdS/CFT correspondence, this gravitational system is dual to a $\mathcal{N} = (4,4)$ two-dimensional conformal field theory living on the boundary of $AdS_3$. The black hole entropy is then computed by the growth of states in the dual two-dimensional conformal field theory via Cardy’s formula [140].

Since then, this success has been generalized to many other types of black holes, in particular, to small black holes which has vanishing horizon in Einstein gravity but acquires finite horizon size due to higher-derivative corrections from string theory, and to some extremal non-supersymmetric black holes and even to near-extremal ones.

The microscopic description of black holes via AdS/CFT correspondence also explains the “holographic principle” of black holes: the information in the bulk of
a black hole can be viewed as being stored in its horizon surface, since the bulk gravitational theory has a dual description in terms of a field theory living on the boundary.

This also simultaneously resolves the black hole information paradox. The dual description of a quantum gravity system in terms of a non-gravitational quantum field theory gives the final verdict that unitary must be preserved and no information will be lost. Hawking radiation as computed in string theory also shows that it is not thermal but instead contains the information of the black hole.

Black holes in string theory are usually coupled with moduli fields, which carry continuous values. The entropy of a black hole depends only on properties of its horizon. Therefore, the entropy of a black hole should depends on the horizon values of its moduli fields \( z^* \), along with other quantized charges \( \{Q\} \):

\[
S = S(Q, z^*) .
\]  

(1.27)

As one evolves the moduli equation of motion from infinity to horizon, naively, the horizon values \( z^* \) should depend on the asymptotic values \( z_0 \), which are continuous.

A zero-temperature (i.e. extremal) black hole is quantum mechanically stable due to absence of Hawking radiation, and its entropy takes discrete values. This means that the entropy of an extremal black hole cannot depend on the asymptotic values of its moduli fields, contrary to the above naive argument.

The puzzle is solved by the black hole “attractor mechanism”: at the horizon of an extremal black hole, the moduli are completely determined by the charges of the black hole, independent of their asymptotic values.

\[
z^* = z^*(Q) \quad \longrightarrow \quad S_{extremal} = S(Q, z^*(Q)) .
\]

(1.28)
Chapter 1: Introduction and Summary

It was first discovered in 1995 for supersymmetric black holes by S. Ferrara, R. Kallosh and A. Strominger [63] and then generalized to non-supersymmetric extremal ones in 2005 by A. Sen [135].

The attractor mechanism is a result of the near-horizon geometry of extremal black holes, rather than supersymmetry. The near-horizon geometry of an extremal black hole has an infinite throat, whereas the one for a non-extremal black hole only has a finite throat. As moduli fields evolve towards the horizon, the infinite throat of an extremal black hole allows them to lose the memory of their initial conditions, whereas the finite throat of a non-extremal black holes forces them to remember their initial conditions.

\[
\begin{align*}
\text{Extremal} & : \quad z^* = z^*(Q), \\
\text{Non-extremal} & : \quad z^* = z^*(Q, z_0). 
\end{align*}
\] (1.29)

Both supersymmetric and non-supersymmetric attractors in a given system correspond to critical points of a common black hole potential function.

Most parts of this thesis will consider zero-temperature black holes, namely black hole attractors, living in a supersymmetric string theory. In particular, the first half of Chapter 2 studies 5D supersymmetric black ring attractors and the second half of Chapter 2 and Chapter 3 studies 5D supersymmetric black holes. Chapter 4 constructs both supersymmetric and non-supersymmetric black hole attractors in four dimensions. The only exception is Chapter 5, which includes finite temperature BTZ black holes in bosonic gravity.
Chapter 1: Introduction and Summary

1.1.3 Outline

The plan of the thesis is as follows. The rest of Chapter 1 presents individual introduction to each chapter of main contents of the thesis, provides additional backgrounds to the ensuing discussions, and gives a brief summary of main contents.

Chapter 2 and Chapter 3 aim at a microscopic description of black objects in five dimensions. Chapter 2 studied the higher derivative corrections for black rings and spinning black holes in five dimensions. It focuses on certain $R^2$ terms found in Calabi-Yau compactification of M-theory. In the case of black rings, for which the microscopic origin of the entropy is generally known, it shows that the higher order macroscopic correction to the entropy matches a microscopic correction. For 5D spinning black holes in M-theory on a Calabi-Yau three-fold, while the microscopic origin of the entropy is unknown, the OSV relation allows us to successfully match the macroscopic correction to the entropy to the one computed from the correction to the topological string amplitudes.

Chapter 3 constructs probe brane configurations that preserve half of the enhanced near-horizon supersymmetry of five-dimensional spinning black holes, whose near-horizon geometry is squashed $AdS_2 \times S^3$. Supersymmetric zero-brane probes stabilized by orbital angular momentum on the $S^3$ are found and shown to saturate a BPS bound. We also find supersymmetric one-brane probes which have momentum and winding around a $U(1)_L \times U(1)_R$ torus in the $S^3$ and in some cases are static with respect to the global time coordinate of the $AdS_2$. Quantizing the moduli space of these classical probe solutions can then provide a microscopic description of 5D rotating black holes.
Chapter 1: Introduction and Summary

Chapter 4 constructs and analyzes generic single-centered and multi-centered black hole attractor solutions in a variety of four-dimensional models which, after Kaluza-Klein reduction, admit a description in terms of 3D gravity coupled to a sigma model whose target space is symmetric coset space. The solutions are in correspondence with certain nilpotent generators of the coset algebra. The non-BPS configurations are found to be drastically different from their BPS counterparts. For example, in $\mathcal{N} = 2$ supergravity coupled to one vector multiplet, the non-BPS single-centered attractor constrains all attractor flows with different asymptotic moduli to flow toward the attractor point along a common tangent direction. The non-BPS multi-centered attractors in these systems are found to enjoy complete freedom in the placement of attractor centers but suffer severe constraints on the allowed D-brane charges, in great contrast to their BPS counterpart,

Chapter 5 studies three dimensional topologically massive gravity (i.e. Einstein gravity deformed by the addition of a gravitational Chern-Simons term) with a negative cosmological constant in asymptotically $AdS_3$ spacetimes. It proves that the theory is unitary and stable only at a special value of Chern-Simons coupling. At this special point, the theory is found to be chiral: both the boundary $CFT$ and BTZ black hole becomes right-moving, and the bulk massive graviton degenerates with the left-moving massless graviton thus can be gauged away. This suggests the existence of a stable, consistent quantum gravity theory at the chiral point which is dual to a holomorphic boundary $CFT_2$.

Chapter 6 studies two-dimensional $\mathcal{N} = 1$ critical string theory with a linear dilaton background. It constructs time-dependent boundary state solutions that cor-
respond to D0-branes falling toward the Liouville wall. It also shows that there exist four types of stable, falling D0-branes (two branes and two anti-branes) in Type 0A projection and two unstable ones in Type 0B projection.

1.2 Higher Derivative Corrections for Black Objects in Five Dimensions

The ten years of spectacular success of string theory in microscopically describing certain supersymmetric black holes provides strong evidence that we are on the right track. However, the many unanswered questions are growing increasingly sharp with time. One such outstanding problem is to find the holographic dual of four-dimensional supersymmetric black holes with all D-brane charges present, thus establishing the $\text{AdS}_2/CFT_1$ correspondence. The two projects summarized in Chapter 2 and Chapter 3 aim at solving this problem. They both focus on five-dimensional black objects. These 5D black objects share many important features with their 4D cousins via the 4D/5D connection, descending from the IIA/M-theory duality. Moreover, the 5D spacetime hosts a much richer spectrum of black objects.

In four dimensions, the solution space of supersymmetric black holes is rather restricted: they all have spherical horizon $S^2$ and zero angular momentum. Moreover, they obey the “No-hair Theorem” (or “Black Hole Uniqueness Theorem”): a classical 4D black hole is characterized by only a few conserved charges, such as mass and electric/magnetic charges.

In five dimensions, the solution space of supersymmetric black objects is greatly
enlarged. First of all, there are two independent angular momenta in 5D. In supersymmetric black hole solutions with a horizon $S^3$, supersymmetry only restricts the two angular momenta to be equal, in contrast to the 4D case where it restricts the angular momentum to vanish. Therefore we can have spinning black holes in 5D (i.e. BMPV black holes) [26]. Secondly, the “No-hair Theorem” breaks down in higher dimensions. The 5D supersymmetric black ring solution has both a ring-like horizon $S^2 \times S^1$ and non-conserved dipole charges as its hairs [59]. Finally, one can form even more exotic solutions by combining these black holes and black rings concentrically [73]. This thesis will focus on 5D spinning black holes and black rings.

Since the higher derivative corrections are present in almost all string compactifications, its incorporation is essential to the precise description of black holes. Furthermore, crucial new physics often emerges only after the higher derivative corrections are included. Chapter 2 studies the higher derivative corrections in the context of the 5D supersymmetric spinning black holes and black rings, from both the supergravity side and the CFT side.

The area law (1.25) is valid only for Einstein gravity. Gravity theories coming from the low energy limit of string theory usually contain higher derivative corrections. In the presence of these higher curvature corrections, the area law needs to be replaced by Wald’s formula, which computes entropy as a Noether surface charge associated with the horizon Killing field in all gravitational systems with diffeomorphism invariance [146]:

$$S_{BH} = 2\pi \int_{Hor} d^{D-2}x \sqrt{h} \frac{\partial L}{\partial R_{\mu\nu\rho\sigma}} \epsilon_{\mu\nu} \epsilon_{\rho\sigma},$$

(1.30)

where $\epsilon_{\alpha\beta}$ is the binormal to the horizon. The integral is taken over an arbitrary cross
section Σ of the horizon.

We specifically considered the effects of following $R^2$ terms found in Calabi-Yau compactification of M-theory.

$$ R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2. $$

(1.31)

In four dimensions, this combination gives the Gauss-Bonnet term

$$ GB_{4D} = \frac{1}{2} \epsilon^{\mu\nu\mu'\nu'} \epsilon^{\rho\sigma\rho'\sigma'} R_{\mu\nu\rho\sigma} R_{\mu'\nu'\rho'\sigma'}. $$

(1.32)

which is a total derivative. In five dimensions, they do not form a total derivative, we will directly evaluate their effects using Wald’s formula.

In the case of black rings, for which the microscopic origin of the entropy is generally known, we found that the higher order macroscopic correction to the entropy matches a microscopic correction. For the 5D rotating black holes in M-theory on a Calabi-Yau three-fold, while the microscopic origin of the entropy is unknown, the OSV relation allowed us to successfully match the macroscopic correction to the entropy to the one computed from the correction to the topological string amplitudes.

1.3 Probe Moduli Space of Rotating Attractors

The near-horizon attractor geometry of a BPS black hole has twice as many supersymmetries as the full asymptotically flat solution. In four dimensions, such geometries admit BPS probe configurations which preserve half of the enhanced supersymmetry of the near-horizon $AdS_2 \times S^2 \times CY_3$ attractor geometry, but break all of the supersymmetries of the original asymptotically flat solution [137]. The quantization of these classical configurations gives rise to the superconformal quantum
mechanics system which is conjectured to be the holographic dual of the IIA string theory on $AdS_2 \times S^2 \times CY_3$ [70]. In particular, the supersymmetric black hole ground states are identified with the chiral primaries of this near-horizon superconformal quantum mechanics, which form the lowest Landau levels that tile the black hole horizon [68]. The counting of the degeneracy of the lowest Landau levels reproduces the Bekenstein-Hawking black hole entropy [69].

Furthermore, a novel feature of these probe brane configurations is that branes and anti-branes antipodally located on the $S^2$ preserve the same supersymmetries. In the dilute gas approximation, the black hole partition function is dominated by the sum over these chiral primary states [71]. An appropriate expansion thus yields a derivation of the OSV relation [119], with branes and anti-branes contributing to the holomorphic and anti-holomorphic parts of the partition function.

These interesting 4D phenomena should all have closely related 5D cousins [70]. In five dimensions, the generic supersymmetric black hole is the BMPV rotating black hole [26]. We are interested in the $\mathcal{N} = 2$ BMPV black hole, which can be constructed by wrapping M2-branes on the holomorphic two-cycles of the Calabi-Yau threefold. Unlike the BMPV black hole in $\mathcal{N} = 4$ and $\mathcal{N} = 8$ compactifications, whose holographic dual have been known for a while, the microscopic description of the $\mathcal{N} = 2$ BMPV black hole have been eluding our search. For some recent progress towards this goal, see [79, 87].

Chapter 3 extends the 4D classical BPS probe analysis of [137] to five dimensions. The 5D problem is considerably enriched by the fact that 5D BMPV BPS black holes can carry angular momentum $J$ and have a $U(1)_L \times SU(2)_R$ rotational isometry
group [26]. BPS zero-brane probes are constructed by wrapping the M2-brane on the holomorphic two-cycles of \( CY_3 \), and are found to orbit the \( S^3 \) using a \( \kappa \)-symmetry analysis. Their location in \( AdS_2 \) depends on the azimuthal angle on \( S^3 \), the background rotation \( J \), and the angular momentum of the probe. The BPS one-branes are constructed by wrapping M5-branes on the holomorphic four-cycles of \( CY_3 \). We find BPS configurations with momentum and winding around a torus generated by a \( U(1)_L \times U(1)_R \) rotational subgroup.\(^6\) A one-brane in five dimensions can carry the magnetic charge dual to the electric charge supporting the BMPV black hole. Interestingly, we find that this allows for static BPS “black ring” configurations, where the angular momentum required for saturation of the BPS bound is carried by the gauge field.

### 1.4 Non-Supersymmetric Attractors in String Theory

Chapter 4 constructs and analyzes the non-supersymmetric black hole attractors, both single-centered and multi-centered, in a large class of 4D \( \mathcal{N} = 2 \) supergravities coupled to vector-multiplets with cubic prepotentials.

Though more realistic than the supersymmetric (BPS) black holes, the non-supersymmetric (non-BPS) ones are far less understood microscopically, due to the absence of the non-renormalization theorem followed from supersymmetry. While the ultimate goal is to microscopically understand all non-BPS black holes using string

\(^{6}\)Inclusion of these states in the partition function of \([71]\) could lead to non-factorizing corrections to the OSV relation.
theory, our first step would be to tackle the extremal (i.e. zero-temperature) non-BPS black holes, since they possess the attractor mechanism which allows one to extrapolate the value of moduli from weak to strong coupling.

The attractor mechanism for supersymmetric (BPS) black holes was discovered in 1995 [63]: at the horizon of a supersymmetric black hole, the moduli are completely determined by the charges of the black hole, independent of their asymptotic values. In 2005, Sen showed that all extremal black holes, both supersymmetric and non-supersymmetric (non-BPS), exhibit attractor behavior [135]: it is a result of the near-horizon geometry of extremal black holes, rather than supersymmetry. Since then, non-BPS attractors have been a very active field of research (see for instance [76, 136, 145, 8, 92, 36, 131, 13, 91, 132, 11, 43, 113]). In particular, a microstate counting for certain non-BPS black holes was proposed in [42]. Moreover, a new extension of topological string theory was suggested to generalize the Ooguri-Strominger-Vafa (OSV) formula so that it also applies to non-supersymmetric black holes [133].

Both BPS and non-BPS attractor points are simply determined as the critical points of the black hole potential $V_{BH}$ [62, 92]. However, it is much easier to solve the full BPS attractor flow equations than to solve the non-BPS ones: the supersymmetry condition reduces the second-order equations of motion to first-order ones. Once the BPS attractor moduli are known in terms of D-brane charges, the full BPS attractor flow can be generated via a harmonic function procedure, i.e., by replacing the charges in the attractor moduli with corresponding harmonic functions:

$$t_{BPS}(x) = t_{BPS}^*(p^I \to H^I(x), q_l \to H_l(x)).$$  \hspace{1cm} (1.33)

In particular, when the harmonic functions $(H^I(x), H_l(x))$ are multi-centered, this
procedure generates multi-centered BPS solutions [18].

The existence of multi-centered BPS bound states is crucial in understanding the microscopic entropy counting of BPS black holes and the exact formulation of OSV formula [48]. One expects that a similarly important role should be played by multi-centered non-BPS solutions in understanding non-BPS black holes microscopically. However, the analytical non-BPS multi-centered black hole solutions with generic background dependence have been elusive — in contrast to the case of BPS attractor flows, the difficulty of solving the second-order non-BPS attractor equations makes the construction of generic non-BPS multi-centered attractor flows a highly non-trivial problem. In fact, even their existence has been in question.

In the BPS case, the construction of multi-centered attractor solutions is a simple generalization of the full attractor flows of single-centered black holes: one needs simply to replace the single-centered harmonic functions in a single-centered BPS flow with multi-centered harmonic functions. However, the full attractor flow of a generic single-centered non-BPS black hole has not been constructed analytically, owing again to the difficulty of solving second-order equations of motion. Ceresole et al. obtained an equivalent first-order equation for non-BPS attractors in terms of a “fake superpotential,” but the fake superpotential can only be explicitly constructed for special charges and asymptotic moduli [34, 98]. Similarly, the harmonic function procedure was only shown to apply to a special subclass of non-BPS black holes, but has not been proven for generic cases [91].

Aiming towards the construction of generic black hole attractors with arbitrary charges and asymptotic moduli, Chapter 4 develops a new framework to encompass
generic black hole attractor solutions, both BPS and non-BPS, single-centered as well as multi-centered, in all models for which the 3D moduli spaces obtained via $c^*$-map are symmetric coset spaces. All attractor solutions in such a 3D moduli space can be constructed algebraically in a unified way. Then the 3D attractor solutions are mapped back into four dimensions to give 4D extremal black holes.

The non-BPS configurations are found to be drastically different from their BPS counterparts. For example, in the particular model that we focused on — $\mathcal{N} = 2$ supergravity coupled to one vector multiplet, the non-BPS single-centered attractor constrains all attractor flows with different asymptotic moduli to flow toward the attractor point along a common tangent direction. And in great contrast to the BPS counterpart, the non-BPS multi-centered attractors in these systems are found to enjoy complete freedom in the placement of attractor centers but suffer severe constraints on the allowed D-brane charges. The constraint on the charges is expected to be released by allowing a coupling between the moduli fields and 3D gravity, thus generating 4D bound state solutions with non-zero angular momentum.

1.5 Chiral gravity in three dimensions

Chapter 5 concerns three dimensional topologically massive gravity (TMG), namely, three dimensional Einstein gravity modified by the addition of a gravitational Chern-Simons action.

Pure Einstein gravity in three dimensions is trivial classically. It has no local propagating degrees of freedom, as can be seen from a degrees-of-freedom counting. The degrees-of-freedom counting of Einstein gravity in D-dimensions receives contri-
butions from the spatial part of the metric and the corresponding momenta, minus the number of constraints from the diffeomorphism symmetry and Bianchi identities:

\[
\text{Spatial metric + Momenta} - \text{Diff} - \text{Bianchi}
= \frac{D(D-1)}{2} + \frac{D(D-1)}{2} - D - D
= D(D - 3)
\]

which vanishes at \( D = 3 \). Therefore there is no gravitational waves traveling in the bulk of 3D spacetime in Einstein gravity.

That the 3D pure Einstein gravity is classically trivial also manifests itself in the fact that 3D Riemann tensor is completely determined by the Ricci tensor:

\[
R_{\mu
u\rho\sigma} = g_{\mu\rho}R_{\nu\sigma} + g_{\nu\sigma}R_{\mu\rho} - g_{\nu\rho}R_{\mu\sigma} - g_{\mu\sigma}R_{\nu\rho} - \frac{1}{2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}R_{\nu\rho})R.
\]  

(1.34)

They both have six degrees of freedom. This means that all solutions of 3D pure Einstein gravity have constant sectional curvature.

One way to render the 3D pure Einstein gravity non-trivial is to quantize the theory. In fact, the existence of BTZ black holes [16] in 3D pure Einstein gravity already signifies that the theory is non-trivial quantum mechanically. In the presence of a negative cosmological constant \( \Lambda \), there exist asymptotically AdS\(_3\) black hole solutions [16] as well as massless gravitons which can be viewed as propagating on the boundary. These BTZ black holes obey the laws of black hole thermodynamics and have an entropy given by the area law. The microscopic origin of the black hole entropy in the classically trivial theory can only be understood in the full quantum version of the theory.
Three dimensional pure gravity can also be rendered non-trivial by adding a gravitational Chern-Simons term. In fact, the Chern-Simons term appears naturally during renormalization of quantum field theory in a three-dimensional gravitational background. It also arises in the compactification of string theory down to three dimensions. The resulting theory has one single massive, propagating graviton degree of freedom at generic Chern-Simons coupling, hence the name “topologically massive gravity” (TMG).\(^7\)

All solutions of Einstein gravity are automatically solutions of TMG. Moreover, with an extra degree of freedom, TMG allows more solutions than its Einstein gravity counterpart. For example, when $\Lambda = 0$, pure Einstein gravity allows only Minkowski spacetime and no black hole solution, whereas TMG has ACL black holes even at $\Lambda = 0$ \([109]\).\(^8\) At $\Lambda < 0$, all solutions of Einstein gravity are locally $AdS_3$; they are $AdS_3$ vacuum and BTZ black holes. In contrast, in addition to $AdS_3$ vacuum and BTZ black holes, TMG with $\Lambda < 0$ also allows Squashed $AdS_3$ solution and “NG/BC” black holes \([116, 82, 24]\).

In Chapter 5, we will focus on theories with negative cosmological constant, and consider only asymptotically $AdS_3$ spacetimes. This will allow us to employ the conjectured $AdS_3/CFT_2$ correspondence to study properties of the bulk theory. Theories with $\Lambda = 0$ is far less interesting than the one with $\Lambda < 0$, and theories with $\Lambda > 0$ has $dS_3$ space as its vacuum, which is metastable and does not have a globally conserved energy, and the conjectured $dS_3/CFT_2$ correspondence is not well understood enough

\(^7\)Note that the naive degrees-of-freedom counting no longer works for 3D TMG since there are now third-time-derivatives in the action.

\(^8\)ACL black hole is a type of 3D non-Einstein black hole solution living in TMG with zero cosmological constant.
to be of much use in understanding the bulk theory.

It is conjectured that all asymptotically $AdS_3$ spacetimes are also locally $AdS_3$, namely, they are $AdS_3$ vacuum solution and BTZ black holes, which belong to Einstein solutions of TMG. Then taking both BTZ black holes and massive gravitons propagating in the $AdS_3$ vacuum into account, we will show that TMG with $\Lambda < 0$ is unstable/inconsistent for generic Chern-Simons coupling: either the BTZ black hole or the massive gravitons would have negative energies. We then showed that the theory is only unitary and stable when the parameters obey: $\mu \ell = 1$. At this special point, the theory has several interesting features:

1. The central charges of the dual boundary $CFT_2$ become

   \[ c_L = 0 , \quad c_R = \frac{3\ell}{G} . \quad (1.35) \]

2. The conformal weights as well as the wave function of the massive graviton degenerate with those of the left-moving weight $(2, 0)$ massless boundary graviton. They are both pure gauge in the bulk, and the gauge transformation parameter does not vanish at infinity.

3. The mass of the massive graviton vanishes.

4. Both the massive graviton and the left-moving massless graviton have zero energy.

5. Both BTZ black holes and the right-moving massless graviton have non-negative energies.

---

9 The Chern-Simons coupling is $1/\mu$

10 Here $\ell$ is the radius of the $AdS_3$ vacuum solution.
6. BTZ black holes become right moving, namely, their mass and angular momentum obey

\[ J = \ell M. \]  

(1.36)

This suggests the existence of a stable, consistent quantum gravity theory at \( \mu \ell = 1 \) which is dual to a holomorphic boundary CFT (i.e. containing only right-moving degrees of freedom) with \( c_R = 3\ell/G \). We conjecture that for a suitable choice of boundary conditions, the zero-energy left-moving graviton excitations at \( \mu \ell = 1 \) can be discarded as pure gauge. We will refer to this theory as 3D chiral gravity. If such a dual CFT does exist and is unitary, an application of Cardy formula can then provide a microscopic derivation of the BTZ black hole entropy [139, 95, 96].

1.6 Time-dependent D-brane Solution in 2D Superstring.

The final chapter — chapter 6 — of this thesis discusses the time-dependent D-brane solutions in two-dimensional superstring theory.

The duality between matrix quantum mechanics and two-dimensional critical string theory in the linear dilaton background is one of many different manifestations of Gauge/Gravity correspondence. In this conjecture, the matrix quantum mechanics is dual to the Liouville field theory (LFT) coupled to the \( c = 1 \) matter field, i.e. the one-dimensional noncritical string theory in flat background. The latter is in turn equivalent to the two-dimensional critical string theory in the linear dilaton background once we identify the Liouville mode of LFT with the spatial dimension of the
critical string, and the worldsheet cosmological constant of the former theory with the amplitude of the tachyon field in the latter one.

In particular, the bosonic Liouville field theory coupled to $c = 1$ matter is dual to the matrix model with the inverse harmonic oscillator potential with matrix eigenvalues filled at only one side of the potential. In the supersymmetrized version of this duality, the $\mathcal{N} = 1$ supersymmetric Liouville field theory (SLFT) coupled to $\hat{c} = 1$ matter field, which is considered as the $\hat{c} = 1$ noncritical Type 0 string theory in the flat 2D target space, is dual to the matrix model with the same inverse harmonic oscillator potential but with matrix eigenvalues filled on both sides of the potential.

The purpose of Chapter 6 is to construct and study the time-dependent D0-brane solutions living in this two-dimensional $\mathcal{N} = 1, \hat{c} = 1$ noncritical Type 0 string theory. Below, we will first give a lightening review on the description of D-branes in terms of boundary states in the closed string sector.

The properties of a D-brane can be described by the boundary condition of the open string attached to it. According to the string Open/Close duality, the boundary condition of the open string corresponds to a certain state (boundary state) in the closed string Hilbert space. This boundary state then serves as a description of the corresponding D-brane in the closed string sector.

For a state in the closed string Hilbert space to be a boundary state, it must satisfy two conditions. The most basic constraint that the boundary states must satisfy comes from the requirement that the corresponding boundary vertex operator preserve the symmetries of the original theory. In the case of the bosonic Liouville field theory, this amounts to requiring conformal invariance which, in the language of
boundary states, translates to the constraints

\[(L_m - \tilde{L}_m)|\alpha\rangle = 0, \quad (1.37)\]

where \(|\alpha\rangle\) labels a boundary state, corresponding to a given boundary condition of open string. Similarly, in the \(\mathcal{N} = 1\) \(\hat{c}_m = 1\) supersymmetric Liouville field theory, a closed string boundary state must satisfy \[3\]

\[(L_m - \tilde{L}_m)|\alpha; \eta, \sigma\rangle = 0, \quad (1.38)\]

where the two additional indices \(\sigma = \text{NS} - \text{NS}\), \(\text{R} - \text{R}\) and \(\eta = \pm\) label the sector and spin structure of the boundary states.

More importantly, a boundary state must satisfy constraints coming from Open/Closed duality of a cylinder diagram. In the open string sector, a cylinder diagram is interpreted as the open string partition function

\[Z_{\alpha\beta}(\tau_o) = \text{Tr}_{H_{\alpha\beta}} [q_{\alpha}^{H_{\alpha}}], \quad (1.39)\]

where \(q_o = e^{2\pi i \tau_o}\) and \(\tau_o\) denotes the open string modulus. \(H_o\) is the open string Hamiltonian; and the trace is taken over the open string spectrum \(\mathcal{H}_{\alpha\beta}\) that satisfies the specified boundary conditions \(\alpha\) and \(\beta\) at the two ends of the open string.

In the string Open/Close duality, the same cylinder diagram can also be interpreted as a two point function in the closed string sector:

\[Z_{\alpha\beta}(\tau_c) = \langle \alpha | (q_{c}^{1/2})^{H_c} | \beta \rangle, \quad (1.40)\]

\[\text{Here we will use bosonic string to illustrate the main points. The computations in Chapter 6 concern superstring, which is slightly more complicated, with two more indices } \sigma \text{ and } \eta \text{ to consider.}\]
where \( q_c = e^{2\pi i \tau_c} \) with \( \tau_c \) denoting the closed string modulus; and \( H_c \) is the closed string Hamiltonian. It computes the evolution of the initial closed string state \( |\beta\rangle \) (which corresponds to the right boundary condition \( \beta \)) to the final closed string state \( \langle \alpha| \) (which corresponds to the left boundary condition \( \alpha \)). The open and closed string moduli are related through worldsheet duality by a modular transformation:

\[
\tau_c = -\frac{1}{\tau_o}.
\]  

(1.41)

The string Open/Close duality is realized through the modular invariance of the open string partition function:

\[
Z_{\alpha\beta}(\tau_o) = Z_{\alpha\beta}(\tau_c) \quad \Rightarrow \quad \text{Tr}_{\mathcal{H}_{\alpha\beta}}[q_o^{H_o}] = \langle \alpha| (q_c^{1/2})^{H_c} |\beta\rangle.
\]  

(1.42)

The above equation can be considered as the definition of boundary states \( |\alpha\rangle \) and \( |\beta\rangle \) — the boundary states are the closed string description of the corresponding D-branes.

Given a set of boundary conditions \( \alpha \) and \( \beta \) at the two ends of an open string, we can use the modular bootstrap to construct the corresponding boundary states \( |\alpha\rangle \) and \( |\beta\rangle \). Now we will review the procedure of modular bootstrap construction.\(^{12}\)

First we expand the boundary states with a set of orthonormal states called Ishibashi states \( |i\rangle \rangle \) \(^{89}\). They are defined to satisfy the additional constraints

\[
\langle i| q_c^{H_c} |j\rangle \rangle = \delta_{ij} \text{Tr}_{\mathcal{H}_i}[q_c^{H_o}] = \delta_{ij} \chi_i(\tau_c),
\]  

(1.43)

where \( i, j \) now label closed string conformal family. \( \mathcal{H}_i \) is spanned by the conformal family corresponding to representation \( i \) of the constraint algebra. \( \chi_i(\tau_c) \) is the char-

\(^{12}\)Again, we will focus on bosonic string here. The computation for \( N = 1, \hat{c}_m = 1 \) supersymmetric Liouville field theory can be found in Chapter 6.
acter of representation $i$ of Virasoro algebra with closed string modulus $\tau_c$. Thus a boundary state $|\alpha\rangle$ can be expanded by Ishibashi states:

$$|\alpha\rangle = \sum_i \Psi_\alpha(i)|i\rangle$$

and the problem of finding $|\alpha\rangle$ is translated into finding the wave function $\Psi_\alpha(i)$.

Expand the closed string two point function (1.40) with a set of Ishibashi states $|i\rangle$:

$$Z_{\alpha\beta}(\tau_c) = \langle \alpha| (q_c^{1/2})\mathcal{H}_c |\beta\rangle = \sum_i \Psi_\alpha^†(i)\Psi_\beta(i)\chi_i(\tau_c) .$$

On the other hand, it was shown by Cardy in [33] that the trace in the open string partition function (1.39) may be further rewritten as

$$Z_{\alpha\beta}(\tau_o) = \text{Tr}_{\mathcal{H}_{\alpha\beta}}[q_o^{\mathcal{H}_o}] = \sum_i n_{\alpha\beta}^i \chi_i(\tau_o) ,$$

where $\chi_i(\tau_o)$ is the character of representation $i$ of Virasoro algebra with open string modulus $\tau_o$. The non-negative integer $n_{\alpha\beta}^i$ counts multiplicity of $\mathcal{H}_i$ in $\mathcal{H}_{\alpha\beta}$, and it depends on the boundary conditions $\alpha, \beta$ at the two ends of the open string. That is, the dependence of open string partition function $Z_{\alpha\beta}(\tau_o)$ on the boundary conditions $\alpha, \beta$ of the open string is only through $n_{\alpha\beta}^i$.

Expressions (1.45) and (1.46) are equal due to Open/Close duality. Therefore, using the modular transformation to relate $\chi(\tau_c)$ and $\chi(\tau_o)$, we can determine the “wave functions” $\Psi_\alpha(i)$, after fixing the additional freedom by noting that these wave functions are one-point functions on the disk and have specific transformation properties under reflection [65]. This is what is known as the modular bootstrap construction. As will be shown in Chapter 6 using the supersymmetrized version of the modular bootstrap construction, one can derive boundary state solutions that
correspond to falling D-branes $\mathcal{N} = 1$, 2D superstring theory in the linear dilaton background.

In the bosonic 2D string in the linear dilaton background with Euclidean time, Lukyanov, Vitchev, and Zamolodchikov showed the existence of a time-dependent boundary state, the so-called paperclip brane. This paperclip brane breaks into two hairpin-shaped branes in the UV region [100]. Under the Wick-rotation from Euclidean time into Minkowski time, the hairpin brane is reinterpreted as the falling D0-brane.

We will show that in $\mathcal{N} = 1$, 2D superstring theory with a linear dilaton background — which we will use interchangeably with $\hat{c}_m = 1 \mathcal{N} = 1$ SLFT — there exists a similar, time-dependent boundary state corresponding to the falling D0-brane. The naive argument for the existence of the falling D0-brane is as follows. Since the mass of the D0-brane is inversely related to the string coupling as

$$m = e^{-\phi},$$

the mass of the D0-brane decreases as it runs along the Liouville direction from the weak coupling region ($\phi \to -\infty$) to the strong coupling region ($\phi \to +\infty$). Thus, if we set a D0-brane free at the weak coupling region, it will roll along the Liouville direction towards the strong coupling region until it is reflected back by the boundary Liouville potential. This is the falling D0-brane solution which can be described by a time-dependent closed string boundary state of the $\mathcal{N} = 1$, 2D superstring.

In the bosonic case, the hairpin brane satisfies symmetries in addition to those of the action (conformal symmetry). The additional symmetry is known as the $\mathcal{W}$-symmetry and is generated by higher spin currents [100]. The hairpin brane is then
constructed from the integral equations that are defined by the $\mathcal{W}$-symmetry. In the $\mathcal{N} = 1$, 2D superstring, it should be possible to use the supersymmeterized version of the $\mathcal{W}$-symmetry to go through a similar construction and find a falling D0-brane. However, we will argue that it can also be obtained by adapting the falling D0-brane solution in $\mathcal{N} = 2$ SLFT [112, 58], to the $\mathcal{N} = 1$, 2D superstring.

We will also show that there exist four types of stable, falling D0-branes (two branes and two anti-branes) in the Type 0A projection and two unstable ones in the Type 0B projection. Type 0, $\mathcal{N} = 1$, 2D superstring theory has a dual description in the language of matrix models. An interesting question then would be to understand these falling D0-branes in the context of the dual matrix model.
Chapter 2

$R^2$ Corrections for 5D Black Holes and Rings

2.1 Introduction

Recently a surprisingly powerful and precise relationship has emerged between higher dimension F-terms in the 4D effective action for $\mathcal{N} = 2$ string theory (as captured by the topological string [23]) and the (indexed) BPS black hole degeneracies [99, 119]. Even more recently [70] a precise relationship has been conjectured between the 4D and 5D BPS black hole degeneracies. This suggests that there should be a direct relationship between higher dimension terms in the 5D effective action and 5D degeneracies which does not employ four dimensions as an intermediate step. Five dimensions is in many ways simpler than four so such a relation would be of great interest. It is the purpose of this work to investigate this issue.

The 4D story benefitted from a well understood superspace formulation [108, 45].
Chapter 2: $R^2$ Corrections for 5D Black Holes and Rings

The relevant supersymmetry-protected terms are integrals of chiral superfields over half of superspace and can be classified. In 5D the situation is quite different (see e.g. [22]). There is no superfield formulation and we do not have a general understanding of the possible supersymmetry-protected terms. In general, the uplift to 5D of most of the 4D F-terms vanishes. However, the area law cannot be the exact answer for the black hole entropy (for one thing it doesn’t give integer numbers of microstates!) so there must be some kind of perturbative supergravity corrections.

As a first step towards a more general understanding, in this work we will study the leading order entropy correction arising from $R^2$ terms, which are proportional to the 4D Euler density. Such terms give the one loop corrections in 4D, and — unlike the higher order terms — do not vanish upon uplift to 5D. They are also of special interest as descendants of the interesting 11D $R^4$ terms [77, 93]. These terms correct the entropy of the both the 5D black ring [59] and the 5D BMPV spinning black hole [26]. We find that the macroscopic black ring correction matches, including the numerical coefficient, a correction expected from the microscopic analysis of [41]. For the BMPV black hole, we find the correction matches, to leading order, one expected from the 4D-5D relation conjectured in [70].

The next section derives the $R^2$ corrections to the 5D entropy as horizon integrals of curvature components using Wald’s formula. Section 2.3 evaluates this formula for the black ring, while section 2.4 evaluates it for BMPV. Section 2.5 contains a brief summary.
2.2 Wald’s Formula in 5D

In this section we will use Wald’s formula to derive an expression for $R^2$ corrections to the 5D entropy.

The Einstein-frame low energy effective action for the compactification of M-theory on a Calabi-Yau threefold $CY_3$ down to five dimensions contains the terms [12]

$$I_0 + \Delta I = -\frac{1}{32\pi^2} \int d^5x \sqrt{|g_5|} R^{(5)} - \frac{1}{2^9 \cdot 3\pi^2} \int d^5x \sqrt{|g_5|} c_{2A} Y^A (R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} - 4R_{\alpha\beta} R^{\alpha\beta} + R^2)$$

in units in which $G_5 = 2\pi$ (for compactification on a circle of unit radius, this choice leads to $G_4 = 1$ and hence facilitates 4D/5D comparisons). Here $Y^A, A = 1, \ldots, n_V$ are scalar components of vector multiplets. They are proportional to the Kähler moduli of $CY_3$, normalized so that

$$D_{ABC} Y^A Y^B Y^C = 1.$$  

(2.2)

c_{2A}$ are the components of the second Chern class of $CY_3$ and $D_{ABC}$ the corresponding intersection numbers. The $R^2$ term in $\Delta I$ arises from dimensional reduction of the much studied $R^4$ term [77, 93] in eleven dimensions. It is also the uplift from four dimensions of an $F$ term whose coefficient is computed by the $N = 2$ topological string on $CY_3$ at one loop order [23].

When we add $R^2$ corrections to the action the entropy is no longer given by the area law; instead, we need to use the more general formula found by Wald [146]

$$S_{BH} = 2\pi \int_{Hor} d^3x \sqrt{h} \frac{\partial L}{\partial R_{\mu\nu\rho\sigma}} \epsilon_{\mu\nu} \epsilon_{\rho\sigma}$$

(2.3)
where, $\epsilon_{\alpha\beta}$ is the binormal to the horizon, defined as the exterior product of two null vectors normal to the horizon and normalized so that $\epsilon_{\alpha\beta}\epsilon^{\alpha\beta} = -2$. We can then identify two types of first-order corrections implied by this formula:

- modifications to the area law due to the additional terms in the action — these terms are evaluated using the zeroth order solutions for the metric and the other fields.

- modification of the area due to the change of the metric on the horizon, which follows from the fact that adding extra terms to the action may change the equations of motion.

In 4D, the second type of modification is absent at leading order for this particular $R^2$ form of $\Delta I$ obtained by reduction of (2.1) [104]. This and the 4D-5D agreement we find to leading order suggest that this may be the case in 5D as well. In order to understand all $R^2$ corrections to the entropy this should be ascertained by direct calculation. In the following we consider only the first type of modification.

The corresponding correction to the entropy is then (see also [110])

$$\Delta S = -\frac{4\pi c_2 \cdot Y}{2^9 \cdot 3\pi^2} \int_{H_{\text{hor}}} d^3 x \sqrt{h} \left( R_{\mu\nu\rho\sigma} \epsilon^{\mu\nu} \epsilon^{\rho\sigma} - 4 R_{\mu\rho} g_{\nu\sigma} \epsilon^{\mu\nu} \epsilon^{\rho\sigma} + R \epsilon_{\mu\nu} \epsilon^{\mu\nu} \right), \quad (2.4)$$

where $h$ is the induced metric on the horizon and the moduli are fixed at their attractor values. In the following we will evaluate this correction for the spinning black hole and black ring solutions.
2.3 Black Ring

The black ring solution was discovered in [59] and its entropy understood from a microscopic perspective in [41]. It represents a supersymmetric solution to 5D supergravity coupled to a number of abelian vector (and hyper)multiplets that describes a charged, rotating black ring. It is characterized by electric charges $q_A$, magnetic dipole charges $p^A$, and the angular momentum around the ring, $J_\psi$. The macroscopic entropy formula for the black ring can be written in the suggestive form

$$S_{BR} = 2\pi \sqrt{\frac{c_L \hat{q}_0}{6}}$$

(2.5)

where, in terms of the macroscopic charges,

$$c_L = 6D = 6D_{ABC}p^Ap^Bp^C,$$

(2.6)

with $D_{ABC}$ being (one sixth) the intersection numbers of the Calabi-Yau, and

$$\hat{q}_0 = -J_\psi + \frac{1}{12}D^{AB}q_Aq_B + \frac{c_L}{24},$$

(2.7)

where $D^{AB}$ is the inverse of $D_{AB} \equiv D_{ABC}p^C$. The microscopic origin of the entropy is from the quantum degeneracy of a 2D CFT with central charge $c_L$ and left-moving momentum $\hat{q}_0$ available for distribution among the oscillators. The last term in (2.7) is ascribed to the left moving zero point energy.

2.3.1 Macroscopic entropy correction

Now we evaluate the correction to the black ring entropy induced by $\Delta I$. Due to the 5D attractor mechanism [90] the moduli take the horizon values

$$Y^A = \frac{p^A}{D^3}.$$

(2.8)
Next, all we need to do is to find the binormal to the horizon for the black ring metric, evaluate the relevant curvature terms at the horizon, and integrate. We obtain
\[ \Delta S_{BR} = \frac{\pi}{6} c_2 \cdot p \sqrt{\frac{q_0}{D}} . \] (2.9)

### 2.3.2 Microscopic entropy correction

The microscopic entropy comes from M5 branes wrapping 4-cycles associated to \( p^A \) in \( CY_3 \). As shown in \([104]\), these are described by a CFT with the left-moving central charge
\[ c_L = 6D + c_2 \cdot p . \] (2.10)

In \([41]\) the leading entropy at large charges was microscopically computed using the leading approximation (2.6) to \( c_L \) at large charges. Subleading modifications should arise from using the exact formula (2.10) in (2.5). This leads to
\[ \Delta S_{BR} = \frac{\pi}{6} c_2 \cdot p \sqrt{\frac{q_0}{D}} + \frac{\pi}{24} c_2 \cdot p \sqrt{\frac{D}{q_0}} + \ldots \] (2.11)

The first term comes from correcting \( c_L \) in (2.5), while the second comes from correcting the zero point shift in (2.7). We see that the macroscopic \( R^2 \) correction matches

\[ 1 \]

Note that in the following we have employed the following relationships between various quantities used in this work, in \([59]\) and in \([26]\):
\[ Q_{emr} = (16\pi G)^{\frac{3}{2}} \mu_{bmpv} = \left( \frac{4G}{\pi} \right)^{\frac{3}{2}} q , \]
\[ q_{emr} = \left( \frac{4G}{\pi} \right)^{\frac{1}{2}} p , \]
\[ J_{emr} = 16\pi J_{bmpv} = 4\pi^2 \mu \omega . \]

The value of Newton’s constant used in \([26]\) is \( G_5 = (16\pi)^{-1} \), so we needed to rescale their metric by \( (16\pi G)^{\frac{3}{2}} \) in order to get ours. Also recall we are setting \( G = 2\pi \) in the text.
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precisely the first term. We do not understand the matching of the second term, but note that it is subleading in the regime \( \hat{q}_0 \gg D \) where Cardy’s formula is valid.

2.4 BMPV Black Hole

Let us now turn now to BMPV — the charged rotating black hole in 5D characterized by electric charges \( q_A \) and angular momenta \( J \) in \( SU(2)_L \). Its leading macroscopic entropy is given by

\[
S_{BMPV} = 2\pi \sqrt{Q^3 - J^2} \tag{2.12}
\]

where

\[
Q^2 = D_{ABC}y^A y^B y^C \tag{2.13}
\]

where the \( y^A \)'s are determined from

\[
q_A = 3D_{ABC}y^B y^C . \tag{2.14}
\]

We find the correction to this entropy following from the application of Wald’s formula to (2.1) to be

\[
\Delta S_{BMPV} = -\frac{\pi}{24} \sqrt{Q^3 - J^2} c_2 \cdot Y \left( \frac{3}{Q} - \frac{J^2}{Q^4} \right) = \frac{\pi}{6} A c_2 \cdot Y \left( \frac{1}{Q} - \frac{A^2}{4Q^4} \right) , \tag{2.15}
\]

where we defined \( A = \sqrt{Q^3 - J^2} \) and the moduli fields take the horizon values\(^2\)

\[
Y^A = \frac{y^A}{Q^2} . \tag{2.16}
\]

In general, the microscopic origin of the entropy for the 5D spinning black holes in M-theory on \( CY_3 \) (unlike for black rings) is not known,\(^3\) so we will not try herein

\(^2\)From now on we will take \( Y^A \) to mean the horizon value of the modulus field \( Y^A \).

\(^3\)It is of course known for \( \mathcal{N} = 4 \) compactifications \([140, 26]\), so it would be interesting to interpret the macroscopic correction for that case.
to understand the microscopic origin of $\Delta S_{BMPV}$. We will however compare it to corresponding corrections in 4D and the topological string partition function. As argued in [70], the exact 5D BMPV entropy is equal to the entropy of the D6-D2-D0 system in 4D, with the same 2-brane charges $q_A$, D6-brane charge $p^0 = 1$, and D0-brane charge $q_0 = 2J$. In the same paper, the following relationship for the partition functions of 5D black holes, 4D black holes and consequently of the topological string — see [119] — was conjectured

$$Z_{5D}(\phi^A, \mu) = Z_{4D}(\phi^A, \phi^0) = \left| Z_{\text{top}} \left( g_{\text{top}} = \frac{8\pi^2}{\mu}, t^A = \frac{2\phi^A}{\mu} \right) \right|^2,$$  \hspace{1cm} (2.17)

where $\phi^A$ are the electric potentials conjugate to $q_A$, while $\phi^0$ is conjugate to $q_0$ in 4D, and $\mu$ to $J$ in 5D. The absolute value in the last expression is defined by keeping $\phi^0$ real. With this in mind, we can start from $F_{\text{top}}$ — the topological string amplitude — and compute the entropy of the BMPV (including first order corrections) as follows.

Up to one-loop order $F_{\text{top}}$ is

$$F_{\text{top}} = \frac{i(2\pi)^3}{g_{\text{top}}^2} D_{ABC} t^A t^B t^C - \frac{i\pi}{12} c_{2A} t^A$$

$$= \frac{i}{\pi} D_{ABC} \phi^A \phi^B \phi^C - \frac{i\pi}{6} c_{2A} \phi^A \frac{\mu}{\mu}.$$  \hspace{1cm} (2.18)

The entropy of the black hole is given by the Legendre transform of

$$\mathcal{F}(\phi^A, \text{Re}\mu) = \ln Z_{BH} = F_{\text{top}} + \bar{F}_{\text{top}}.$$  \hspace{1cm} (2.19)

To first order we have

$$\mathcal{F} = -\frac{1}{\pi^2} D_{ABC} \phi^A \phi^B \phi^C - \frac{\pi^2}{6} c_{2A} \phi^A \frac{\text{Re}\mu}{(\frac{\text{Re}\mu}{2\pi})^2 + 1},$$  \hspace{1cm} (2.20)
which gives

\[ q_A = \frac{1}{\pi^2} \frac{3D_{ABC}\phi^B\phi^C - \frac{\pi^2}{6}c_{2A}}{(\frac{Re\mu}{2\pi})^2 + 1} \]  \hspace{1cm} (2.21)

\[ J = -\frac{Re\mu}{2\pi^4} \frac{D_{ABC}\phi^A\phi^B\phi^C - \frac{\pi^2}{6}c_{2A}\phi^A}{((\frac{Re\mu}{2\pi})^2 + 1)^2} \]

and therefore

\[ S = 2\pi\sqrt{Q^3 - J^2}(1 + \frac{1}{12} \frac{c_{2A}Y^A}{Q} + \ldots) \]  \hspace{1cm} (2.22)

where the \ldots stand for higher order corrections in \( |g_{top}|^2 = 16\pi^2A^2/Q^3 \).

We see that to the 5D \( R^2 \) corrections (2.15) to the entropy do not exactly match the 4D corrections (2.22). This is possible of course because dimensional reduction of the 5D \( R^2 \) gives the 4D \( R^2 \) term plus more terms involving 4D field strengths. However we also see that the mismatch is subleading in the expansion in \( g_{top} \), and we can therefore conclude that the 5D \( R^2 \) term captures the subleading correction to the area law.

### 2.5 Summary

We have shown that higher dimension corrections to the 5D effective action do give corrections to the black hole/black ring entropy just as in 4D, but that the 5D situation is currently under much less control than the 4D one. Some leading order computations were performed and found to give a partial match between macroscopic and microscopic results. We hope these computations will provide useful data for finishing the 5D macro/micro story.
Chapter 3

Supersymmetric Probes in a Rotating 5D Attractor

3.1 Introduction

The near-horizon attractor geometry of a BPS black hole has twice as many supersymmetries as the full asymptotically flat solution. In four dimensions, such geometries admit BPS probe configurations which preserve half of the enhanced supersymmetry of the near-horizon $AdS_2 \times S^2 \times CY_3$ attractor geometry, but break all of the supersymmetries of the original asymptotically flat solution [137]. The quantization of these classical configurations gives rise to the superconformal quantum mechanics system which is conjectured to be the holographic dual of the IIA string theory on $AdS_2 \times S^2 \times CY_3$ [70]. In particular, the supersymmetric black hole ground states are identified with the chiral primaries of this near-horizon superconformal quantum mechanics, which form the lowest Landau levels that tile the black hole.
horizon [68]. The counting of the degeneracy of the lowest Landau levels reproduces
the Bekenstein-Hawking black hole entropy [69].

Furthermore, a novel feature of these probe brane configurations is that branes
and anti-branes antipodally located on the $S^2$ preserve the same supersymmetries.
In the dilute gas approximation, the black hole partition function is dominated by
the sum over these chiral primary states [71]. An appropriate expansion thus yields
a derivation of the OSV relation [119], with branes and anti-branes contributing to
the holomorphic and anti-holomorphic parts of the partition function.

These interesting 4D phenomena should all have closely related 5D cousins [70].
In five dimensions, the generic supersymmetric black hole is the BMPV rotating
black hole [26]. We are interested in the $\mathcal{N} = 2$ BMPV black hole, which can be
constructed by wrapping M2-branes on the holomorphic two-cycles of the Calabi-Yau
threefold. Unlike the BMPV black hole in $\mathcal{N} = 4$ and $\mathcal{N} = 8$ compactifications,
whose holographic dual has been known for a while, the microscopic description of
the $\mathcal{N} = 2$ BMPV black hole has been eluding our search. For some recent progress
towards this goal, see [79, 87].

The present paper extends the 4D classical BPS probe analysis of [137] to five
dimensions. The 5D problem is considerably enriched by the fact that 5D BMPV BPS
black holes can carry angular momentum $J$ and have a $U(1)_L \times SU(2)_R$ rotational
isometry group [26]. BPS zero-brane probes are constructed by wrapping the M2-
brane on the holomorphic two-cycles of $CY_3$, and are found to orbit the $S^3$ using
a $\kappa$-symmetry analysis. Their location in $AdS_2$ depends on the azimuthal angle on
$S^3$, the background rotation $J$, and the angular momentum of the probe. The BPS
one-branes are constructed by wrapping M5-branes on the holomorphic four-cycles of $CY_3$. We find BPS configurations with momentum and winding around a torus generated by a $U(1)_L \times U(1)_R$ rotational subgroup.\footnote{Inclusion of these states in the partition function of [71] could lead to non-factorizing corrections to the OSV relation.} A one-brane in five dimensions can carry the magnetic charge dual to the electric charge supporting the BMPV black hole. Interestingly, we find that this allows for static BPS “black ring” configurations, where the angular momentum required for saturation of the BPS bound is carried by the gauge field.

### 3.2 Review of the BMPV Black Hole

The generic five-dimensional $\mathcal{N} = 2$ supersymmetric rotating black hole arises from M2-branes wrapping holomorphic two-cycles of a Calabi-Yau threefold $X$. It is characterized by electric charges $q_A$, $A = 1, 2, \ldots, b_2(X)$, and the angular momentum $J$ in $SU(2)_{\text{left}}$. The metric is [26]

$$ds^2 = -\left(1 + \frac{Q}{r^2}\right)^{-2} \left[dt + \frac{J}{2r^2} \sigma_3\right]^2 + \left(1 + \frac{Q}{r^2}\right) \left(dr^2 + r^2 d\Omega_3^2\right), \quad (3.1)$$

$$d\Omega_3^2 = \frac{1}{4} \left[ d\theta^2 + d\phi^2 + d\psi^2 + 2 \cos \theta d\psi d\phi \right] = \frac{1}{4} \sum_{i=1}^{3} (\sigma_i)^2, \quad (3.2)$$

where the ranges of the angular parameters are

$$\theta \in [0, \pi], \quad \phi \in [0, 2\pi], \quad \psi \in [0, 4\pi]. \quad (3.3)$$
\( \sigma_i \) are the right-invariant one-forms:\(^2\)

\[
\begin{align*}
\sigma_1 &= -\sin \psi d\theta + \cos \psi \sin \theta d\phi , \\
\sigma_2 &= \cos \psi d\theta + \sin \psi \sin \theta d\phi , \\
\sigma_3 &= d\psi + \cos \theta d\phi ,
\end{align*}
\]

(3.4)

and we choose Planck units \( l_5 = \left( \frac{4G_5}{\pi} \right)^{1/3} = 1 \). The graviphoton charge \( Q \) is determined via the equations

\[
Q^2 = D_{ABC} y^A y^B y^C ,
\]

(3.5)

\[
q_A = 3D_{ABC} y^B y^C ;
\]

(3.6)

with \( D_{ABC} \) the intersection form on \( X \).

The near-horizon limit \( (r \to 0) \) of the metric is

\[
ds^2 = - \left[ \frac{r^2}{Q} dt + \frac{J}{2Q} \sigma_3 \right]^2 + Q \frac{dr^2}{r^2} + Q d\Omega^2_3 .
\]

(3.7)

Rescaling \( t \) to absorb \( Q \), defining \( \sin^2 B = \frac{J^2}{Q^2} \) and \( r^2 = 1/\sigma \), we obtain the metric in Poincaré coordinates:

\[
ds^2 = \frac{Q}{4} \left[ -\frac{dt}{\sigma} + \sin B \sigma_3 \right]^2 + \frac{d\sigma^2}{\sigma^2} + \frac{\sigma_1}{\sigma^2} + \frac{\sigma_2}{\sigma^2} + \frac{\sigma_3}{\sigma} \right].
\]

(3.8)

The graviphoton field strength in these coordinates is

\[
F_{[2]} = dA_{[1]} , \quad A_{[1]} = \frac{\sqrt{Q}}{2} \left[ \frac{1}{\sigma} dt + \sin B \sigma_3 \right] .
\]

(3.9)

\(^2\)The \( SU(2) \) rotation matrix is parameterized as:

\[
e^{i \frac{2\pi}{7} \psi} e^{i \frac{2\pi}{7} \phi} e^{i \frac{2\pi}{7} \phi} = \begin{pmatrix}
\cos \frac{\theta}{2} e^{i(\psi+\phi)/2} & \sin \frac{\theta}{2} e^{i(\psi-\phi)/2} \\
-\sin \frac{\theta}{2} e^{-i(\psi-\phi)/2} & \cos \frac{\theta}{2} e^{-i(\psi+\phi)/2}
\end{pmatrix}.
\]
It was shown in [47] that the black hole entropy is given by the quantum eigenstates with respect to the global, rather than the Poincaré, time of the near horizon $AdS_2$. Therefore, we will mostly work in the global coordinates $(\tau, \chi, \theta, \phi, \psi)$. Using the coordinate transformation between the global coordinates and Poincaré ones:

\[
\begin{align*}
t &= \frac{\cos B \cosh \chi \sin \tau}{\cosh \chi \cos \tau + \sinh \chi}, \\
\sigma &= \frac{1}{\cosh \chi \cos \tau + \sinh \chi}, \\
\psi_{\text{Poincaré}} &= \psi_{\text{global}} + 2 \tan B \tanh^{-1} \left( e^{-\chi} \tan \frac{\tau}{2} \right),
\end{align*}
\]

we obtain the metric in the global coordinates:

\[
ds^2 = \frac{Q}{4} \left[ -\cosh^2 \chi d\tau^2 + d\chi^2 + (\sin B \sinh \chi d\tau - \cos B \sigma_3)^2 + \sigma_1^2 + \sigma_2^2 \right], \quad (3.10)
\]

in which

\[
A_{[1]} = \frac{\sqrt{Q}}{2} [\cos B \sinh \chi d\tau + \sin B \sigma_3]. \quad (3.11)
\]

The near horizon geometry of the BMPV black hole is a kind of squashed $AdS_2 \times S^3$. The near-horizon isometry supergroup is $SU(1,1|2) \times U(1)_{\text{left}}$, where the bosonic subgroup of $SU(1,1|2)$ is $SU(1,1) \times SU(2)_{\text{right}}$ [74]. When $J = 0$, $U(1)_{\text{left}}$ is promoted to $SU(2)_{\text{left}}$ and the full $SO(4) \cong SU(2)_{\text{right}} \times SU(2)_{\text{left}}$ rotational invariance is restored. The unbroken rotational symmetries for $J \neq 0$ are generated by the Killing vectors

\[
\xi_3^L = \partial_\psi, \quad (3.12)
\]
and

\[
\xi_1^R = \sin \phi \partial_\theta + \cos \phi (\cot \theta \partial_\phi - \csc \theta \partial_\psi), \\
\xi_2^R = \cos \phi \partial_\theta - \sin \phi (\cot \theta \partial_\phi - \csc \theta \partial_\psi), \\
\xi_3^R = \partial_\phi.
\] (3.13)

The supersymmetries arise from Killing spinors $\epsilon$ which are the solutions of the equation

\[
\left[ d + \frac{1}{4} \omega_{ab} \Gamma^{ab} + \frac{i}{8} \left( e^a \Gamma^{bc} F_{bc} - 4 e^a \Gamma^{ab} F_{ab} \right) \right] \epsilon = 0.
\] (3.14)

To solve this in global coordinates we choose the vielbein

\[
e^0 = \sqrt{Q} \left[ \cosh (\sin B \cos B \psi) \cosh \chi d\tau + \sinh (\sin B \cos B \psi) d\chi \right], \\
e^1 = \sqrt{Q} \left[ \sinh (\sin B \cos B \psi) \cosh \chi d\tau + \cosh (\sin B \cos B \psi) d\chi \right], \\
e^2 = \sqrt{Q} \left[ - \sin (\cos^2 B \psi) d\theta + \cos (\cos^2 B \psi) \sin \theta d\phi \right], \\
e^3 = \sqrt{Q} \left[ \cos (\cos^2 B \psi) d\theta + \sin (\cos^2 B \psi) \sin \theta d\phi \right], \\
e^4 = \sqrt{Q} \left[ - \sin B \sinh \chi d\tau + \cos B \sigma_3 \right].
\] (3.15)

The Killing spinors are then [9, 10]

\[
\epsilon = e^{\left[ - \frac{1}{2} (\sin B \cos B \Gamma^{01} + \cos^2 B \Gamma^{23}) \psi \right]} e^{\left[ + \frac{1}{4} (\cos B \Gamma^{24} + i \sin B \Gamma^{2}) \theta \right]} e^{\left[ - \frac{1}{2} (\cos B \Gamma^{34} + i \sin B \Gamma^{3}) \phi \right]} \\
e^{\left[ + \frac{1}{2} (\sin B \Gamma^{04} - i \cos B \Gamma^{0}) \chi \right]} e^{\left[ - \frac{1}{2} (\sin B \Gamma^{14} - i \cos B \Gamma^{1}) \tau \right]} \epsilon_0 \\
\equiv S \epsilon_0,
\] (3.16)

where $\epsilon_0$ is any spinor with constant components in the frame (3.15).
As a comparison, for Poincaré coordinates we choose the vielbein
\[ e^0 = \frac{\sqrt{Q}}{2} \left[ \frac{dt}{\sigma} + \sin B \sigma_3 \right], \quad e^1 = \frac{\sqrt{Q}}{2} \frac{d\sigma}{\sigma}, \]
\[ e^2 = \frac{\sqrt{Q}}{2} \sigma_1, \quad e^3 = \frac{\sqrt{Q}}{2} \sigma_2, \quad e^4 = \frac{\sqrt{Q}}{2} \sigma_3. \] (3.17)

The Killing spinors are [74]
\[ \epsilon^+ = \frac{1}{\sqrt{\sigma}} R(\theta, \phi, \psi) \epsilon_0^+, \] (3.18)
\[ \epsilon^- = \left[ \sqrt{\sigma}(1 - \sin B \Gamma^0) - \frac{t}{\sqrt{\sigma}} \Gamma^{01} \right] R(\theta, \phi, \psi) \epsilon_0^-, \] (3.19)

where
\[ R(\theta, \phi, \psi) = e^{-\frac{1}{2} \Gamma^{23} \psi} e^{\frac{1}{2} \Gamma^{24} \theta} e^{-\frac{1}{2} \Gamma^{23} \phi}, \]
\[ i \Gamma^0 \epsilon_0^\pm = \pm \epsilon_0^\pm, \] (3.20)

for constant \( \epsilon_0^\pm \).

### 3.3 Supersymmetric Probe Configurations

In this section, we find classical brane trajectories which preserve some supersymmetries of the rotating attractor (3.7). The worldvolume action has a local \( \kappa \)-symmetry (parameterized by \( \kappa \)) as well as a spacetime supersymmetry transformation (parameterized by \( \epsilon \)) which acts nonlinearly. A spacetime supersymmetry is preserved if its action on the worldvolume fermions \( \Theta \) can be compensated by a \( \kappa \) transformation [19, 21]:
\[ \delta_\epsilon \Theta + \delta_\kappa \Theta = \epsilon + (1 + \Gamma)\kappa(\sigma) = 0, \] (3.21)
where $\Gamma$ is given in various cases analyzed below. This gives the condition

$$(1 - \Gamma)\epsilon = 0,$$  \hspace{1cm} (3.22)$$

which must be solved for both the Killing spinor and the probe trajectory.

### 3.3.1 Zero-brane probe

The zero-brane can be obtained by wrapping M2-branes on the holomorphic two-cycles of the Calabi-Yau threefold $X$. It carries electric charges $v_A$, $A = 1, 2, \ldots, b_2(X)$. For the zero-brane the (bosonic part of the) $\kappa$-symmetry projection operator is

$$\Gamma = \frac{1}{\sqrt{h_{00}}} \tilde{\Gamma}_0 ,$$  \hspace{1cm} (3.23)$$

where $h$ and $\tilde{\Gamma}_0$ are the pull-backs of the metric and Dirac matrix onto the worldline of the zero-brane, respectively:

$$h_{00} = \partial_0 X^\mu \partial_0 X^\nu G_{\mu\nu} ,$$  \hspace{1cm} (3.24)$$

$$\tilde{\Gamma}_0 = \partial_0 X^\mu e_\mu^a \Gamma_a .$$  \hspace{1cm} (3.25)$$

#### Global coordinates

First, let’s look at the global coordinates. In the static gauge, where we set the worldvolume time $\sigma^0$ equal to the global time $\tau$, the $\kappa$-symmetry operator is

$$\Gamma = \frac{1}{\sqrt{h_{00}}} \frac{dX^\mu}{d\tau} e_\mu^a \Gamma_a .$$  \hspace{1cm} (3.26)$$

To solve for the classical trajectory of a supersymmetric zero-brane, we plug the Killing spinors (3.16) into the $\kappa$-symmetry condition (3.22) of the supersymmetric zero-brane. A zero-brane following a classical trajectory, given by
(χ(τ), θ(τ), φ(τ), ψ(τ)), is supersymmetric if, in the notation of (3.16),

\[
\frac{1}{\sqrt{h_0}} \frac{dX^\mu}{d\tau} e^a_\mu S^{-1} \Gamma_a S \epsilon_0 = \epsilon_0 ,
\]

(3.27)

for some constant \(\epsilon_0\), where \(S = S(\chi, \tau, \theta, \phi)\). The explicit prefactors are

\[
S^{-1} e_0^a \Gamma_a S = \sqrt{Q} \left[ (\cosh \chi \cos \tau \cos^2 B + \sin \theta \cos \phi \sin^2 B) / \Gamma^0 \\
+ i \cosh \chi \sin \tau \cos B / \Gamma^{01} - i \cos \theta \sin B / \Gamma^{02} - i \sin \theta \sin \phi \sin B / \Gamma^{03} \\
+ i (\cosh \chi \cos \tau - \sin \theta \cos \phi) \sin B / \Gamma^{04} \right],
\]

\[
S^{-1} e_1^a \Gamma_a S = (-1) \sqrt{Q} \left[ \sin \theta \cos \phi \cos \tau / \Gamma^1 \\
- \sin \tau \sin B e^{1/2} (\cos B \Gamma^{34} + i \sin B \Gamma^3) / \phi e^{-\left(\cos B \Gamma^{24} + i \sin B \Gamma^2\right)} / \theta \\
e^{1/2} (\cos B \Gamma^{34} + i \sin B \Gamma^3) / \phi / \Gamma^4 \\
- i \cos \tau \cos \theta \sin B / \Gamma^{12} - i \cos \tau \sin \theta \sin B / \Gamma^{13} \\
+ i e^{\left(\sin B \Gamma^{14} - i \cos B \Gamma^1\right)} / \tau \sinh \chi \cos B / \Gamma^{01} \\
+ i (\cosh \chi - \sin \theta \cos \phi \cos \tau) \cos B (\sin B / \Gamma^{14} - i \cos B \Gamma^1) \right],
\]

\[
S^{-1} e_2^a \Gamma_a S = (-1) \sqrt{Q} \left[ \cosh \chi \cos \tau \cos \phi / \Gamma^3 - \cosh \chi \cos \tau \sin \phi \cos B / \Gamma^4 \\
+ e^{\left(\cos B \Gamma^{34} + i \sin B \Gamma^3\right)} / \phi (i \sinh \chi \cos B / \Gamma^{03} \\
- i \cosh \chi \sin \tau \cos B / \Gamma^{13} + i \cos \theta \sin B / \Gamma^{23} \\
+ i (\cosh \chi \cos \tau \cos \phi - \sin \theta \sin \phi) \sin B / \Gamma^{34} + i \sin B / \Gamma^3 \right),
\]

\[
S^{-1} e_3^a \Gamma_a S = (-1) \sqrt{Q} \left[ (\cosh \chi \cos \tau \cos^2 B + \sin \theta \cos \phi \sin^2 B) / \Gamma^2 \\
+ i \sinh \chi \cos B / \Gamma^{02} - i \cosh \chi \sin \tau \cos B / \Gamma^{12} - i \sin \theta \sin \phi \sin B / \Gamma^{23} \\
+ i (\cosh \chi \cos \tau - \sin \theta \cos \phi) \sin B / \Gamma^{24} \right],
\]
We first see that a probe static in the global time $\tau$ cannot be supersymmetric. For such a probe we have $\frac{d\chi}{d\tau} = \frac{d\theta}{d\tau} = \frac{d\phi}{d\tau} = \frac{d\psi}{d\tau} = 0$ and the $\kappa$-symmetry condition reduces to

$$\frac{1}{\sqrt{-1 - \cos^2 B \sinh^2 \chi}} \cdot \left[(\cosh \chi \cos \tau \cos^2 B + \sin \theta \cos \phi \sin^2 B)\Gamma^0ight.\left. + i \cosh \chi \sin \tau \cos B \Gamma^{01} - i \cos \theta \sin B \Gamma^{02} - i \sin \theta \sin \phi \sin B \Gamma^{03}\right) + i (\cosh \chi \cos \tau - \sin \theta \cos \phi) \sin B \cos B \Gamma^{04}]\epsilon_0 = \epsilon_0 .$$

The terms in this equation proportional to $\cos \tau$, $\sin \tau$, and 1 must all vanish separately, which is clearly impossible. The lack of such configurations is not surprising, because angular momentum must be nonzero for a nontrivial BPS configuration.

Now we allow the probe to orbit around the $S^3$. Solving the $\kappa$-symmetry condition (3.22) using (3.28) for Killing spinors obeying

$$\Gamma^{02}\epsilon_0 = \mp \epsilon_0 ,$$

we find the supersymmetric trajectory at a generic $(\chi, \theta, \psi)$ to be

$$\frac{d\chi}{d\tau} = \frac{d\theta}{d\tau} = \frac{d\psi}{d\tau} = 0 , \quad \frac{d\phi}{d\tau} = \pm 1 .$$

This is a probe orbiting along the $\phi$-direction.

The constraint on the Killing spinor (3.30) projects out half of the components of $\epsilon_0$, i.e. the orbiting zero-brane probe is a half-BPS configuration. We will show
in the next subsection, using the BPS bound, that this supersymmetric trajectory is unique up to rotations.

**A BPS bound**

The worldline action of a zero brane probe, with mass $m$ and the electric charge $q$, can be written as

$$S = -m \int \sqrt{h} d\sigma^0 + q \int A_{[1]},$$

(3.32)

where $A_{[1]}$ is the 1-form gauge field (3.11). The zero-brane obtained by wrapping M2-branes on the holomorphic two-cycles of the Calabi-Yau threefold $X$ carries electric charges $v_A, A = 1, 2, \ldots, b_2(X)$. For the supersymmetric branes we are considering, $m = q = \frac{v_A y^A}{\sqrt{Q/2}}$.

In global coordinates with $\sigma^0 = \tau$, the Lagrangian of the system is

$$\mathcal{L} = \frac{\sqrt{Q}}{2} \{-m \sqrt{\cosh^2 \chi - \dot{\chi}^2} - [\sin B \sinh \chi - \cos B(\dot{\psi} + \cos \theta \dot{\phi})]^2 - \dot{\theta}^2 - \sin^2 \theta \dot{\phi}^2$$

$$+ m [\cos B \sinh \chi + \sin B(\dot{\psi} + \cos \theta \dot{\phi})]\}.$$

(3.33)

The corresponding Hamiltonian is

$$H = \cosh \chi \left\{ P_\chi^2 + P_\theta^2 + \left( \frac{\cos \theta P_\phi - P_\psi}{\sin \theta} \right)^2 + P_\phi^2 + \left( \frac{\sin B P_\psi - \frac{\sqrt{Q}}{2} m}{\cos B} \right)^2 \right\}$$

$$+ \sinh \chi \left( \frac{\sin B P_\psi - \frac{\sqrt{Q}}{2} m}{\cos B} \right),$$

(3.34)
where the momenta are

\[
\begin{align*}
P_{\chi} &= \frac{m\sqrt{Q}}{2\sqrt{h}} \dot{\chi}, \\
P_{\theta} &= \frac{m\sqrt{Q}}{2\sqrt{h}} \dot{\theta}, \\
P_{\phi} &= \frac{m\sqrt{Q}}{2} \left[ \frac{1}{\sqrt{h}} \left( -\cos B \cos \theta \sinh \chi - \cos B(\dot{\psi} + \cos \theta \dot{\phi}) + \frac{\sin^2 \theta \dot{\phi}}{\sqrt{\cos^2 B \sinh^2 \chi + 1}} \right) \\
&\quad + \sin B \cos \theta \right], \\
P_{\psi} &= \frac{m\sqrt{Q}}{2} \left[ \frac{1}{\sqrt{h}} \left( -\cos B \sin B \sinh \chi - \cos B(\dot{\psi} + \cos \theta \dot{\phi}) \right) + \sin B \right],
\end{align*}
\]

(3.35)

and

\[
\begin{align*}
h &= \cosh^2 \chi - \dot{\chi}^2 - [\sin B \sinh \chi - \cos B(\dot{\psi} + \cos \theta \dot{\phi})]^2 - \dot{\theta}^2 - \sin^2 \theta \dot{\phi}^2. \\
& \quad (3.36)
\end{align*}
\]

The unbroken rotational symmetries lead to the conserved charges:

\[
\begin{align*}
J_{\text{right}}^1 &= \sin \phi P_{\theta} + \cos \phi (\cot \theta P_{\phi} - \csc \theta P_{\psi}), \\
J_{\text{right}}^2 &= \cos \phi P_{\theta} - \sin \phi (\cot \theta P_{\phi} - \csc \theta P_{\psi}), \\
J_{\text{right}}^3 &= P_{\phi}, \\
J_{\text{left}}^3 &= P_{\psi}.
\end{align*}
\]

(3.37)

It is easy to see that there are no static solutions. They would have to minimize the potential energy according to

\[
0 = \frac{\partial H}{\partial \chi} = \frac{\sqrt{Q}}{2} m \cos B \cosh \chi \left( \frac{\cos B \sinh \chi}{\sqrt{\cos^2 B \sinh^2 \chi + 1}} - 1 \right),
\]

(3.38)

which has no solutions for finite \(\chi\). Physically, the probe is accelerated to \(\chi = \pm \infty\).

Now we allow the probe to orbit. Solutions of this type can be stabilized by the angular potential. The supersymmetric configuration turns out to be at constant
radius in the $AdS_2$, i.e. $P_\chi = 0$. The Hamiltonian is minimized with respect to $\chi$ when
\begin{equation}
\tanh \chi = -\frac{1}{\sqrt{P_\theta^2 + \left(\frac{\cos \theta P_\phi - P_\psi}{\sin \theta}\right)^2 + P_\phi^2 + \left(\frac{\sin B P_\psi - \sqrt{Q^2 m}}{\cos B}\right)^2}} \left(\frac{\sin B P_\psi - \sqrt{Q^2 m}}{\cos B}\right).
\end{equation}
(3.39)

The value of $H$ at the minimum is
\begin{equation}
H_{\text{min}} = \sqrt{P_\theta^2 + \left(\frac{\cos \theta P_\phi - P_\psi}{\sin \theta}\right)^2 + P_\phi^2} = |\vec{J}_{\text{right}}|,
\end{equation}
(3.40)

where $|\vec{J}_{\text{right}}|^2 = (J_1^\text{right})^2 + (J_2^\text{right})^2 + (J_3^\text{right})^2$. This implies the BPS bound
\begin{equation}
H \geq |\vec{J}_{\text{right}}|
\end{equation}
(3.41)
for generic $\chi$.

Up to spatial rotations, we may always choose static BPS solutions to satisfy
\begin{equation}
H = J_3^\text{right} = \pm P_\phi, \quad J_1^\text{right} = J_2^\text{right} = 0.
\end{equation}
(3.42)

This implies
\begin{equation}
P_\theta = 0, \quad \cos \theta P_\phi = P_\psi.
\end{equation}
(3.43)

Hence, the azimuthal angle is determined by the ratio of left and right angular momenta:
\begin{equation}
\cos \theta = \frac{J_3^\text{left}}{J_3^\text{right}}.
\end{equation}
(3.44)

We can rewrite $\dot{\phi}$ and $\dot{\psi}$ in terms of $P_\phi$ and $P_\psi$. With $\chi = \dot{\theta} = 0$,
\begin{align}
\dot{\phi} &= \frac{\cosh \chi \left(\frac{P_\phi - \cos \theta P_\psi}{\sin^2 \theta}\right)}{\sqrt{P_\theta^2 + \left(\frac{\cos \theta P_\phi - P_\psi}{\sin \theta}\right)^2 + P_\phi^2 + \left(\frac{\sin B P_\psi - \sqrt{Q^2 m}}{\cos B}\right)^2}},
\end{align}
(3.45)
\begin{align}
\dot{\psi} &= \frac{\cosh \chi \left[\tan B \left(\frac{\sin B P_\psi - \sqrt{Q^2 m}}{\cos B}\right) - \left(\frac{\cos \theta P_\phi - P_\psi}{\sin \theta}\right)\right] + \tan B \sinh \chi}{\sqrt{P_\theta^2 + \left(\frac{\cos \theta P_\phi - P_\psi}{\sin \theta}\right)^2 + P_\phi^2 + \left(\frac{\sin B P_\psi - \sqrt{Q^2 m}}{\cos B}\right)^2}}.
\end{align}
(3.46)
Eliminate $\chi$ through (3.39),

$$
\dot{\phi} = \frac{1}{\sqrt{P_\phi^2 + \left(\frac{\cos \theta P_\phi - P_\psi}{\sin \theta}\right)^2 + P_\phi^2}} \left(\frac{P_\phi - \cos \theta P_\psi}{\sin^2 \theta}\right),
$$

$$
\dot{\psi} = \frac{1}{\sqrt{P_\phi^2 + \left(\frac{\cos \theta P_\phi - P_\psi}{\sin \theta}\right)^2 + P_\phi^2}} \left(\frac{P_\psi - \cos \theta P_\phi}{\sin^2 \theta}\right).
$$

Plug in (3.43), the solution is

$$
\dot{\theta} = 0, \quad \dot{\phi} = \pm 1, \quad \dot{\psi} = 0,
$$

for which $(P_\phi, P_\psi)$ are

$$
P_\psi = \pm \frac{\sqrt{Q}}{2} m \frac{\cos \theta}{\cos B \sinh \chi \pm \sin B \cos \theta},
$$

$$
P_\phi = \pm \frac{\sqrt{Q}}{2} m \frac{1}{\cos B \sinh \chi \pm \sin B \cos \theta}.
$$

The energy of the particle following this trajectory is equal to $\pm P_\phi$:

$$
H = \frac{\sqrt{Q}}{2} m \frac{1}{\cos B \sinh \chi \pm \sin B \cos \theta} = \pm P_\phi.
$$

We see that the solution with $\dot{\phi} = 1$ ($\dot{\phi} = -1$) corresponds to a chiral (anti-chiral) BPS configuration.

Therefore, we have confirmed that the supersymmetric trajectories (3.31) obtained by solving the $\kappa$-symmetry condition correspond to the BPS states.

**Poincaré coordinates**

In Poincaré coordinates and static gauge $\sigma^0 = t$, the $\kappa$-symmetry condition for a static probe is

$$
\frac{1}{\sqrt{-\frac{1}{\sigma^2}}} \left[ -\frac{1}{\sigma} \Gamma^0 \right] \epsilon = i \Gamma^0 \epsilon = \epsilon.
$$
This equation is solved by simply taking $\epsilon = \epsilon^+ = \frac{1}{\sqrt{\sigma}} R(\theta, \phi, \psi) \epsilon_0^+$. Again, we find a half-supersymmetric solution, although the broken supersymmetries are different than in the global case. It can be seen that there are no supersymmetric orbiting trajectories in Poincaré time.

### 3.3.2 One-brane probe

In this subsection, we find some supersymmetric one-brane configurations. The one-brane probe is constructed by wrapping the M5-brane on a holomorphic 4-cycle of the CY$_3$. We consider a specific Ansatz with no worldvolume electromagnetic field and with the one-brane geometry:

\[
\begin{align*}
\tau &= \sigma^0, \\
\phi &= \dot{\phi} \sigma^0 + \phi' \sigma^1, \\
\psi &= \dot{\psi} \sigma^0 + \psi' \sigma^1, \quad (3.54)
\end{align*}
\]

where $(\sigma^0, \sigma^1)$ are worldvolume coordinates, and $\dot{\phi}$, $\dot{\psi}$, $\phi'$ and $\psi'$ are all taken to be constant. Note that since $(\psi, \phi)$ are the orbits of $(J^3_L, J^3_R)$, they may be viewed as one-brane momentum-winding modes on the torus generated by $(J^3_L, J^3_R)$. This torus degenerates to a circle at the loci $\theta = \{0, \pi\}$. One-branes of the form (3.54) at these loci are therefore static (up to reparametrizations).
Chapter 3: Supersymmetric Probes in a Rotating 5D Attractor

With no electromagnetic field the $\kappa$-symmetry condition is\(^3\)

$$\frac{1}{2} \epsilon^{ij} \tilde{\Gamma}_{ij} \epsilon = \epsilon , \quad (3.55)$$

where $h$ and $\tilde{\Gamma}_i$ are the pull-backs of the 5D metric and gamma matrices onto the one-brane worldsheet. With the Ansatz (3.54), we have explicitly

$$\begin{align*}
\tilde{\Gamma}_0 & = \Gamma_\tau + \dot{\phi} \Gamma_\phi + \dot{\psi} \Gamma_\psi , & (3.56) \\
\tilde{\Gamma}_1 & = \phi' \Gamma_\phi + \psi' \Gamma_\psi , & (3.57) \\
\frac{1}{2} \epsilon^{ij} \tilde{\Gamma}_{ij} & = \frac{1}{2\sqrt{\det h}} [\phi' \Gamma_{\tau\phi} + \psi' \Gamma_{\tau\psi} + (\dot{\phi} \psi' - \dot{\psi} \phi') \Gamma_{\phi\psi}] , \quad (3.58)
\end{align*}$$

and

$$\begin{align*}
h_{00} & = \frac{Q}{4} \{- \cosh^2 \chi + [\sin B \sinh \chi - \cos B(\dot{\psi} + \cos \theta \dot{\phi})]^2 + \sin^2 \theta \dot{\phi}^2 \} , \\
h_{11} & = \frac{Q}{4} \{\cos^2 B(\psi' + \cos \theta \phi')^2 + \sin^2 \theta \phi'^2 \} , \quad (3.59) \\
h_{01} & = \frac{Q}{4} \{[\sin B \sinh \chi - \cos B(\dot{\psi} + \cos \theta \dot{\phi})](- \cos B(\psi' + \cos \theta \phi') + \sin^2 \theta \dot{\phi} \phi') \}
\end{align*}$$

and hence

$$\begin{align*}
\det h & = \left(\frac{Q}{4}\right)^2 \{\cosh^2 \chi [\cos^2 B(\psi' + \cos \theta \phi')^2 + \sin^2 \theta \phi'^2] \\
& \quad - \sin^2 \theta [\sin B \sinh \chi \phi' - \cos B(- \psi' \dot{\phi} + \phi' \dot{\psi})]^2 \} . \quad (3.60)
\end{align*}$$

It is simplest to analyze the $\kappa$-symmetry condition in the form

$$S^{-1} \frac{1}{2} \epsilon^{ij} \tilde{\Gamma}_{ij} S \epsilon_0 = \epsilon_0 . \quad (3.61)$$

\(^3\)There is a simple kappa-symmetric action in six dimensions, but not in five. In 5D we expect an extra scalar field along with the transverse coordinates to fill out the supermultiplet. For the case of the M5-brane wrapping a Calabi-Yau 4-cycle, the scalar in the effective one-brane arises as a mode of the antisymmetric tensor field. The Ansatz of this section corresponds to taking this extra scalar to be a constant.
The rotated gamma matrices appearing in this expression are explicitly

\[
S^{-1}_\tau \Gamma_{\phi} S = -\frac{Q}{4} [(\cosh^2 \chi \cos^2 B + \sin^2 \theta \sin^2 B)\Gamma^{02} - \imath (\cosh \chi \cos \tau \cos^2 B + \sin \theta \cos \phi \sin^2 B \\
- \imath \cosh \chi \sin \tau \cos BT^1 + \imath \sin \theta \sin \phi \sin BT^3 \\
- \imath (\cosh \chi \cos \tau - \sin \theta \cos \phi) \sin B \cos BT^4)(\cos \theta \sin BT^0 + \sinh \chi \cos BT^2)] ,
\]

\[
S^{-1}_\tau \Gamma_{\psi} S = \frac{Q}{4} \cos B \{- \cosh^2 \chi \cos \theta \cos BT^{02} \\
+ \cos B \sinh \chi [\imath \cosh \chi \sin \theta \cos \tau \cos \phi \Gamma^4 + \cosh \chi \sin \theta \sin \phi \sin \tau \Gamma^{13} \\
- \cosh \chi \sin \theta \cos \tau \sin \phi (\sin BT^{34} - \imath \cos BT^3) \\
+ \cosh \chi \sin \theta \cos \phi \sin \tau (\cos BT^{14} + \imath \sin BT^1)] \\
-(\cos^2 B \cosh^2 \chi \sin \theta \cos \phi + \sin^2 B \cosh \chi \cos \tau) \Gamma^{04} \\
- \sin B \cos B \cosh \chi \sinh \chi \cos \tau \cos \theta \Gamma^{24} \\
- \cosh \chi \sin \tau \sin BT^{01} + \cos B \cosh \chi \sinh \chi \cos \theta \sin \tau \Gamma^{12} \\
- \cosh^2 \chi \sin \theta \sin \phi \cos BT^{03} \\
- \imath \cosh \chi (\cosh \chi \sin \theta \cos \phi - \cos \tau) \sin B \cos BT^0 \\
+ \imath \cos^2 B \cosh \chi \sinh \chi \cos \tau \cos \theta \Gamma^{2} \} ,
\]
\[ S^{-1}\Gamma_{\phi\psi}S \] 
\[ = \frac{Q}{4} \cos B \{ + \sinh \chi \sin^2 \theta \sin B \Gamma^{02} \]
\[ + \sin B \cos \theta [i \cosh \chi \sin \theta \cos \phi \cos \tau \Gamma^4 + \cosh \chi \sin \theta \sin \phi \sin \tau \Gamma^{13} \]
\[ + \cosh \chi \sin \theta \cos \phi \sin \tau (\cos B \Gamma^{14} + i \sin B \Gamma^1) \]
\[ - \cosh \chi \sin \theta \sin \phi \cos \tau (\sin B \Gamma^{34} - i \cos B \Gamma^3) \] 
\[ - \sin B \cos B \sinh \chi \sin \theta \cos \theta \cos \phi \Gamma^{04} \]
\[ + (\cosh \chi \sin^2 \theta \cos \tau \sin^2 B + \sin \theta \cos \phi \cos^2 B) \Gamma^{24} \]
\[ - \cosh \chi \sin^2 \theta \sin \tau \sin B \Gamma^{12} \]
\[ - \sin B \cos \theta \sin \phi \sinh \chi \Gamma^{03} + \sin \theta \sin \phi \cos B \Gamma^{23} \]
\[ - i \sin^2 B \sinh \chi \sin \theta \cos \theta \cos \phi \Gamma^0 \]
\[ - i \sin \theta (\cosh \chi \sin \theta \cos \tau - \cos \phi) \cos B \sin B \Gamma^2 \} . \]

This all simplifies at points obeying
\[ \sinh \chi = \pm \tan B \cos \theta \] 
(3.65)

when \(-\psi' \dot{\phi} + \phi' \dot{\psi} = \pm \psi'\). Under these conditions
\[ \sqrt{\text{det} h} = \frac{Q}{4} (\phi' + \cos \theta \psi') , \] 
(3.66)

and
\[ S^{-1}[\phi' \Gamma_{\tau\phi} + \psi' \Gamma_{\tau\psi} + (\dot{\phi} \psi' - \dot{\psi} \phi') \Gamma_{\phi\psi}]S \]
\[ = \frac{Q}{4} [-(\phi' + \cos \theta \psi') \Gamma^{02} + (\phi' \hat{D}_1 + \psi' \hat{D}_2) (\Gamma^0 \pm \Gamma^2)] , \] 
(3.67)
where

\[ \hat{\mathcal{D}}_1 = i \cos \theta \sin B \left[ \cosh \chi \cos \tau \cos^2 \theta + \sin \theta \cos \phi \sin^2 \theta \right] \]
\[ - i \cosh \chi \sin \tau \cos B \Gamma^1 + i \sin \theta \sin \phi \sin B \Gamma^3 \]
\[ - i(\cosh \chi \cos \tau - \sin \theta \cos \phi) \sin B \cos B \Gamma^4 \big] \text{,} \]
\[ \hat{\mathcal{D}}_2 = - \cos B (\cos^2 B \sin \theta \cos \phi + \sin^2 B \cosh \chi \cos \tau) \Gamma^4 + \cos B \sin B \cosh \chi \sin \tau \Gamma^1 \]
\[ + \cos^2 B \sin \theta \sin \phi \Gamma^3 - i \sin B \cos^2 B (\sin \theta \cos \phi - \cosh \chi \cos \tau) \big] . \quad (3.68) \]

So far we have not chosen which supersymmetries are to be preserved. We take those generated by spinors obeying \( \Gamma^{02} \epsilon_0 = \pm \epsilon_0 \), or equivalently \( \Gamma^2 \epsilon_0 = \mp \Gamma^0 \epsilon_0 \). In this case, the last term in (3.67) can be dropped and the supersymmetry conditions are satisfied.

To summarize, any configuration satisfying

\[ - \psi' \phi' + \phi' \psi = \pm \psi' , \quad \dot{\chi} = \dot{\theta} = 0 , \]
\[ \sinh \chi = \pm \tan B \cos \theta \]

preserves those supersymmetries corresponding to

\[ \Gamma^{02} \epsilon_0 = \pm \epsilon_0 . \quad (3.70) \]

Other BPS configurations preserving other sets of supersymmetries can be obtained by \( SL(2, R) \times SO(4) \) rotations of these ones.

Note that, as for the zero-branes, there are generic solutions for any \( \theta \). These include \( \theta = \{0, \pi\} \), which correspond to static one-branes because the \((\psi, \phi)\) torus degenerates to a circle along these loci. Static solutions are possible because a one-brane probe in 5D couples magnetically to the dual of the spacetime gauge field \( F_{[2]} \) of (3.11) hence there is nonzero angular momentum carried by the fields.
3.4 Conclusion

In this chapter, we constructed supersymmetric brane probe solutions in the squashed $AdS_2 \times S^3$ near-horizon geometry of the BMPV black hole.

We expect that the quantization of the moduli space of these classical configurations will provide a microscopic description of the five-dimensional $N = 2$ rotating black holes. This will be carried out in the future work.
Chapter 4

Non-Supersymmetric Attractor

Flow in Symmetric Spaces

4.1 Introduction

Soon after the attractor mechanism was first discovered in supersymmetric (BPS) black holes [63], it was reformulated in terms of motion on an effective potential for the moduli [62]. Ferrara et al demonstrated that the critical points of this potential correspond to the attractor values of the moduli. More recently, several groups used the effective potential to show that non-supersymmetric (non-BPS) extremal black holes can also exhibit the attractor mechanism, thereby creating a new and exciting field of research [92, 76]. Many connections between non-BPS attractors and other active areas of string theory soon revealed themselves. Andrianopoli et al found that both BPS and non-BPS black holes embedded in a supergravity with a symmetric moduli space can be studied using the same formalism, and they uncovered many
intricate relations between the two [11, 43]. Dabholkar, Sen and Trivedi proposed a microstate counting for non-BPS black holes (albeit subject to certain constraints [42]). Saraikin and Vafa suggested that a new extension of topological string theory generalizes the Ooguri-Strominger-Vafa (OSV) formula such that it is also valid for non-supersymmetric black holes [133]. Studying non-BPS attractors could also give insight into non-supersymmetric flux vacua. Given all these possible applications, it is important to characterize non-BPS black holes as fully as possible.

There has been a great deal of progress in understanding the near-horizon region of these non-BPS attractors. The second derivative of the effective potential at the critical point determines whether the black hole is an attractor, and the location of the critical point yields the values of the moduli at the horizon; in this way, one can compute the stability and attractor moduli for all models with cubic prepotential [145, 113]. However, the effective potential has only been formulated for the leading-order terms in the supergravity lagrangian. If one wants to include higher-derivative corrections, one can instead use Sen’s entropy formalism, which incorporates Wald’s formula, to characterize the near-horizon geometry in greater generality [135]. Sen’s method has led to many new results [132, 13, 131, 36, 8, 136]. The tradeoff is that this method cannot be used to determine any properties of the solution away from the horizon.

The BPS attractor flow is constructed from the attractor value $z_{BPS}^* = z_{BPS}^*(p^I, q_I)$ by simply replacing the D-brane charges with the corresponding harmonic functions:

$$z_{BPS}(\vec{x}) = z_{BPS}^*(p^I \rightarrow H^I(\vec{x}), q_I \rightarrow H_I(\vec{x})) \quad ,$$

(4.1)
where the harmonic functions are
\[
\begin{pmatrix}
H^I(\vec{x}) \\
H_I(\vec{x})
\end{pmatrix} = \begin{pmatrix}
h^I \\
h_I
\end{pmatrix} + \frac{1}{|\vec{x}|} \begin{pmatrix}
p^I \\
q_I
\end{pmatrix}.
\]
Moreover, this procedure can be applied to construct the multi-centered BPS attractor flow that describes the supersymmetric black hole bound state \([18]\), where the harmonic functions are generalized to have multiple centers:
\[
\begin{pmatrix}
H^I(\vec{x}) \\
H_I(\vec{x})
\end{pmatrix} = \begin{pmatrix}
h^I \\
h_I
\end{pmatrix} + \sum_i \frac{1}{|\vec{x} - \vec{x}_i|} \begin{pmatrix}
(p^I)_i \\
(q_I)_i
\end{pmatrix}.
\]
It is conjectured in \([91]\) that the non-BPS flow can be generated in the same fashion, namely, by replacing the charges in the attractor value with the corresponding harmonic functions. However, as will be proven in this chapter, this procedure does not work for systems with generic charge and asymptotic moduli.\(^1\)

In principle, one could construct the full non-BPS flow (the black hole metric, together with the attractor flow of the moduli) by solving the equation of motion derived from the Lagrangian. However, this is a second-order differential equation and only reduces to a first-order equation upon demanding the preservation of supersymmetry. Ceresole et al have written down an equivalent first-order equation in terms of a “fake superpotential,” but the fake superpotential can only be explicitly constructed for special charges and asymptotic moduli \([34, 98]\). The most generic non-BPS equation of motion is complicated enough that it has not yet been solved. Similarly, multi-centered non-BPS black holes have not been studied.

Our goal is to construct the full flow for non-BPS stationary black holes in four dimensions. Instead of directly solving the equation of motion, we reduce the action

\(^1\)This has also been shown in [98]
on the timelike isometry and dualize all 4D vectors to scalars. The new moduli space $M_{3D}$ contains isometries corresponding to all the charges of the black hole, and the black hole solutions are simply geodesics on $M_{3D}$. This method was introduced in [29] and has been used to construct static and rotating black holes in heterotic string theory [39, 40] and to study the classical BPS single-centered flow and its radial quantization [81, 114].

In this chapter, we work in two specific theories of gravity, but we expect that this method can be used for any model whose $M_{3D}$ is symmetric. The basic technique is reviewed in more detail in Section 4.2. In Section 4.3, we show how this method works in a simple case: the toroidal compactification of $D$-dimensional pure gravity. Section 4.4 serves as an introduction to single-centered attractor flow in $\mathcal{N} = 2$ supergravity coupled to one vector multiplet, and Sections 4.5 and 4.6 are dedicated to constructing the full flows for both BPS and non-BPS single-centered black holes with generic charges. We find that they are generated by the action of different classes of nilpotent elements in the coset algebra. Both types of flows are shown to reach the correct attractor values at the horizon. In Section 4.7, the procedure is generalized straightforwardly to construct both BPS and non-BPS multi-centered solutions. We use a metric ansatz with a flat spatial slice and we are able to recover the BPS bound states described by Bates and Denef. Using the same ansatz, we are able to build non-BPS multi-centered solutions. Unfortunately, solutions generated this way turn out to always have charges at each center which are mutually local. The last section reviews our conclusions and suggests possibilities for future work.
4.2 Framework

Here we outline the method we will use to construct black hole solutions. This method was first described in [29]. We first reduce a general gravity action from four dimensions down to three, and derive the equation of motion. We then specialize to certain theories which have a 3D description in terms of a symmetric coset space. In such situations, we can easily find solutions to the equation of motion. The solutions are geodesics (or generalizations thereof) on the 3D moduli space and they are generated by elements of the coset algebra.

4.2.1 3D moduli space

We will study stationary solutions in a theory with gravity coupled to scalar and vector matter. Let the scalars be $z^i$ and the vectors be $A^I$. Then the most general ansatz for a stationary solution in four dimensions is:

$$ds^2 = -e^{2U} (dt + \omega)^2 + e^{-2U} g_{ab} dx^a dx^b,$$

$$F^I = dA^I = d\left( A_0^I (dt + \omega) + A^I \right),$$

where $a, b = 1, 2, 3$ label the spatial directions and bold font denotes three-dimensional fields and operators. Since none of the fields are time-dependent, we can compactify on the time isometry and reduce to three-dimensional space $\mathcal{M}_{3D}$. This procedure is called the $c^*$-map. In three dimensions, a vector is Hodge-dual to a scalar. The equations of motion for $\omega$ and the gauge fields allow us to define the dual scalars $\phi_\omega$ and $\phi_{A^I}$. 
Chapter 4: Non-Supersymmetric Attractor Flow in Symmetric Spaces

We then obtain the 3D lagrangian in terms of only scalars

\[ \mathcal{L} = \frac{1}{2} \sqrt{g} \left( -\frac{1}{\kappa} R + \partial_a \phi^m \partial^a \phi^n g_{mn} \right), \quad (4.4) \]

where \( \phi^n \) are the moduli fields

\[ \phi^n = \{U, \bar{z}^i, \bar{z}^\bar{i}, \phi_\omega, A_0^I, \bar{A}_I \} \quad (4.5) \]

and \( g_{ab} \) is the space time metric and \( g_{mn} \) is the metric of a manifold \( \mathcal{M}_{3D} \). The system is 3D gravity minimally coupled to a nonlinear sigma model with moduli space \( \mathcal{M}_{3D} \). Next, we will find the equation of motion in this theory.

### 4.2.2 Attractor flow equation

The equation of motion of 3D gravity is Einstein’s equation:

\[ R_{ab} - \frac{1}{2} g_{ab} R = \kappa T_{ab} = \kappa (\partial_a \phi^m \partial_b \phi^n g_{mn} - \frac{1}{2} g_{ab} \partial_c \phi^m \partial^c \phi^n g_{mn} ), \quad (4.6) \]

and the equation of motion of the moduli is:

\[ \nabla_a \nabla^a \phi^n + \Gamma^m_{np} \partial_a \phi^m \partial^a \phi^p = 0 . \quad (4.7) \]

For simplicity, we consider only the case where the 3D spatial slice is flat (it is guaranteed to be flat only for extremal single-centered black holes). Then the dynamics of the moduli are decoupled from that of the 3D gravity:

\[ R_{ab} = 0 \implies \partial_a \phi^m \partial_b \phi^n g_{mn} = 0 . \quad (4.8) \]

In the multi-centered case, we need to solve the full equations for the moduli as functions of the 3D coordinates \( \vec{x} \). For single-centered solutions, the moduli only
depend on $r$; to satisfy the above equations, the motion of the moduli must follow null geodesics inside $\mathcal{M}_{3D}$.

A generic null geodesic flows to the boundary of the moduli space $\mathcal{M}_{3D}$. A single-centered attractor flow is defined as a null geodesic that terminates at a point on the $U \to -\infty$ boundary and in the interior region with respect to all other coordinates. This is guaranteed for the BPS attractor by the constraints imposed by the supersymmetry. To find the single-centered non-supersymmetric attractor flow, one needs to find a way to construct null geodesics and a constraint that picks out the ones that stop at this specific component of the boundary. In the next section, we will show that we can do this for models with special properties, and the method can be easily generalized to find the multi-centered attractor solution.

4.2.3 Models with symmetric moduli space

The problem of finding such a constraint in a generic model is not easy. To simplify, we study any model whose moduli space is a symmetric homogeneous space: $\mathcal{M}_{3D} = G/H$. When $\mathcal{M}_{3D}$ is a homogeneous space, the isometry group $G$ acts transitively on $\mathcal{M}_{3D}$. $H$ denotes the isotropy group, which is the maximal compact subgroup of $G$ when one compactifies on a spatial isometry down to (1, 2) space, or the analytical continuation of the maximal compact subgroup of $G$ when one compactifies on the time isometry down to (0, 3) space. The Lie algebra $g$ has the Cartan decomposition: $g = h \oplus k$ where

$$[h, h] = h, \quad [h, k] = k.$$ (4.9)
When $G$ is semi-simple, the homogeneous space is symmetric, and

$$[k, k] = h.$$  \hfill{(4.10)}

The models with symmetric moduli space includes: $D$-dimensional gravity toroidally compactified to four dimensions, certain models of 4D $\mathcal{N} = 2$ supergravity coupled to vector-multiplet, and all 4D $\mathcal{N} > 2$ extended supergravity. The entropy of the last two classes is U-duality invariant. In the present work, we will only consider the first two classes, namely, the $D$-dimensional gravity toroidally compactified to four dimensions, and the 4D $\mathcal{N} = 2$ supergravity coupled to $n_V$ vector-multiplet.

The left-invariant current is

$$J = M^{-1}dM = J_k + J_h,$$  \hfill{(4.11)}

where $M$ is the coset representative, and $J_k$ is the projection of $J$ onto the coset algebra $k$. The lagrangian density of the sigma-model with target space $G/H$ is given by $J_k$ as:

$$L = \text{Tr}(J_k \wedge *_3 J_k).$$  \hfill{(4.12)}

The geodesic of the homogeneous space written in terms of the coset representative is simply

$$M = M_0 e^{kr/2} \quad \text{with} \quad k \in k,$$  \hfill{(4.13)}

where $M_0$ parameterizes the initial point, and the $\frac{1}{2}$ is for later convenience. A null geodesic has zero length:

$$|k|^2 = 0.$$  \hfill{(4.14)}

Therefore, in a homogeneous space, we can find the null geodesics that end at an
attractor point by imposing the appropriate constraint on the null elements of the coset algebra.

Since $M$ is defined up to the action of the isotropy group $\mathbf{H}$, in order to read off the moduli fields from $M$ in an $\mathbf{H}$-independent way, we construct the symmetric matrix using the metric signature matrix $S_0$:

$$S \equiv M S_0 M^T .$$

(4.15)

In all systems considered in the present work, $\mathbf{H}$ is the maximal orthogonal subgroup of $\mathbf{G}$ with the correct signature:

$$H S_0 H^T = S_0 \quad \text{for } \forall H \in \mathbf{H} .$$

(4.16)

That is, the isotropy group $\mathbf{H}$ preserves the symmetric metric matrix $S_0$. Therefore, $S$ is invariant under $M \to MH$ with $H \in \mathbf{H}$. Moreover, as the isotropy group $\mathbf{H}$ acts transitively on the space of matrices with a given signature, the space of possible $S$ is the same as the symmetric space $\mathbf{G}/\mathbf{H}$. That is, the moduli of $\mathbf{G}/\mathbf{H}$ can be combined into the symmetric matrix $S$. And the current of $S$ is

$$J_S = S^{-1} dS .$$

(4.17)

It is easy to perform the projection onto the coset algebra $\mathbf{k}$. The (generalized) orthogonality condition of the isotropy group $\mathbf{H}$ can be expressed in terms of the subalgebra element $h$ which is in $H = e^h$ as

$$h S_0 + S_0 h^T = 0 \quad \forall h \in h .$$

(4.18)

In other words, $(h S_0)$ is anti-symmetric: $(h S_0)^T = -(h S_0)$. Thus the coset algebra, being the compliment of $h$, can be defined as the $\mathbf{k}$ with $(k S_0)$ being symmetric:
$(kS_0)^T = (kS_0)$, i.e.

$$k^T = S_0^{-1}kS_0 \quad \forall k \in k .$$  \hspace{1cm} (4.19)

Therefore, the projection of an element $g$ in $g$ onto the coset algebra $k$ is:

$$g_k = g + S_0 g^T S_0^{-1} .$$  \hspace{1cm} (4.20)

For the left-invariant current $J = M^{-1}dM$, the projection onto $k$ is:

$$J_k = J + S_0 J^T S_0^{-1} .$$  \hspace{1cm} (4.21)

It is straightforward to show that the current constructed from $S$ is related to the projected left-invariant current $J_k$ by:

$$J_S = S^{-1}dS = 2(S_0 M^T)^{-1}J_k(S_0 M^T) .$$  \hspace{1cm} (4.22)

The lagrangian in terms of $S$ is thus $L = \frac{1}{4} \text{Tr}(J_S \wedge \star_3 J_S)$. That is, the lagrangian density is

$$\mathcal{L} = \frac{1}{4} \text{Tr}(S^{-1} \nabla S \cdot S^{-1} \nabla S) ,$$  \hspace{1cm} (4.23)

which is invariant under the action of the isometry group $G$:

$$S \rightarrow G^{-1}SG \quad \text{where} \quad G \in G$$  \hspace{1cm} (4.24)

and whose conserved current is:

$$J = S^{-1} \nabla S ,$$  \hspace{1cm} (4.25)

where we have dropped the subscript $S$ in $J_S$, since we will only be dealing with this current from now on. The equation of motion is the conservation of the current:

$$\nabla \cdot J = \nabla \cdot (S^{-1} \nabla S) = 0 .$$  \hspace{1cm} (4.26)
We now specialize to the single-centered solutions: they correspond to geodesics in the coset manifold. The spherical symmetry allows the 3D metric to be parameterized as

$$ds_3^2 = C(r)^2 dx^2.$$  \hspace{1cm} (4.27)

Then the equations of motion involve the operator $d_r r^2 C(r) d_r$, and reduce to geodesic equations in terms of a parameter $\tau$ such that

$$\frac{dr}{d\tau} = r^2 C(r).$$  \hspace{1cm} (4.28)

The function $C(r)$ is then determined from the equations of motion of 3D gravity. The equations of motion can be written as

$$\frac{d}{d\tau} (S^{-1} \frac{dS}{d\tau}) = 0.$$  \hspace{1cm} (4.29)

In the extremal limit the geodesics become null, the 3D metric is flat and

$$\tau = -\frac{1}{r}.$$  \hspace{1cm} (4.30)

In the search for multi-centered extremal solutions, where the spherical symmetry is absent, it is very convenient to restrict to solutions with a flat 3D metric. This is consistent with the equations of motion as long as the 3D energy momentum tensor is zero everywhere:

$$T_{ab} = \text{Tr}(J_a J_b) = 0.$$  \hspace{1cm} (4.31)

The coupled problem with generic non-flat 3D metric is much harder, and exact solutions are hard to find unless a second Killing vector is present.

Since different values of the scalars at infinity are easily obtained by a $G$ transformation, to start with, we will consider the flow starting from $M_0 = 1$, and generalize
to generic asymptotic moduli later. For a single-centered solution, the flow of $M$ is $M = M_0 e^{k\tau/2}$. Since all the coset representatives can be brought into the form $e^g$ with some $g \in k$ by an $H$-action, we can write $M_0 = e^{g/2}$, so $M = e^{g/2} e^{k\tau/2}$. And the flow of $S$ is
\[ S(\tau) = e^{g/2} e^{k\tau} e^{g/2} S_0. \] (4.32)
The charges of the solution are read from the conserved currents
\[ J(r) = S^{-1} \nabla S = \frac{S_0 e^{-g/2} k e^{g/2} S_0}{r^2} \hat{r}. \] (4.33)

4.3 Toroidal Reduction of $D$-dimensional Pure Gravity

Now we use the method introduced in the previous section to analyze pure gravity toroidally compactified down to four dimensions. We explain why the attractor flow generator, $k$, needs to be nilpotent, and we find the Jordan forms of $k_2$ and $k$. Using this information, we construct single-centered attractor flows. We then generalize to multi-centered black holes in pure gravity and show that these solutions have mutually local charges and no intrinsic angular momentum.

4.3.1 Kaluza-Klein reduction

The simplest example of a system that admits a 3D description in terms of a sigma model on a symmetric space is pure gravity in $D$ dimensions, compactified on
a $D - 4$ torus. The KK reduction to 4D parameterizes the metric as
\[ ds_D^2 = \rho_{pq}(dy^p + A^p_\mu dx^\mu)(dy^q + A^q_\nu dx^\nu) + \frac{1}{\sqrt{\det \rho}}ds_4^2, \quad 1 \leq p, q \leq D - 4. \] (4.34)

Here $y^p$ are the torus coordinates, $x^\mu$ the coordinates on $R^3$, and $\rho_{pq}$ the metric of the torus. A 4D metric with one timelike Killing spinor is then parameterized as
\[ ds^2_4 = -u(dt + \omega_idx^i)^2 + \frac{1}{u}ds_3^2, \] (4.35)

where $u = e^{2U}$, to connect with the parametrization in the later part of the chapter; and $i = 1, 2, 3$ denote the 3D space coordinates.

The two expressions combine as
\[ ds_D^2 = G_{ab}(dy^a + \tilde{\omega}^a_idx^i)(dy^b + \tilde{\omega}^b_jdx^j) + \frac{1}{-\det G}ds_3^2, \quad 0 \leq a, b \leq D - 4. \] (4.36)

Here $y^a$ are the torus coordinates plus time, $x^i$ coordinates on $R^3$ and
\[ G = \begin{pmatrix} \rho_{pq} & \rho_{pr}A^r_0 \\ A^r_0\rho_{rq} & A^r_0\rho_{rs}A^s_0 - \frac{u}{\sqrt{\det \rho}} \end{pmatrix}, \] (4.37)

and $\tilde{\omega}^a = (\tilde{\omega}^p, \tilde{\omega}^0)$ is:
\[ \tilde{\omega}^p = (A^p_i - A^0_0\omega_i)dx^i, \quad \tilde{\omega}^0 = \omega. \] (4.38)

If the forms $\tilde{\omega}^a$ are dualized to scalars $\alpha_a$ as
\[ d\alpha_a = -\det G G_{ab} *_3 d\tilde{\omega}^b, \] (4.39)

the various scalars can be combined into a symmetric unimodular $(D - 2) \times (D - 2)$ matrix
\[ S = \begin{pmatrix} G_{ab} + \frac{1}{\det G}\alpha_a\alpha_b & \frac{1}{\det G}\alpha_a \\ \frac{1}{\det G}\alpha_b & \frac{1}{\det G} \end{pmatrix}. \] (4.40)
In terms of the 4D fields that is
\[ S = \begin{pmatrix}
\rho_{pq} - \frac{1}{u \sqrt{\det \rho}} \alpha_p \alpha_q & \rho_{pr} A^0_r - \frac{1}{u \sqrt{\det \rho}} \alpha_p \alpha_0 & -\frac{1}{u \sqrt{\det \rho}} \alpha_p \\
A^0_r \rho_{rq} - \frac{1}{u \sqrt{\det \rho}} \alpha_0 \alpha_q & A^0_r \rho_{rs} A^s_0 - \frac{u}{\sqrt{\det \rho}} - \frac{1}{u \sqrt{\det \rho}} \alpha_0 \alpha_0 & -\frac{1}{u \sqrt{\det \rho}} \alpha_0 \\
-\frac{1}{u \sqrt{\det \rho}} \alpha_q & -\frac{1}{u \sqrt{\det \rho}} \alpha_0 & 0
\end{pmatrix}. \tag{4.41}
\]

The equations of motion derive from the lagrangian density \( \mathcal{L} = Tr \nabla S S^{-1} \nabla S S^{-1} \), invariant under \( S \rightarrow U^T SU \) for any \( U \) in \( SL(D - 2) \). As this \( SL(D - 2) \) action is transitive on the space of matrices with a given signature, the space of possible \( S \) is the same as the symmetric space \( SL(D - 2)/SO(D - 4, 2) \). Notice that the signature of the stabilizer \( SO(D - 4, 2) \) is appropriate for the reduction from \( (D - 1, 1) \) to \( (3, 0) \) signature. The usual reduction from \( (D - 1, 1) \) to \( (2, 1) \) would give a \( SL(D - 2)/SO(D - 2) \), while the Euclidean reduction from \( (D, 0) \) to \( (3, 0) \) gives \( SL(D - 2)/SO(D - 3, 1) \) \[88\].

The coset representative under the left \( SO(D - 4, 2) \) action can be described in terms of a set of vielbeins:
\[ e^A = E^A_a (dy^a + \omega^a_i dx^i) \quad , \quad e^I = \frac{1}{\det M} e^I_{(3)} \tag{4.42} \]
as
\[ M = \begin{pmatrix}
E^A_a & 0 \\
\frac{1}{\det E} \alpha_{ab} & \frac{1}{\det E}
\end{pmatrix}. \tag{4.43} \]

Then the symmetric \( SO(D - 4, 2) \) invariant matrix
\[ S = MS_0 M^T \tag{4.44} \]
can be used to read off the solution more easily. Without loss of generality, we can take \( S_0 \) to be the signature matrix:
\[ S_0 = \text{Diag}(\eta, -1) = \text{Diag}(1, \cdots, 1, -1, -1) \tag{4.45} \]
The equations of motion are equivalent to the conservation of the $SL(D - 2)$ currents $J = S^{-1}dS$. Some of those currents correspond to the usual gauge currents in 4D: the first $D - 4$ elements of the last column $J_{i,D-2}$ are the KK monopole currents, the first $D - 4$ elements of the row before the last $J_{D-3,i}$ are the KK momentum currents and the element $J_{D-3,D-4}$ is the current for the 3D gauge field $\omega$. Regular 4D solutions must have zero sources for this current, otherwise $\omega$ will not be single valued.

4.3.2 Nilpotency

We now show that all the attractor flows are generated by the nilpotent generators in the coset algebra. To get extremal black hole solutions with a near horizon $AdS_2 \times S^2$, the function $u$ must scale as $r^2$ as $r$ goes to zero while the scalars go to a constant. This makes $S$ diverge as $\frac{1}{r^2}$. The most natural way for $S$ to diverge as $\tau^2$ for large $\tau$ is that $k$ is nilpotent, with

$$k^3 = 0 .$$

(4.46)

This will be the crucial condition through the whole work.

4.3.3 A toy example: Hyperkähler Euclidean metrics in 4D

Not every null geodesic corresponds to extremal black hole solutions. Let’s consider a simple example: hyperkähler euclidean metrics in 4D.

Although this example is not about a black hole, it is still quite instructive. The
3D sigma model is $SL(2)/SO(1,1)$, i.e. $AdS_2$. The coset representative is written as

$$M = \begin{pmatrix}
  u^{1/2} & 0 \\
  a & 1/2 \\
 -a/2 & 1/2 
\end{pmatrix}$$

and the symmetric invariant

$$S = \begin{pmatrix}
  u - a^2/u \\
  -a/u \\
  -a/u \\
  -1/u 
\end{pmatrix}.$$  

A geodesic is the exponential of a Lie algebra element in the orthogonal to the stabilizer. The stabilizer $SO(1,1)$ is generated by $\sigma^1$. A null geodesic is hence the exponential of $k = \sigma^3 \pm i\sigma^2$. This is a nilpotent matrix, $k^2 = 0$, hence $M = 1 + \tau k/2$.

Take:

$$k = \sigma^3 + i\sigma^2 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix};$$

then

$$M = \begin{pmatrix} 1 + \tau/2 & \tau/2 \\ -\tau/2 & 1 - \tau/2 \end{pmatrix};$$

and we can read off the geodesic solution from the invariant

$$S = M^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M = \begin{pmatrix} 1 + \tau & \tau \\ \tau & -1 + \tau \end{pmatrix}.$$ 

Hence

$$u = \frac{1}{1 - \tau}, \quad a = -\frac{\tau}{1 - \tau}.$$ 

Dualizing $a$, we get

$$*d\omega = -\frac{da}{u^2} = \frac{1}{r^2} dr, \quad u^{-1} = 1 + \frac{1}{r}.$$
The 4D Euclidean metric is just the Taub-NUT metric. Notice that the multi-centered Taub-NUT generalization of the metric is obtained by replacing $\tau$ above with some harmonic function $\sum_i \frac{q^i}{|x-x_i|}$. The sigma model equations of motion are equivalent to the conservation of the current $J = S^{-1} \nabla S$, and if $S$ is given as above with $\tau = \tau(\vec{x})$ then the equation of motion are

$$\nabla^2 \tau(\vec{x}) = 0.$$  \hfill (4.54)

### 4.3.4 Single-centered black holes in pure gravity

**Constructing the flow generator $k$**

Now we look in detail at the single-centered black holes in pure gravity. Notice that as $u$ goes to zero $S$ tends asymptotically to a rank one matrix

$$S = -\frac{1}{u \sqrt{\det \rho}} \begin{pmatrix} \alpha_p \alpha_q & \alpha_p \alpha_0 & \alpha_p \\ \alpha_0 \alpha_q & \alpha_0 \alpha_0 & \alpha_0 \\ \alpha_q & \alpha_0 & 1 \end{pmatrix},$$  \hfill (4.55)

hence the matrix $k^2$ should also have rank 1. By inspection of $S$ it is clear that a $k^2$ of rank higher than one gives a geodesic for which the matrix elements of $\rho$ also diverge as $\tau^2$ so that the scalar fields do not converge to fixed attractor values.

Notice that if $k$ is nilpotent, then $S$ is a polynomial in $\frac{1}{r}$, and the various scalars in the solution will all be simple functions of $r$ for such extremal solutions!

The explicit form for $k$ in terms of the charges is then straightforward to write. Consider the Jordan form of $k^2$: as it is nilpotent, the eigenvalues are all zero. As it is rank one, it has one single indecomposable block of size two: \[
\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]
written as

\[ k^2 = -\eta v^T f(P, Q) \]  

(4.56)

with \( v \) null in the metric \( \eta \), and \( f(P, Q) \) any degree-two homogeneous function of the charge \((P, Q)\). This form is chosen so that \( v \) does not scale with the charge \((P, Q)\).

Then \( k \) must have a Jordan form with all eigenvalues zero, one block of size 3:

\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\]

and possibly some other extra blocks of size two. Alternatively, there is a subspace \( V \) annihilated by \( k \), a subspace \( V' \) whose image under \( k \) sits in \( V \) and has the same dimension as \( V' \), and a single vector \( w \) such that \( kw \in V' \) and is non-zero.

\[ kw \subset V', \quad kV' \subset V, \quad kV = 0. \]  

(4.57)

From the symmetry of \( \eta K \), it follows that the space \( kV' \) is made of null vectors only, and that \( kw \) is orthogonal to it. Because \( \eta k^2 = -vv^T \), \( (kw)^T \eta kw \) is negative. Because of the signature of \( \eta \) it is straightforward to see that \( V' \) can be of dimension at most one; hence there are no blocks of size 2 in the Jordan form of \( k \).

Taking this into account, the final form of \( k \) is simply

\[ k = \eta vw^T + \eta wv^T \]  

(4.58)

where \( v \) and \( w \) are two orthogonal \((D-2)\)-dimensional vectors with \( v \) being null and \( w \) having norm \(-f(P, Q)\):}

\[ w\eta v = 0, \quad v\eta v = 0, \quad w\eta w = -f(P, Q). \]  

(4.59)

Using the fact that in \( k \), the first \( D - 4 \) elements of the last column \( K_{i,D-2} \) are the magnetic charges, and the first \( D - 4 \) elements of the row before the last \( K_{D-3,i} \) are
the electric charges, and the element $K_{D-3,D-4}$ is the Taub-NUT charge, which has to vanish, we have $2(D - 4) + 1 = 2D - 7$ conditions. Together with the three constraints coming from the norms and orthogonality condition (4.59), they can be used to solve for the $2D - 4$ degrees of freedom in $(v, w)$.

The full solution of $(v, w)$ requires one to solve some degree-four equations, hence we will leave it in a slightly implicit form. Let $(p, q)$ be two $(D-4)$-dimensional vectors proportional to the magnetic and electric charges, so that the magnetic charge and the electric charge $(P, Q)$ of the 4D gauge fields are

$$P = \sqrt{p^2 + p \cdot q} \cdot p, \quad Q = \sqrt{q^2 + p \cdot q} \cdot q.$$  \hspace{1cm} (4.60)

And we choose $f(P, Q)$ to be

$$f(P, Q) = p \cdot q.$$  \hspace{1cm} (4.61)

The solution of $v$ and $w$ written in terms of $(p, q)$ is

$$v = \frac{1}{\sqrt{p \cdot q}} \begin{pmatrix} q + p \\ -\sqrt{q^2 + p \cdot q} \\ -\sqrt{p^2 + p \cdot q} \end{pmatrix}, \quad w = \frac{\sqrt{p \cdot q}}{2} \begin{pmatrix} q - p \\ -\sqrt{q^2 + p \cdot q} \\ \sqrt{p^2 + p \cdot q} \end{pmatrix},$$  \hspace{1cm} (4.62)

and $k$ can be written as

$$k = \begin{pmatrix} qq^T - pp^T & -\sqrt{q^2 + p \cdot q} q & \sqrt{p^2 + p \cdot q} p \\ \sqrt{q^2 + p \cdot q} q^T & -(q^2 + p \cdot q) & 0 \\ -\sqrt{p^2 + p \cdot q} p^T & 0 & p^2 + p \cdot q \end{pmatrix}.$$  \hspace{1cm} (4.63)

Full flow

First, for the full flow starting from $M_0 = 1$, the scalars for the attractor solution generated by this $k$ can be read off from $S(\tau) = e^{k\tau}$, by comparing with the form of
\[ S \text{ in terms of the 4D fields:} \]
\[ u^{-2} = [1 + (p^2 + p \cdot q)(\tau + \frac{p \cdot q}{2} \tau^2)][1 + (q^2 + p \cdot q)(\tau + \frac{p \cdot q}{2} \tau^2)] \quad (4.64) \]

and
\[ \rho = 1 + \frac{(qq^T - pp^T)\tau + [(p^2 + p \cdot q)qq^T - \frac{p \cdot q}{2} (p + q)(p + q)^T] \tau^2}{1 + (p^2 + p \cdot q)(\tau + \frac{p \cdot q}{2} \tau^2)} \quad (4.65) \]

Notice that as \( \tau \to \infty \), \( \tau^{-2} e^{-2U} \) has the correct limit \( \frac{P \cdot Q}{2} \), which is the entropy where \( P \) and \( Q \) are the physical electric and magnetic charges.

To generalize to arbitrary asymptotic moduli, \( M(\tau) = e^{g/2} e^{k\tau/2} \), and the flow of \( S \) is (4.32), which can be written as \( S(\tau) = e^{K(\tau)} S_0 \), where \( K(\tau) \) is a matrix function.

From now on, we use lower case \( k \) to denote the coset algebra that generates the attractor flow, and capital \( K \) to denote the function which we exponentiate directly to produce the solution.

We will choose \( K(\tau) \) to have the same properties as the generator \( k \):
\[ K^3(\tau) = 0 \quad \text{and} \quad K^2(\tau) \text{ rank one.} \quad (4.66) \]

The equations of motion \( \nabla \cdot (S^{-1} \nabla S) = 0 \) then simplify considerably with this ansatz.

If one further requires that the subspace image of \( K^2(\tau) \) remains constant everywhere, such that
\[ K^2(\tau) \nabla K(\tau) = \nabla K(\tau) K^2(\tau) = 0 \quad (4.67) \]
then the current reduces to
\[ J = S^{-1} \nabla S = S_0 \left( \nabla K(\tau) + \frac{1}{2} [\nabla K(\tau), K(\tau)] \right) S_0 \quad (4.68) \]
and the equations of motion are
\[ \nabla^2 K(\tau) + \frac{1}{2} [\nabla^2 K(\tau), K(\tau)] = 0 \quad (4.69) \]
which is solved by a harmonic $K(\tau)$.

It might appear hard to find a $K(\tau)$ that is harmonic and satisfies all the required constraints. However, by remembering that the constraints dictate $K(\tau)$ to have the form:

$$K(\tau) = \eta V W^T + \eta W V^T,$$

where $V$ is null and does not scale with the charge $(P, Q)$, and $W$ orthogonal to $V$ everywhere, one can simply pick a constant null vector $V = v'$ and a harmonic vector $W(\tau)$ everywhere orthogonal to $v'$:

$$W(\tau) = w'\tau + m \quad \text{with} \quad v' \cdot W(\tau) = 0.$$  \hspace{1cm} (4.71)

Here $m$ is a $(D-2)$-vector and contains the information of asymptotic moduli. Thus an appropriate $K(\tau)$ is built:

$$K(\tau) = k'\tau + g$$  \hspace{1cm} (4.72)

where

$$k' = \eta v' w'^T + \eta w' v'^T, \quad g = \eta v' m^T + \eta m v'^T.$$  \hspace{1cm} (4.73)

Now we need to solve for $(v', w')$ for the same charge $(P, Q)$ but in the presence of $m$. The form of $g$ guaranteed that

$$[k', g] = 0,$$  \hspace{1cm} (4.74)

where we have used the fact that $v'$ is null and $w'$ is orthogonal to $v'$. Therefore, shifting the starting point of moduli does not change the current as a function of $(v, w)$:

$$J(v', w') = S_0 \left( \frac{k'}{r^2} \right) S_0 = S_0 \left( \frac{\eta v'(w')^T + \eta w'(v')^T}{r^2} \right) S_0.$$  \hspace{1cm} (4.75)
Thus, the solution of \((v', w')\) in terms of charges solved from the current does not change as we vary the starting point of the flow, i.e. they do not depend on the asymptotic moduli:

\[
\begin{align*}
    v'(Q) &= v(Q) \\
    w'(Q) &= w(Q)
\end{align*}
\]  
(4.76)

In summary, the flow with arbitrary starting point is simply generated by

\[
K(\tau) = \eta v W(\tau)^T + \eta W(\tau)v^T \quad \text{with} \quad W = w \tau + m ,
\]  
(4.77)

where \((v, w)\) only depend on the charges \((P, Q)\) and \(m\) gives the asymptotic moduli.

**Example: 5D pure gravity compactified on a circle.**

Consider for example the case of extremal black holes in \(D = 5\) pure gravity compactified on a circle. The 3D sigma model is \(SL(3)/SO(1, 2)\). The symmetric invariant is

\[
S_{gr} = \begin{pmatrix}
    \rho - \frac{1}{u\sqrt{\rho}}\alpha_1\alpha_1 & \rho A_0 - \frac{1}{u\sqrt{\rho}}\alpha_1\alpha_0 & -\frac{1}{u\sqrt{\rho}}\alpha_1 \\
    \rho A_0 - \frac{1}{u\sqrt{\rho}}\alpha_0\alpha_0 & \rho(A_0)^2 - \frac{u}{\sqrt{\rho}} - \frac{1}{u\sqrt{\rho}}A_0\alpha_0 & -\frac{1}{u\sqrt{\rho}}\alpha_0 \\
    -\frac{1}{u\sqrt{\rho}}\alpha_1 & -\frac{1}{u\sqrt{\rho}}\alpha_0 & -\frac{1}{u\sqrt{\rho}}
\end{pmatrix}.
\]  
(4.78)

Then we can calculate \(S = S_0 e^{k\tau}\) using \((4.63)\) and compare the result to \(S_{gr}\) above to solve for all the scalars. We find that

\[
\begin{align*}
    e^{-2U} &= \sqrt{[1 + (q^2 + pq)(\tau + \frac{pq}{2}\tau^2)][1 + (p^2 + pq)(\tau + \frac{pq}{2}\tau^2)]} \quad (4.79) \\
    \rho &= \frac{1 + (q^2 + pq)(\tau + \frac{pq}{2}\tau^2)}{1 + (p^2 + pq)(\tau + \frac{pq}{2}\tau^2)}
\end{align*}
\]  
(4.80)

when starting from the identity. If we allow arbitrary \(g\) the flow is too complicated to write explicitly here, but the attractor value of \(\rho\) is the same: \(q/p\).
4.3.5 Multi-centered solutions in pure gravity

In the context of pure gravity compactified on a torus, we can also give some examples of multi-centered solutions in the same spirit as the ones for BPS solutions in $\mathcal{N} = 2$ supergravity, though some important features of the latter are not present here.

We are interested in solutions given in terms of harmonic functions which can generalize the single-centered extremal solutions presented above. Similar to the single-centered case, we exponentiate a matrix function $K(\vec{x})$:

$$S(\vec{x}) = e^{K(\vec{x})}S_0 .$$  \hspace{1cm} (4.81)

We will choose $K(\vec{x})$ to have the same properties of the generator $k$:

$$K^3(\vec{x}) = 0 \quad \text{and} \quad K^2(\vec{x}) \text{ rank one} .$$  \hspace{1cm} (4.82)

Using a similar argument to the single-centered flow, we require that the subspace image of $K^2(\vec{x})$ remains constant everywhere, such that

$$K^2(\vec{x})\nabla K(\vec{x}) = \nabla K(\vec{x})K^2(\vec{x}) = 0 ,$$  \hspace{1cm} (4.83)

then the equations of motion are

$$\nabla^2 K(\vec{x}) + \frac{1}{2}[\nabla^2 K(\vec{x}), K(\vec{x})] = 0 ,$$  \hspace{1cm} (4.84)

which is solved by a harmonic $K(\vec{x})$.

A multi-centered $K(\vec{x})$ that is harmonic and satisfies all the required constraints can then be built in the same way as the single-centered one:

$$K(\vec{x}) = \eta v W(\vec{x})^T + \eta W(\vec{x})v^T ,$$  \hspace{1cm} (4.85)
where $v$ is the same constant null vector as in $k$, and $W(\vec{x})$ is the multi-centered harmonic function:

$$W(\vec{x}) = \sum_i \frac{w_i}{|\vec{x} - \vec{x}_i|} + m$$

(4.86)

where $\vec{x}_i$ is the position of the $i^{th}$ center, and $w_i$ is determined by the charges at the $i^{th}$ center, and $m$ is related to the moduli at infinity. Requiring $W(\vec{x})$ to be orthogonal to $v$ everywhere gives the following constraints on the $\{w_i, m\}$: First, taking $\vec{x}$ to infinity, it gives

$$v \cdot m = 0$$

(4.87)

Second, $w_i$ are orthogonal to $v$:

$$v \cdot w_i = 0$$

(4.88)

In addition to the constraint from the zero Taub-NUT charge condition $-bz - cy = 0$, this makes the space of each possible $w_i$ only $(D-4)$-dimensional. That is, though $W$ is a $(D-2)$-vector, one has only $(D-4)$ independent harmonic functions to work with, because of the orthogonality to $v$ and the requirement of no timelike NUT charges. This makes the solution relatively boring.

The multi-centered solution in pure gravity does not have the characteristic features of the typical BPS multi-centered solution in $\mathcal{N} = 2$ supergravity, where many centers with relatively non-local charges form bound states which carry intrinsic angular momentum.

The basic reason is that when the ansatz $K(\vec{x}) = \eta v W^T(\vec{x}) + \eta W(\vec{x})v^T$ is used, the second term of the conserved currents $J = S_0(\nabla K + \frac{1}{2}[\nabla K, K])S_0$ drops out. The first result is that the charges of the various centers in the solution can be read off directly from $(v, w_i)$, and they do not depend on the positions, charges of the other
centers. Thus, there is no constraint on the position of each center as in the $\mathcal{N} = 2$ BPS multi-centered solution: centers can be moved around freely.

Moreover, the condition of no timelike Taub-NUT charge is a linear constraint on the charges at each center which results in a static 4D solution, as $\ast d\omega = 0$ leads to $\omega = 0$. Therefore, no angular momentum is present.

### 4.4 Attractor Flows in $G_2(2)/(SL(2, \mathbb{R}) \times SL(2, \mathbb{R}))$

We now tackle a more complicated subject: $\mathcal{N} = 2$ supergravity coupled to one vector multiplet. First, we reduce the theory down to three dimensions and derive the metric for the resulting moduli space, which is the coset $G_2(2)/(SL(2, \mathbb{R}) \times SL(2, \mathbb{R}))$.\(^2\) We then discuss the Cartan and Iwasawa decompositions of the group $G_2(2)$, which we use to construct the coset algebra and translate the flow of coset representative into the flow of the moduli fields, respectively. We then specify the representation of $G_2(2)$ we will be working with, and describe the form of attractor flow generators in this representation.

#### 4.4.1 The moduli space $\mathcal{M}_{3D}$

The 3D moduli space for $\mathcal{N} = 2, d = 4$ supergravity coupled to $n_V$ vector multiplets is well-studied, for example in [46, 35, 127]. Some of the main results are compiled in Appendix A. Here we briefly review the essential points.

\(^2\)Other work on this coset space has appeared recently, including [25, 37, 80].
The bosonic part of the action is:

\[
S = -\frac{1}{16\pi} \int d^4x \sqrt{|g^{(4)}} \left[ R - 2g_{ij}dz^i \wedge \ast_4dz^j - F^I \wedge G_I \right],
\]

where \( I = 0, 1, \ldots, n_V \), and \( G_I = (\text{Re} \mathcal{N})_{IJ} F^J + (\text{Im} \mathcal{N})_{IJ} \ast F^J \). For a model endowed with a prepotential \( F(X) \),

\[
\mathcal{N}_{IJ} = F_{IJ} + 2i \frac{\text{Im} F \cdot X_I (\text{Im} F \cdot X)_J}{X \cdot \text{Im} F \cdot X},
\]

where \( F_{IJ} = \partial_I \partial_J F(X) \). We reduce to three dimensions, dualizing the vectors \((\omega, A^I)\) to the scalars \((\sigma, B_I)\), and renaming \( A^I_0 \) as \( A^I \). The metric of \( \mathcal{M}_{3\text{D}}^* \) is then

\[
ds^2 = \text{d}U \cdot \text{d}U + \frac{1}{4} e^{-4U} (\text{d}\sigma + A^I \text{d}B_I - B_I \text{d}A^I) \cdot (\text{d}\sigma + A^I \text{d}B_I - B_I \text{d}A^I) + g_{ij}(z, \bar{z}) \text{d}z^i \cdot \text{d}\bar{z}^j + \frac{1}{2} e^{-2U} [(\text{Im} \mathcal{N}^{-1})^{IJ}(\text{d}B_I + \mathcal{N}_{IK} \text{d}A^K) \cdot (\text{d}B_J + \overline{\mathcal{N}}_{KL} \text{d}A^L)].
\]

It is a para-quaternionic-Kähler manifold. Since the holonomy is reduced from \( SO(4n_V + 4) \) to \( Sp(2, \mathbb{R}) \times Sp(2n_V + 2, \mathbb{R}) \), the vielbein has two indices \((\alpha, A)\) transforming under \( Sp(2, \mathbb{R}) \) and \( Sp(2n_V + 2, \mathbb{R}) \), respectively. The para-quaternionic vielbein is the analytical continuation of the quaternionic vielbein computed in [61]. The explicit form is given in Appendix A.

For \( n_V = 1 \), \( X^I = (X^0, X^1) \). For our purpose, we choose the prepotential

\[
F(X) = -\frac{(X^1)^3}{X^0}.
\]

The metric of \( \mathcal{M}_{3\text{D}}^* \) with one-modulus is (4.91) with \( g_{zz} = \frac{3}{4y^2} \) and \( \mathcal{N} \) and \( (\text{Im} \mathcal{N})^{-1} \) being

\[
\mathcal{N} = \begin{pmatrix}
-2(x - iy)(x + iy)^2 & 3x(x + iy) \\
3x(x + iy) & -3(2x + iy)
\end{pmatrix}, \quad \text{Im} \mathcal{N}^{-1} = -\frac{1}{y^3} \begin{pmatrix} 1 & x \\ x & 3x^2 + y^2 \end{pmatrix}.
\]
The isometries of the $\mathcal{M}_{3D}^*$ descend from the symmetries of the 4D system. The gauge symmetries in 4D give shifting isometries of $\mathcal{M}_{3D}^*$, whose associated conserved charges are:

$$q_I d\tau = J_{A^I} = P_{A^I} - B_I P_\sigma, \quad p_I d\tau = J_{B_I} = P_{B_I} + A^I P_\sigma, \quad k d\tau = J_{\sigma} = P_\sigma$$

(4.93)

where the momenta $\{P_\sigma, P_{A^I}, P_{B_I}\}$ are

$$P_\sigma = \frac{1}{2} e^{-4U} (d\sigma + A^I dB_I - B_I dA^I),$$

$$P_{A^I} = e^{-2U} [(\text{Im} N)_{IJ} dA^J + (\text{Re} N)_{IJ} (\text{Im} N^{-1})^{JK} (dB_K + (\text{Re} N)_{KL} dA^L)] - B_I P_\sigma,$$

$$P_{B_I} = e^{-2U} [(\text{Im} N^{-1})^{IJ} (dB_J + (\text{Re} N)_{JK} dA^K)] + A^I P_\sigma.$$  

(4.94)

Here $\tau$ is the affine parameter defined as $d\tau \equiv - \ast_3 \sin \theta d\theta d\phi$. $(p^0, p^1, q_1, q_0)$ are the D6-D4-D2-D0 charges, and $k$ the Taub-NUT charge. A non-zero $k$ gives rise to closed time-like curves, so we will set $k = 0$ from now on.

Note that the time translational invariance in 4D gives rise to the conserved current

$$J_U = P_U + 2\sigma J_\sigma + A^I J_{A^I} + B^I J_{B^I},$$

(4.95)

where $P_U = 2dU$. The corresponding conserved charge is the ADM mass: $2M_{ADM} d\tau = J_U$.

### 4.4.2 Extracting the coordinates from the coset elements

The metric (4.91) for the case $n_V = 1$ describes an eight-dimensional manifold with coordinates $\phi^a = \{u, x, y, \sigma, A^0, A^1, B_1, B_0\}$. This manifold is the coset space $G_{2(2)}/(\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}))$. The root diagram for the Cartan decomposition of $G_{2(2)}$ is shown in Figure 4.1.
The six roots that lie on the horizontal and vertical axes $\{L^{\pm}_h, L^3_h, L^\pm_v, L^3_v\}$ are the six non-compact generators of the subgroup $H = SL(2, \mathbb{R})_h \times SL(2, \mathbb{R})_v$:

$$[L^3_{h/v}, L^\pm_{h/v}] = \pm L^\pm_{h/v}, \quad [L^+_{h/v}, L^-_{h/v}] = 2L^3_{h/v}, \quad (4.96)$$

and the two vertical columns of eight roots $\{a_{\alpha A}\}$ are the basis of the subspace $K$. 

$\{a_{1A}, a_{2A}\}$ for each $A$ is a spin-1/2 doublet under the horizontal $SL(2, \mathbb{R})$:

$$[L^3_h, \begin{pmatrix} a_{1A} \\ a_{2A} \end{pmatrix}] = \begin{pmatrix} -\frac{1}{2}a_{1A} \\ \frac{1}{2}a_{2A} \end{pmatrix}, \quad [L^+_h, \begin{pmatrix} a_{1A} \\ a_{2A} \end{pmatrix}] = \begin{pmatrix} 0 \\ a_{1A} \end{pmatrix},$$
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\[
\left[ L_h^{-}, \begin{pmatrix} a_{1A} \\ a_{2A} \end{pmatrix} \right] = \begin{pmatrix} -a_{2A} \\ 0 \end{pmatrix}.
\]

And \{a_{\alpha 1}, a_{\alpha 2}, a_{\alpha 3}, a_{\alpha 4}\} for each \(\alpha\) span a spin-3/2 representation of the vertical \(SL(2, \mathbb{R})\):

\[
\left[ L_3^v, \begin{pmatrix} a_{\alpha 1} \\ a_{\alpha 2} \\ a_{\alpha 3} \\ a_{\alpha 4} \end{pmatrix} \right] = \begin{pmatrix} -3a_{\alpha 1} \\ -\frac{3}{2}a_{\alpha 2} \\ \frac{1}{2}a_{\alpha 3} \\ \frac{3}{2}a_{\alpha 4} \end{pmatrix},
\]

\[
\left[ L_+^v, \begin{pmatrix} a_{\alpha 1} \\ a_{\alpha 2} \\ a_{\alpha 3} \\ a_{\alpha 4} \end{pmatrix} \right] = \begin{pmatrix} 0 \\ 3a_{\alpha 1} \\ 2a_{\alpha 2} \\ a_{\alpha 3} \end{pmatrix},
\]

\[
\left[ L_-^v, \begin{pmatrix} a_{\alpha 1} \\ a_{\alpha 2} \\ a_{\alpha 3} \\ a_{\alpha 4} \end{pmatrix} \right] = \begin{pmatrix} -a_{\alpha 2} \\ -2a_{\alpha 3} \\ -3a_{\alpha 4} \\ 0 \end{pmatrix}.
\]

All the commutators can be easily read off from the Root diagram (4.1), we will only write down the following ones which will be useful later.

\[
[a_{11}, a_{14}] = -\frac{1}{3}[a_{12}, a_{13}] = -4L_h^+ \quad [a_{21}, a_{24}] = -\frac{1}{3}[a_{22}, a_{23}] = -4L_h^-.
\]

Being semisimple, the algebra of \(G_{2(2)}\) has the Iwasawa decomposition \(g = h \oplus a \oplus n\), where \(a\) is the maximal abelian subspace of \(k\), and \(n\) is the nilpotent subspace of the positive root space \(\Sigma^+\) of \(a\). In Figure 4.2, we show the Iwasawa decomposition of \(G_{2(2)}\).

The two Cartan generators in \(a\) are \{\(u, y\)\}, and \{\(x, \sigma, A^0, A^1, B_1, B_0\)\} span a nilpotent subspace \(n\): \(n^7 = 0\) for \(n \in n\). \(a\) and \(n\) together generate the solvable subgroup \(Solv\) of \(G\), which act transitively on \(M_{3D} = G_{2(2)}/SL(2, \mathbb{R}) \times SL(2, \mathbb{R})\). In
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\[
\hat{A}_0 \quad \hat{A}_1 \quad \hat{B}_0 \quad \hat{B}_1
\]

\[
\hat{x} \quad \hat{y} \quad \hat{u} \quad \hat{a}
\]

Figure 4.2: Root Diagram of Isometry of \( M_{3D} = G_{2(2)}/(SL(2, \mathbb{R}) \times SL(2, \mathbb{R})) \).

\{u, y, x, \sigma, A^0, A^1, B_1, B_0\} generates the solvable subgroup.

In particular, \( y \) generates the rescaling of \( y \), and \( \{u, x, \sigma, A^0, A^1, B_1, B_0\} \) generates the translation of \( \{U, x, \sigma, A^0, A^1, B_1, B_0\} \). The moduli space \( M_{3D} \) can be parameterized by the solvable elements:

\[
\Sigma(\phi) = e^{Uu+\ln y} e^{x+\sigma} e^{A^I+B_1+B_0+\sigma}. \tag{4.98}
\]

The origin of the moduli space

\[
a = A^0 = A^1 = B_1 = B_0 = 0 \quad x = 0 \quad y = u = 1 \tag{4.99}
\]

correspond to \( \Sigma(\phi) = 1 \).
In Fig 4.2, the isometries are plotted according to their eigenvalues under the two Cartan generators $u$ and $y$ [46]. $\{u, y\}$ are related to $a_{\alpha A}$ by\footnote{The matrix representation of $u$ and $y$ are:

$$u = \text{Diag}[0, \frac{1}{2}, -\frac{1}{2}, 0, -\frac{1}{2}, \frac{1}{2}, 0] \quad y = \text{Diag}[1, -\frac{1}{2}, -\frac{1}{2}, -1, \frac{1}{2}, \frac{1}{2}, 0]$$

\footnotetext[3]{The matrix representation of $u$ and $y$ are:

$$u = \text{Diag}[0, \frac{1}{2}, -\frac{1}{2}, 0, -\frac{1}{2}, \frac{1}{2}, 0] \quad y = \text{Diag}[1, -\frac{1}{2}, -\frac{1}{2}, -1, \frac{1}{2}, \frac{1}{2}, 0]$$

}\n
$$u = -\frac{1}{8}[(a_{11} + a_{24}) - (a_{13} + a_{22})] \quad y = \frac{1}{8}[3(a_{11} + a_{24}) + (a_{13} + a_{22})]. \quad (4.100)$$

The three generators $\{\sigma, u, \hat{\sigma}\}$ on the horizontal axis and $\{x, y, \hat{x}\}$ on the vertical axis form the horizontal and vertical $SL(2, \mathbb{R})$, respectively. The vertical $SL(2, \mathbb{R})$ generates the duality invariance. Denote the two vertical columns of eight isometries as

$$\begin{pmatrix}
\xi_{21} & \xi_{11} \\
\xi_{22} & \xi_{12} \\
\xi_{23} & \xi_{13} \\
\xi_{24} & \xi_{14}
\end{pmatrix} \equiv \begin{pmatrix}
-\hat{A}^0 & B_0 \\
-3\hat{A}^1 & 3B_1 \\
\hat{B}_1 & -A^1 \\
-\hat{B}_0 & A^0
\end{pmatrix},$$

$\{\xi_{1A}, \xi_{2A}\}$ for each $A$ span a spin-$1/2$ representation of the horizontal $SL(2, \mathbb{R})$, and $\{\xi_{\alpha 1}, \xi_{\alpha 2}, \xi_{\alpha 3}, \xi_{\alpha 4}\}$ for each $\alpha$ span a spin-$3/2$ representation of the vertical $SL(2, \mathbb{R})$.

Parameterizing the coset representative $M$ as the solvable elements, the symmetric matrix $S$ can be written in terms of the eight coordinates $\phi^n$, from the solvable elements $\Sigma$

$$S(\phi) = \Sigma(\phi)S_0\Sigma(\phi)^T. \quad (4.101)$$

The coordinates are read off from the symmetric matrix $S$.\footnote{The matrix representation of $u$ and $y$ are:

$$u = \text{Diag}[0, \frac{1}{2}, -\frac{1}{2}, 0, -\frac{1}{2}, \frac{1}{2}, 0] \quad y = \text{Diag}[1, -\frac{1}{2}, -\frac{1}{2}, -1, \frac{1}{2}, \frac{1}{2}, 0]$$

\footnotetext[3]{The matrix representation of $u$ and $y$ are:

$$u = \text{Diag}[0, \frac{1}{2}, -\frac{1}{2}, 0, -\frac{1}{2}, \frac{1}{2}, 0] \quad y = \text{Diag}[1, -\frac{1}{2}, -\frac{1}{2}, -1, \frac{1}{2}, \frac{1}{2}, 0]$$}
4.4.3 Nilpotency of the attractor flow generator $k$.

The near-horizon geometry of the 4D attractor is $AdS_2 \times S^2$, i.e.

$$e^{-U} \to \sqrt{V_{BH}} \tau \quad \text{as} \quad \tau \to \infty .$$

(4.102)

In terms of the variable $u \equiv e^{2U}$

$$u \to \frac{1}{V_{BH}} \tau^{-2} \quad \text{as} \quad \tau \to \infty .$$

(4.103)

The solvable element is

$$M = e^{U u + \cdots} \sim u^{\frac{1}{2}} u .$$

(4.104)

As the flow goes to the near-horizon, $u \to 0$

$$M(\tau) \sim u^{-\ell/2} \sim \tau^\ell ,$$

(4.105)

where $-\ell$ is the lowest eigenvalue of $u$. That is, $M(\tau)$ is a polynomial function of $\tau$.

On the other hand, since the geodesic flow is generated by $k$ via

$$M(\tau) = M(0)e^{k\tau/2} ,$$

(4.106)

i.e., $M(\tau)$ is an exponential function of $\tau$. To reconcile the two statements, $k$ must be nilpotent:

$$k^{\ell+1} = 0 .$$

(4.107)

That is, the element in $k$ that generates the attractor flow is nilpotent. Moreover, by looking at the weights of the fundamental representation of $G_2$, we see that $\ell = 2$

$$k_+^3 = 0 .$$

(4.108)
Moreover, the nilpotency of the attractor flow generators guarantees that it is null:

$$k^3 = 0 \implies (k^2)^2 = 0 \implies Tr(k^2) = 0 ,$$

(4.109)

which means that \(k\) is null.

### 4.4.4 Properties of attractor flow

The scalar moduli space is parameterized by a symmetric \(7 \times 7\) matrix \(S\) which sits in \(G_2(2)\), i.e. preserves a non-degenerate three form \(w_{ijk}\) such that

\[
\eta_{is} = w_{is}w_{stu}w_{mn}e^{jkmno}\]

is a metric with signature \((4, 3)\) normalized so that \(\eta^2 = 1\).

To facilitate the comparison with the pure 5D gravity case we decompose \(7\) as \(3 \oplus 3 + 1\) of \(SL(3)\) and pick as non-zero components of \(w\) the \(3 \wedge 3 \wedge 3, \bar{3} \wedge \bar{3} \wedge \bar{3}\) and \(3 \otimes \bar{3} \otimes 1\) as

\[
w = dx_1 \wedge dx_2 \wedge dx_3 + dy_1 \wedge dy_2 \wedge dy_3 - \frac{1}{\sqrt{2}}dx_a \wedge dy_a \wedge dz .
\]

(4.110)

The resulting expression for \(\eta\) is

\[
\eta = dx_a dy_a - dz^2 .
\]

(4.111)

We know that \(k\) must be an element of \(G_2(2)\), hence also of \(SO(4, 3)\). As in the pure gravity case, we choose the representation such that

\[
S_0kS_0 = k^T , \quad S_0k^2S_0 = (k^2)^T .
\]

(4.112)

In this base a \(G_2(2)\) Lie algebra element is given as

\[
k = \begin{pmatrix}
A_{i1}^{j1} & \epsilon_{i1j2k}v^k & \sqrt{2}w_{i1} \\
\epsilon^{i2j1k}w_k & -A_{j2}^{i2} & -\sqrt{2}v^{i2} \\
-\sqrt{2}v^{j1} & \sqrt{2}w_{j2} & 0
\end{pmatrix}.
\]

(4.113)
Here $A$ is a traceless $3 \times 3$ matrix. $S$ is a symmetric element in $G_{2(2)}$ with signature \{1, -1, -1, 1, -1, -1, 1\}, i.e. $S = MS_0M^T$ with

$$S_0 = \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & \eta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \text{Diag}(1, -1, -1, 1, -1, -1, 1) \quad (4.114)$$

where $\eta_1$ is the one for pure gravity.

If the gauge field is turned off, then $S$ is block diagonal

$$S|_{F=0} = \begin{pmatrix} S_{gr} & 0 & 0 \\ 0 & S_{gr}^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (4.115)$$

where $S_{gr}$ is the same as the one for pure 5D gravity. Turning on a non-zero 5D vector field corresponds to a more general $S$:

$$S = e^{k_1} (S|_{F=0}) e^{k_1} \quad (4.116)$$

with $k_1$ a $G_{2(2)}$ Lie algebra matrix with $w_1$ equal to the fifth component of the gauge field, $v^2$ equal to the time component of the gauge field and $w_3$ equal to the scalar dual to the three-dimensional part of the gauge field.

In this representation, $(x, y)$ can be extracted from the symmetric matrix $S$ via:

$$x(\tau) = -\frac{S_{35}(\tau)}{S_{33}(\tau)}, \quad y^2 = \frac{S_{33}(\tau)S_{55}(\tau) - S_{35}(\tau)^2}{S_{33}(\tau)^2} \quad (4.117)$$

And $u$ via:

$$u = \frac{1}{\sqrt{S_{33}(\tau)S_{55}(\tau) - S_{35}(\tau)^2}} \quad (4.118)$$
The 4D gauge currents sit in \( J = S^{-1} \nabla S \), where \( J_{12} (J_{31}) \) is again the electric(magnetic) current for the \( KK \) photon, \( J_{32} \) the timelike NUT current, and \( J_{72} (J_{51}) \) the electric(magnetic) current for the reduction of the 5D gauge field.

\[
J_{32} = -2J_\sigma \quad J_{12} = \sqrt{2}J_{A^0} \quad J_{72} = \frac{2}{3}J_{A^1} \quad J_{51} = -\sqrt{2}J_{B_1} \quad J_{31} = \sqrt{2}J_{B_0}.
\]

Moreover,
\[
J_{22} - J_{33} = 2J_U.
\]

We use \( Q \) to denote the charge matrix, where it relates to the D-brane charge \( \{ p^0, p^1, q_1, q_0 \} \) and the vanishing NUT charge \( k \) by
\[
(Q_{31}, Q_{51}, Q_{72}, Q_{12}) = (\sqrt{2}p^0, -\sqrt{2}p^1, \frac{2}{3}q_1, \sqrt{2}q_0), \quad Q_{32} = -2k = 0.
\]

Since \( k \) is nilpotent: \( k^3 = 0 \),
\[
S = e^{k\tau} S_0 = (1 + k\tau + \frac{1}{2}k^2\tau^2)S_0.
\]

The \( AdS_2 \times S^2 \) near-horizon geometry of the 4D attractor dictates \( u = \frac{1}{V_{BH}} \tau^{-2} \) as \( \tau \to \infty \). Therefore, the flow generator \( k \) can be obtained by
\[
k^2 = 2V_{BH}|_\ast (uS|_{u \to 0})S_0.
\]

Computing \( k^2 \) using \( S \) constructed from the solvable elements \( \Sigma(\phi) \) shows that \( k^2 \) is of rank two, its Jordan form has two blocks of size 3.\(^4\) It can be written as
\[
k^2 = \sum_{a,b=1,2} v_a v_b^T c_{ab} S_0
\]

\(^4\)In fact, the real \( G_{2(2)} \) group has two third-degree nilpotent orbits, and it can be shown algebraically that in both orbits, \( k^2 \) is of rank two and has Jordan form with two blocks of size 3 [38].
with \( v_a \) null and orthogonal to each other: \( v_a \cdot v_b \equiv v_a^T S_0 v_b = 0 \), and \( c_{ab} \) depends on the particular choice of \( k \). Thus \( k \) can be expressed as:

\[
k = \sum_{a=1,2} (v_a w_a^T + w_a v_a^T) S_0
\]

(4.125)

where each \( w_a \) is orthogonal to both \( v_a \): \( w_a \cdot v_a = 0 \), and \( w_a \) satisfy

\[
w_a \cdot w_b = c_{ab}.
\]

(4.126)

Parallel to the pure gravity case, the single-centered attractor flow is constructed as \( S(\tau) = e^{K(\tau)} S_0 \), where we choose \( K(\tau) \) to have the same properties as the generator \( k \):

\[
K^3(\tau) = 0 \quad \text{and} \quad K^2(\tau) \text{ rank two.}
\]

(4.127)

This determines \( K(\tau) = k \tau + g \) where

\[
k = \sum_{a=1,2} [v_a w_a^T + w_a v_a^T] S_0 \quad \text{and} \quad g = \sum_{a=1,2} [v_a m_a^T + m_a v_a^T] S_0
\]

(4.128)

where the two 7-vectors \( m_a \)'s are orthogonal to \( v_a \) and contain the information of asymptotic moduli. Using \([[k, g], g] = 0\), the current is reduced to

\[
J = \frac{S_0 (k + \frac{1}{2} [k, g]) S_0}{r^2}
\]

(4.129)

from which we obtain \( v_a \) and \( w_a \) in terms of the charges and the asymptotic moduli.

### 4.5 Flow Generators in the \( G_{2(2)}/(SL(2, \mathbb{R}) \times SL(2, \mathbb{R})) \) Model

We now explicitly construct the generators of single-centered attractor flows. We start with the BPS flow which is associated with a specific combination of the coset
algebra generators $a_{\alpha A}$. It can be derived from the condition of preservation of supersymmetry. We then construct the non-BPS attractor flow generator in analogy with the BPS one. In Section 4.5.2, we write $k_{BPS}$ and $k_{nonBPS}$ in terms of the $v_a$ and $w_a$ vectors. This form will be especially helpful in generalizing to the multi-centered case.

4.5.1 Construction of flow generators

Constructing $k_{BPS}$ using supersymmetry

To describe BPS trajectories it is useful to remember that the stabilizer of $S$ in $G_{2(2)}$ is $SO(1, 2) \times SO(1, 2)$, corresponding to the elements of $G_{2(2)}$ which are antisymmetric after multiplication by $S_0$. Geodesics are exponentials of elements that are symmetric after multiplication by $S_0$. Such elements sit in a $(2, 4)$ representation of $SO(1, 2) \times SO(1, 2)$. A BPS trajectory is highest weight for the first $SO(1, 2)$. Labelling the symmetric generators as $a_{\alpha A}$ under the two $SO(1, 2)$ groups, a BPS trajectory is generated by

$$k_{BPS} = a_{\alpha A} C^A \dot{z}^\alpha.$$  \hspace{1cm} (4.130)

The twistor $z$ and the coefficients $C^A$ are fixed in terms of the charges of the extremal BPS black hole and the condition of zero time-like NUT charge.

To see why this is true, expand the coset element $k_{BPS}$ that generates the BPS attractor flow using $a_{\alpha A}$:

$$k_{BPS} = a_{\alpha A} C^{\alpha A}$$ \hspace{1cm} (4.131)

where $C^{\alpha A}$ are conserved along the flow. On the other hand, the conserved currents in the homogeneous space are constructed by projecting the one-form valued Lie algebra
$g^{-1} \cdot dg$ onto $k$, which gives the vielbein in the symmetric space:

$$g^{-1}dg|_k = a_{\alpha A} V^{\alpha A} \quad (4.132)$$

where $V^{\alpha A}$ is conserved:

$$\frac{d}{d\tau} \left( V^{\alpha A} \dot{\phi}^a \right) = 0 . \quad (4.133)$$

Therefore, the expansion coefficients of $k_{BPS}$ are

$$C^{\alpha A} = V^{\alpha A} \dot{\phi}^a . \quad (4.134)$$

In terms of the vielbein, the supersymmetry condition that gives the BPS geodesics are written as [127]:

$$V^{\alpha A} z_\alpha = 0 . \quad (4.135)$$

That is:

$$V^{\alpha A} \dot{\phi}^a z_\alpha = 0 \quad \implies \quad C^{\alpha A} z_\alpha = 0 . \quad (4.136)$$

Define $z^\alpha = \epsilon^{\alpha \beta} z_\beta$,

$$C^{\alpha A} = C^{A} z^\alpha . \quad (4.137)$$

Therefore, the coset element $k_{BPS}$ is expanded by the coset algebra basis $a_{\alpha A}$ as

$$k_{BPS} = a_{\alpha A} C^{A} z^\alpha . \quad (4.138)$$

Note that $k_{BPS}$ has five parameters $(C^A, z)$ where $A = 1, \ldots, 4$. As will be shown later, $z$ can actually be determined by $(C^A)$ and moduli at infinity. So the geodesic generated by $k_{BPS}$ is indeed a four-parameter family. It is easy to show that $k_{BPS}$ is null, but more importantly, it is nilpotent:

$$k_{BPS}^3 = 0 . \quad (4.138)$$

As will be shown later, $k_{BPS}$ indeed gives the correct BPS attractor flow.
**Constructing** $k_{NonBPS}$

To construct the non-BPS attractor flow, one needs to find an element in the coset algebra distinct from $k_{BPS}$ that satisfies:

$$k_{NonBPS}^3 = 0 .$$  \hfill (4.139)

The hint again comes from the BPS generator. Note that $k_{BPS} = a_{\alpha A} P^A z^\alpha$ can be written as:

$$k_{BPS} = e^{-zL_h} k_{BPS}^0 e^{zL_h} ,$$  \hfill (4.140)

where $k_{BPS}^0$ spans only the right four coset generators $a_{1A}$:

$$k_{BPS}^0 = a_{1A} C^A .$$  \hfill (4.141)

That is, $k_{BPS}$ is generated by starting with the element spanning the four generators annihilated by the horizontal $SL(2)$ raising operator $L_h^+$, then conjugating with the horizontal $SL(2)$ lowering operator $L_h^-$. And it is very easy to show that $(k_{BPS}^0)^3 = 0$ which proves $(k_{BPS})^3 = 0$.

In $G_2(2)/SL(2,\mathbb{R})^2$, there are two third-degree nilpotent generators in total [38]. And since there are only two $SL(2,\mathbb{R})$‘s inside $H$, a natural guess for a non-BPS solution is to look at vectors with fixed properties under the second $SL(2,\mathbb{R})$ group. An interesting condition is to have positive charge under some rotation of $L_v^3$, i.e. an $SL(2,\mathbb{R})$ rotation of $\sum_{A=1,2} a_{\alpha A} C^{\alpha A}$. Therefore, this suggests us to start with the element spanning the four generators annihilated by the square of the vertical $SL(2,\mathbb{R})$ raising operator $(L_v^+)^2$ and then conjugate with the vertical $SL(2,\mathbb{R})$ lowering operator $L_v^-:

$$k_{NonBPS}(z) = e^{-zL_v} k_{NonBPS}^0 e^{zL_v} ,$$  \hfill (4.142)
where

\[ k_{\text{NonBPS}}^0 = a_{\alpha a} C^{\alpha a} \quad \text{where} \quad \alpha, a = 1, 2. \tag{4.143} \]

And one can show that: \((k_{\text{NonBPS}}^0)^3 = 0\) which proves \((k_{\text{NonBPS}}^0)^3 = 0\). Moreover, \((k_{\text{NonBPS}})^2\) is rank two.

As long as one can pick the coefficients \(C^{\alpha A}\) and the twistor \(z\) that describes the \(SO(1,2)\) direction to be such that the time NUT charge is zero, this generator will give nice non-BPS extremal black holes. All the known non-BPS solutions may be recovered this way, and more, as this construction gives absolute freedom to pick the charges and moduli at infinity for the black hole (clearly for certain values of charges and moduli the solution will crash into a naked singularity, but this is to be expected from comparison with the BPS case).

### 4.5.2 Properties of flow generators

**Properties of \(k_{\text{BPS}}\)**

We now turn to solving for \(v_a\) and \(w_a\) in (4.125) in terms of \(C^A\) and \(z\). First, from (4.124) we know that the null space of \(k^2\) is five-dimensional and the \(v_a\) span the two-dimensional complement of this null space. For \(k_{\text{BPS}} = a_{\alpha A} C^A z^\alpha\) the null space of \((k_{\text{BPS}})^2\) does not depend on \(C^A\). Therefore, the \(v_a\) depend only on the twistor \(z = z^2/z^1\).

Recall that we are using the basis where \(k\) has the form (4.113). From inspection of \(k_{\text{BPS}}^2\), we find that \((v_1, v_2)\) can always be chosen to have the form:\(^5\)

\[
\begin{align*}
v_1 &= (V_1, -\eta_1 V_1, 0) & v_2 &= (-V_2, \eta_1 V_2, \sqrt{2})
\end{align*}
\tag{4.144}
\]

\(^5\)When solving for \((v, w)\), there are some freedom on the choice of \((v_1, v_2)\) and \((w_1, w_2)\): firstly, a
where \( \eta_1 \) is a 3D metric of signature \((1, -1, -1)\), and \( V_1, V_2 \) are two three-vectors with

\[
V_1 \cdot V_1 = 0, \quad V_1 \cdot V_2 = 0, \quad V_2 \cdot V_2 = -1. \tag{4.145}
\]

Since any linear combination of \((v_1, v_2)\) forms a new set of \((v_1, v_2)\), this means in particular that any \( v_2 + cv_1 \) gives a new \( v_2 \). Looking at the forms of \((v_1, v_2)\), we see that \( V_2 \) is defined up to a shifting of \( V_1 \) as \( V_2 = V_2^0 - cV_1 \).

An explicit computation shows that \( V_1 \) and \( V_2 \) are given by the twistor \( z \) and \( u \) as

\[
V_1 = \begin{pmatrix}
(z^1)^2 + (z^2)^2 \\
(z^1)^2 - (z^2)^2 \\
2z^1z^2
\end{pmatrix}, \quad V_2 = \frac{1}{z^1u^2 - z^2u^1} \begin{pmatrix}
z^1u^1 + z^2u^2 \\
z^1u^1 - z^2u^2 \\
z^1u^2 + z^2u^1
\end{pmatrix}, \tag{4.146}
\]

where the twistor \( u = \frac{u^2}{u^1} \) is related to \( c \) by

\[
u = -\frac{1 + 2cz}{1 - 2cz}z. \tag{4.147}
\]

rotational freedom

\[
(v_1, v_2) \rightarrow (v_1, v_2) \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \quad \text{and} \quad (w_1, w_2) \rightarrow (w_1, w_2) \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}
\]

where \( R \) is orthogonal: \( RR^T = 1 \). Secondly, a rescaling freedom:

\[
v_a \rightarrow rv_a \quad \text{and} \quad w_a \rightarrow \frac{1}{r}w_a.
\]
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The twistor representation\(^6\) of \(V_1\) and \(V_2\) are

\[
V_1^{\alpha\beta} = 2z^\alpha z^\beta \quad V_2^{\alpha\beta} = z^\alpha u^\beta + z^\beta u^\alpha
\]

(4.148)

where we have used the rescaling freedom to set \(z^1 u^2 - z^2 u^1\) to be 1. Note that for the BPS case, the twistor \(u\) is totally arbitrary.

Now we solve for \(w_a\). The condition that \(w_a\) are orthogonal to \(v_a\) dictates that they have the form:

\[
w_1 = (W_1, \eta_1 W_1, 0) \quad w_2 = (W_2, \eta_1 W_2, 0)
\]

(4.149)

where \(W_1\) and \(W_2\) are linearly independent, and are related to the charges by \(w_a \cdot w_b = c_{ab}\):

\[
W_1 \cdot W_1 = \frac{1}{2} c_{11} \quad W_1 \cdot W_2 = \frac{1}{2} c_{12} \quad W_2 \cdot W_2 = \frac{1}{2} c_{22}.
\]

(4.150)

Recall that \(V_2\) is defined up to a shift by \(V_1\): \(V_2 = V_2^0 - cV_1\). The consequence is that \(W_1\) is defined up to a shift by \(W_2\): \(W_1 = W_1^0 + cW_2\). Note that the numerical factors in front of \(V_1\) and \(W_2\) are opposite. Write down \((W_1^0, W_2)\) in terms of \((C^A, z)\):

\[
W_1^0 = \frac{1}{4z}
\begin{pmatrix}
(C^2 + C^4) + (C^1 + C^3)z \\
(C^2 - C^4) + (C^1 - C^3)z \\
2C^3 + 2C^2 z
\end{pmatrix}, \quad
W_2 = \frac{1}{2}
\begin{pmatrix}
-(C^2 + C^4) + (C^1 + C^3)z \\
-(C^2 - C^4) + (C^1 - C^3)z \\
-2C^3 + 2C^2 z
\end{pmatrix}.
\]

(4.151)

---

\(^6\)With the inner product of three-vectors defined as

\[
v_a \cdot v_b = v_a^T \eta_1 v_b.
\]

The twistor representation of a three-vector \(v = (x, y, z)\) is

\[
\sigma_v = x \sigma_0 + y \sigma_3 + z \sigma_1 = \begin{pmatrix}
x + y \\
z \\
x - y
\end{pmatrix}.
\]

Its length is

\[v^T \eta_1 v = \det(\sigma_v) = x^2 - y^2 - z^2.\]
The twistor representations of $W_1$ and $W_2$ are

$$W_1 = \begin{pmatrix} C^1u^2 - C^2u^1 & C^2u^2 - C^3u^1 \\ C^2u^2 - C^3u^1 & C^3u^2 - C^4u^1 \end{pmatrix}, \quad W_2 = \begin{pmatrix} C^1z^2 - C^2z^1 & C^2z^2 - C^3z^1 \\ C^2z^2 - C^3z^1 & C^3z^2 - C^4z^1 \end{pmatrix}.$$  

(4.152)

Define the totally symmetric $P^{\alpha\beta\gamma}$:

$$P^{111} = C^1, \quad P^{112} = C^2, \quad P^{122} = C^3, \quad P^{222} = C^4.$$  

(4.153)

Then the three-vectors $(W_1, W_2)$ span the four dimensional space

$$(W_1^\alpha, W_2^\alpha)^{BPS} = (P^{\alpha\beta\gamma}u_\gamma, P^{\alpha\beta\gamma}z_\gamma).$$  

(4.154)

**Properties of $k_{NonBPS}$**

The form of $v_a$ for the non-BPS case is only slightly different from the BPS case: the two vectors $v_a$ can be chosen to have the form:

$$v_1 = (V_1, \eta_1 V_1, 0), \quad v_2 = (V_2, -\eta_1 V_2, \sqrt{2})$$  

(4.155)

where $V_1, V_2$ are two three-vectors satisfying the same condition as the BPS ones (4.145). Again, the vectors $V_1$ and $V_2$ can be written as (4.146), and the twistor representations are given in (4.148) with one major difference: $u$ is no longer arbitrary, but is determined by $C^{\alpha A}$ as:

$$u = \frac{u^2}{u^1} = \frac{C^{22}}{C^{12}}.$$  

(4.156)

The form of $(w_1, w_2)$ are also slightly different from the BPS one (4.149)

$$w_1 = (W_1, -\eta_1 W_1, 0), \quad w_2 = (W_2, \eta_1 W_2, 0).$$  

(4.157)
The \((W_1, W_2)\) can be written in terms of \((C^\alpha, z)\) thus:

\[
W_1 = \frac{1}{2(C^{122}z^2 - C^{112}z^2)}, \\
\left(\begin{array}{c}
(C^{11}C^{22} - C^{12}C^{21})z^2 + (C^{22})^2 + [C^{11}C^{22} - C^{12}C^{21} + (C^{12})^2] \\
-(C^{11}C^{22} - C^{12}C^{21})z^2 + (C^{22})^2 + [C^{11}C^{22} - C^{12}C^{21} + (C^{12})^2] \\
2[(C^{11}C^{22} - C^{12}C^{21})z + C^{12}C^{22}]
\end{array}\right),
\]

\[
W_2 = -\frac{1}{2} \left(\begin{array}{c}
z[C^{11}z^2 + (3C^{12} - C^{21})z - 2C^{22}] + [C^{11}z + C^{12} - C^{21}] \\
-3z[C^{11}z^2 + (3C^{12} - C^{21})z - 2C^{22}] + [C^{11}z + C^{12} - C^{21}] \\
2[C^{11}z^2 + (2C^{12} - C^{21})z - C^{22}]
\end{array}\right).
\]

In terms of \(u = \frac{u^2}{u^1} = \frac{C^{22}}{C^{11}}\), the twistor representation of \(W_1\) and \(W_2\) are:

\[
W_1^{\alpha\beta} = u^\alpha u^\beta + (C^{11}u^2 - C^{12}u^1)z^\alpha z^\beta, \quad \quad (4.161)
\]

\[
W_2^{\alpha\beta} = (z^\alpha u^\beta + u^\alpha z^\beta) + (C^{21} - C^{11}z - 3u^1)z^\alpha z^\beta. \quad \quad (4.162)
\]

As a consequence, the precise value of \(u\) is an extra constraint on the vectors \(w_a\), and there is only a three-dimensional space of them, with \((W_1, W_2)\) a linear combination of \((0, V_1), (V_1, 0)\) and \((u^\alpha u^\beta, 2V_2)\):

\[
(W_1, W_2)_{NonBPS} = m(0, V_1) + n(V_1, 0) + \ell(u^\alpha u^\beta, 2V_2). \quad \quad (4.163)
\]

### 4.6 Single-centered Attractor Flows in \(G_{2(2)}/(SL(2, \mathbb{R}) \times SL(2, \mathbb{R}))\) Model

Now that we have completely characterized the generators of single-centered attractor flow, we can lift the geodesics to four-dimensional black hole solutions. After
some calculational effort, we find that the BPS solution is given in terms of harmonic functions. Next, we show that the non-BPS case is qualitatively different, and the final solutions cannot be formulated so simply.

4.6.1 BPS attractor flow

Lifting a geodesic to 4D

We have already noted that the flow starting from generic asymptotic moduli \((x_0, y_0)\) is generated by \(M(\tau) = e^{(k\tau + g)/2}\), with \(g\) defined before in (4.128). The matrix \(g\) has the same form as \(k\). Therefore, in the BPS case, it has the expansion

\[
g_{\text{BPS}} = a_{\alpha A} z^\alpha G^A, \tag{4.164}
\]

where the twistor \(z\) is the same as the one in \(k_{\text{BPS}}\). The flow of \((x, y)\) and \(u\) can be extracted from the symmetric matrix \(S(\tau) = M(\tau)S_0M(\tau)^T\) via (4.117) and (4.118).

Using \(k_{\text{BPS}}^3 = 0\) and \(g^3 = 0\), \(S(\tau) = S_0 e^{k\tau + g}\) is a quadratic function of \(\tau\):

\[
\begin{align*}
S_{55}(\tau)_{\text{BPS}} & = \alpha_B(\tau) + \beta_B(\tau) - 1, \\
S_{35}(\tau)_{\text{BPS}} & = \gamma_B(\tau) + \delta_B(\tau), \\
S_{33}(\tau)_{\text{BPS}} & = \epsilon_B(\tau) + \zeta_B(\tau) - 1, \tag{4.165}
\end{align*}
\]
where \( \alpha_B(\tau), \gamma_B(\tau), \epsilon_B(\tau) \) are quadratic functions of \( \tau \), and \( \beta_B, \delta_B, \zeta_B \) are linear functions of \( \tau \):

\[
\begin{align*}
\alpha_B(\tau) &= z^2(H^2H^4 - (H^3)^2) + z(H^2H^3 - H^1H^4) + (H^1H^3 - (H^2)^2), \\
\beta_B(\tau) &= (H^2 - H^4)z + (H^3 - H^1), \\
\gamma_B(\tau) &= -\frac{1}{2}((H^1H^4 - H^2H^3)(z^2 - 1) + 2(H^2(H^2 + H^4) - H^3(H^1 + H^3))z), \\
\delta_B(\tau) &= -\frac{1}{2}((H^1 + H^3)z - (H^2 + H^4)), \\
\epsilon_B(\tau) &= (H^1H^3 - (H^2)^2)z^2 + (H^1H^4 - H^2H^3)z + (H^2H^4 - (H^3)^2), \\
\zeta_B(\tau) &= 2(H^2z + H^3),
\end{align*}
\]

where \( H^A \) is a linear function of \( \tau \) defined as \( H^A(\tau) \equiv C_A^\tau + G^A \).

The attractor values are reached when \( \tau \to \infty \) along the geodesic:

\[
\begin{align*}
x_{\text{BPS}}^* &= \frac{(k^2S_0)_{35}}{(k^2S_0)_{33}}, \\
y_{\text{BPS}}^* &= \sqrt{(k^2S_0)_{33}(k^2S_0)_{55} - (k^2S_0)_{35}^2} \\
&\quad \frac{(k^2S_0)_{33}}{(k^2S_0)_{33}},
\end{align*}
\]

and

\[
u_{\text{BPS}}^* = \frac{1}{\sqrt{(k^2S_0)_{33}(k^2S_0)_{55} - (k^2S_0)_{35}^2}}.
\]

The asymptotic moduli \((x_0, y_0)\) can be expressed in terms of \((G^A, z)\) by extraction from \( S = e^gS_0 \).

Using this technique, one can construct all BPS single-centered black holes. The charges of each black hole can be read off from the current \( J \) using (4.119). One example is given in Figure 4.3, where we parametrically plot \( x(\tau) \) and \( y(\tau) \) for a BPS black hole with charges \((p^0, p^1, q_1, q_0) = (5, 2, 7, -3) \) and attractor point \((x^*, y^*) = (0.329787, 0.788503)\).\footnote{The discriminant of the charge \((5, 2, 7, -3)\) is positive, so this is indeed a BPS solution.}
**4D solution for given set of charges**

To get the solution for a specific set of charges requires more effort. In this section, we present the analytical result for any set of charges \((p^I, q_I)\).

The ten parameters in \(k_{BPS}\) and \(g\) are \(\{z, u, C^A, G_A\}\), among which the twistor \(u\) is arbitrary, corresponding to the freedom of the shift by \((v_1, w_2)\) in the definition of \((v_2, w_1)\): \((v_2, w_1) \rightarrow (v_2 + cv_1, w_1 - cw_2)\). The remaining true parameters are enough to parameterize the four D-brane charges \((p^I, q_I)\) and the arbitrary asymptotic moduli \((x_0, y_0)\) under the condition of vanishing Taub-NUT charge and fixing \(u_0 = 1\). We now solve for \(k_{BPS}\) and \(g\) for the given D-brane charges and \((x_0, y_0)\), using the eight constraints, namely, 4 charges and zero Taub-NUT charge plus 3 asymptotic moduli, to fix \(C^A\) and \(G^A\), leaving the other twistor \(z\) unfixed.\(^8\)

For the sake of simplicity, we will denote \(k_{BPS}\) by \(k\) for the rest of this section. Then the current \(J(Q) = \frac{k T}{\tau}\) gives the five coupled equations:

\[
Q = S_0(k + \frac{1}{2}[k, g])S_0 .
\]

(4.169)

In order to show that the BPS flow can be expressed in terms of harmonic functions:

\[
H(\tau) = Q\tau + h \quad \text{with} \quad Q = (p^I, q_I) \quad \text{and} \quad h = (h^I, h_I) ,
\]

(4.170)

we will solve \(g\) in terms of \(h\) instead of \((x_0, y_0)\). The four \(h\)'s relate to the asymptotic moduli by

\[
x_0 = x^*(Q \rightarrow h) \quad y_0 = y^*(Q \rightarrow h) \quad u_0 = u^*(Q \rightarrow h) ,
\]

(4.171)

and there is one extra degree of freedom to be fixed later.

\(^8\)The twistor \(z\) can be left unfixed because we will not specify the asymptotic values of the scalars with translational invariance, namely, \(\{a, A^I, B_I\}\). Fixing them can fix the twistor \(z\).
To evaluate \([k, g]\), we first use the commutation relation (4.97) to obtain

\[
[a_1 A C^A, a_1 B G^B] = \langle C, G \rangle (-4L^+_h),
\]

(4.172)

where the product between \(C^A\) and \(G^A\) is defined as \(\langle C, G \rangle \equiv C^1 G^4 - 3C^2 G^3 + 3C^3 G^2 - C^4 G^1\). Then twisting Eq (4.172) with the twistor \(z\) as in (4.140) gives the commutator of \(k\) and \(g\) with the same twistor \(z\):

\[
[k, g] = [a_\alpha A z^\alpha C^A, a_\beta B z^\beta G^B] = \langle C, G \rangle \Theta,
\]

(4.173)

where \(\Theta\) is defined as \(\Theta \equiv -\frac{4}{1+z} e^{-zL^-_h} L^+_h e^{zL^-_h}\). On the other hand, using (4.128),

\[
[k, g] = (v_2 v_1^T - v_1 v_2^T) S_0 (w_2 \cdot m_1 - w_1 \cdot m_2).
\]

(4.174)

\(\Theta\) can also be written as \(\Theta = (v_2 v_1^T - v_1 v_2^T) S_0\), and we can check that \((w_2 \cdot m_1 - w_1 \cdot m_2) = \langle C, G \rangle\).

First, separate from \(G^A\) the piece which has the same dependence on \((h, z)\) as \(C^A\) on \((Q, z)\):

\[
G^A = G^A_h + E^A \quad \text{with} \quad G^A_h \equiv C^A(Q \rightarrow h, z).
\]

(4.175)

That is, \(g\) contains two pieces:

\[
g = g_h + \Lambda \quad \text{with} \quad g_h = a_\alpha A z^\alpha G^A_h \quad \text{and} \quad \Lambda = a_\alpha A z^\alpha E^A.
\]

(4.176)

We need to solve for \(E^A\).

There are three constraints from (4.171). The \((x_0, y_0)\) and \(u_0\) are extracted from the symmetric matrix \(S = e^S_0\) via (4.117) and (4.118). On the other hand, requiring
(4.171) gives \((x_0, y_0, u_0)\) in terms of \(h\):

\[
x_0 = -\frac{(g_h^2 S_0)_{35}}{(g_h^2 S_0)_{33}},
\]

\[
y_0 = \frac{\sqrt{(g_h^2 S_0)_{33}(g_h^2 S_0)_{55} - (g_h^2 S_0^0)_{2}^2}}{(g_h^2 S_0)_{33}},
\]

\[
u_0 = \frac{1}{\sqrt{(g_h^2 S_0)_{33}(g_h^2 S_0)_{55} - (g_h^2 S_0^0)_{35}^2}}.
\]

Therefore, defining \(\Pi \equiv (e^g - \frac{g_h^2}{2})S_0\), \(\Pi\) has to satisfy three constraints:

\[
\Pi_{33} = \Pi_{35} = \Pi_{55} = 0
\]

in order for (4.177) to hold for arbitrary \(h\). Using the unfixed degree of freedom in \(h\)'s to set \(\langle C, G_h \rangle = 0\), (4.169) becomes

\[
Q = S_0(k + \frac{1}{2}[k, \Lambda])S_0.
\]

The zero Taub-NUT charge condition in (4.179) imposes the fourth constraint on \(\Lambda\): the \((3,2)\)-element of \(S_0(k + \frac{1}{2}[k, \Lambda])S_0\) for arbitrary \(k\) has to vanish. Combining with (4.178), we have 4 constraints to fix \(E^A\) to be:

\[
E^1 = -E^3 = -\frac{1}{1 + z^2}, \quad E^2 = -E^4 = \frac{z}{1 + z^2}.
\]

The remaining 4 conditions in the coupled equations (4.179) determine \(C^A\) in the BPS generator \(k_{BPS} = a_\alpha A^\alpha C^A\) to be

\[
C^1 = \sqrt{2-q_0 - q_1 z - 3p^1 z^2 + p^0 z^3} \quad \frac{1}{(1 + z^2)^2},
\]

\[
C^2 = \sqrt{2-q_1^3 - (2p^1 - v_0)z + (p^0 + 2q_1^3)z^2 + p^1 z^3} \quad \frac{1}{(1 + z^2)},
\]

\[
C^3 = \sqrt{2-p^1 + (p^0 + 2q_1^3)z + (2p^1 - v_0)z^2 - q_1^3 z^3} \quad \frac{1}{(1 + z^2)},
\]

\[
C^4 = \sqrt{2p^0 + 3p^1 z - q_1 z^2 + q_0 z^3} \quad \frac{1}{(1 + z^2)^2}.
\]
The $G^A_h$ are then determined by $G^A_h = a_\alpha A^z A^C (Q \rightarrow h, z)$. Using the solution of $C^A$ and $G^A_h$, we see the product $\langle C^A, G^A_h \rangle$ is proportional to the symplectic product of $(p', q_I)$ and $(h_I, h_I)$:

$$\langle C^A, G^A_h \rangle = \frac{2}{1 + z^2} < Q, h > \quad \text{where} \quad < Q, h > \equiv p^0 h_0 + p^1 h_1 - q_1 h^1 - q_0 h^0 .$$

(4.182)

The condition $\langle C^A, G^A_h \rangle = 0$ is then the integrability condition on $h$:

$$< Q, h > = p^0 h_0 + p^1 h_1 - q_1 h^1 - q_0 h^0 = 0 .$$

(4.183)

Substituting the expressions of $C^A$ and $G^A_h$ in terms of $(p_I, q_I)$ into (4.165), we obtain the BPS attractor flow in terms of the charges $(p_I, q_I)$. In particular, the attractor values are

$$x^*_{BPS} = - \frac{p^0 q_0 + p^1 q_1}{2[(p^1)^2 + p^0 q_1^2]} , \quad y^*_{BPS} = \frac{\sqrt{J_4(p^0, p^1, q_1, q_0)}}{2[(p^1)^2 + p^0 q_1^2]} ,$$

(4.184)

where $J_4(p^0, p^1, q_1, q_0)$ is the quartic $E_7(7)$ invariant:

$$J_4(p^0, q_1, q_0) = 3(p^1 q_1)^2 - 6(p^0 q_0)(p^1 q_1) - (p^0 q_0)^2 - 4(p^1)^3 q_0 + 4p^0 (q_1)^3 ,$$

(4.185)

thus $J_4(p^0, p^1, q_1, q_0)$ is the discriminant of charge. The attractor values match those from the compactification of Type II string theory on diagonal $T^6$, with $q_1 \rightarrow \frac{q_1}{3}$. The attractor value of $u$ is

$$u^*_{BPS} = \frac{1}{\sqrt{J_4(p^0, p^1, q_1, q_0)}} .$$

(4.186)

The constraint on $h$ from $u_0 = 1$ is then $J_4(h^0, h^1, h^1, h_0) = 1$.

Now we will show that the geodesic we constructed above indeed reproduces the attractor flow given by replacing charges by the corresponding harmonic functions in the attractor moduli. Using the properties of $\Lambda$, we have proved that, in terms of
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$k$ and $g$, the flow of $(x, y)$ can be generated from the attractor value by replacing $k$ with the harmonic function $k\tau + g_h$:

$$x(\tau) = x^*(k \rightarrow k\tau + g_h), \quad y(\tau) = y^*(k \rightarrow k\tau + g_h). \quad (4.187)$$

Since $k$ and $g_h$ have the same twistor $z$, this is equivalent to replacing the $C^A$ with the harmonic function $C^A\tau + G^A_h$ while leaving the twistor $z$ fixed:

$$x(\tau) = x^*(C^A \rightarrow C^A\tau + G^A_h, z), \quad y(\tau) = y^*(C^A \rightarrow C^A\tau + G^A_h, z). \quad (4.188)$$

Since $C^A$ is linear in $Q$ and $G^A_h$ linear in $h$, and since $z$ drops out after plugging in the solution of $C^A$ in terms of $(Q, z)$ and $G^A_h$ in terms of $(h, z)$, this proves that the flow of $(x_0, y_0)$ is given by replacing the charges in the attractor moduli by the corresponding harmonic functions:

$$x_{BPS}(\tau) = x^*_{BPS}(Q \rightarrow Q\tau + h), \quad y_{BPS}(\tau) = y^*_{BPS}(Q \rightarrow Q\tau + h). \quad (4.189)$$

The integrability condition $< Q, h > = 0$, in terms of $H = Q\tau + h$, is

$$< H, dH > = 0. \quad (4.190)$$

4.6.2 Non-BPS attractor flow

Lifting a geodesic to 4D

Similar to the BPS attractor flow, the non-BPS flow is generated by $M(\tau) = e^{(k\tau+g)/2}$, and $(x, y)$ can be extracted from the symmetric matrix $S(\tau)$ by (4.117) and the relevant elements of $S(\tau)$ are given by (4.165). The only difference is that now $\{\alpha_B, \beta_B, \gamma_B, \delta_B, \epsilon_B, \zeta_B\}$ are changed into the non-BPS counterparts
\{\alpha_{NB}, \beta_{NB}, \gamma_{NB}, \delta_{NB}, \epsilon_{NB}, \zeta_{NB}\}, which can be written in terms of \(H_{\alpha a} \equiv C_{\alpha a}^\tau + G_{\alpha a}\) and \(z\):

\[
\begin{align*}
\alpha_{NB}(\tau) &= -(H^{21}H^{12} - H^{22}H^{11})(z^2 - 1)^2 + (H^{22})^2 z^2 - 2zH^{12}H^{22} + (H^{12})^2, \\
\beta_{NB}(\tau) &= (z^2 - 1)(H^{11} - H^{21}z) - 3z^2 H^{22} + 2zH^{12} + H^{22}, \\
\gamma_{NB}(\tau) &= (z^2 - 1)(2z(H^{11}H^{22} - H^{12}H^{21}) + H^{22}H^{12}) + z((H^{22})^2 - (H^{12})^2), \\
\delta_{NB}(\tau) &= \frac{1}{2}(z(1 + z^2)H^{11} + z^2(3H^{12} - H^{21}) - 2H^{22}z + (H^{12} - H^{21})), \\
\epsilon_{NB}(\tau) &= (4H^{11}H^{22} - H^{12}(H^{12} + 4H^{21}))z^2 - 2H^{12}H^{22}z - (H^{22})^2, \\
\zeta_{NB}(\tau) &= 2(H^{11}z^2 + (2H^{12} + H^{21})z + H^{22}). \tag{4.191}
\end{align*}
\]

Note that \(\frac{H^{22}}{H^{12}} = u\) is fixed, independent of \(\tau\). The non-BPS flow written in terms of \((H_{\alpha a}, z)\) has the same simple form as the BPS flow, i.e. the scalars are rational functions with both the numerator and denominator being only quadratic. This is due to the nilpotency of the generator: \(k^3 = 0\). Again, the attractor values are reached when \(\tau \to \infty\), and the asymptotic moduli can be expressed in terms of \((G_{\alpha a}, z)\) by extraction from \(S = e^gS_0\).

Unlike the BPS case, there are only eight parameters in \(k_{NonBPS}\) and \(g_{NonBPS}\): the two twistors \((z, u)\) and \((C_{\alpha a}, G_{\alpha a})\) under the constraints that

\[
u = \frac{C_{22}}{C_{12}} = \frac{G_{22}}{G_{12}}. \tag{4.192}
\]

Therefore, while \(k_{BPS}\) and \(g_{BPS}\) can parameterize arbitrary \((p^I, q_I)\) and \((x_0, y_0)\) while leaving \((z, u)\) free, all the parameters in \(k_{NonBPS}\) and \(g_{NonBPS}\), including \((z, u)\), will be fixed.
The attractor point in terms of $C^{\alpha a}$ is

\[
x^{*}_{\text{NonBPS}} = \frac{\gamma_{NB}(H^{\alpha a} \rightarrow C^{\alpha a})}{\epsilon_{NB}(H^{\alpha a} \rightarrow C^{\alpha a})}, \quad (4.193)
\]

\[
y^{*}_{\text{NonBPS}} = \frac{\sqrt{\alpha_{NB}(H^{\alpha a} \rightarrow C^{\alpha a})\epsilon_{NB}(H^{\alpha a} \rightarrow C^{\alpha a})} - \gamma^2_{NB}(H^{\alpha a} \rightarrow C^{\alpha a})}{\epsilon_{NB}(H^{\alpha a} \rightarrow C^{\alpha a})}, \quad (4.194)
\]

with $z$ given by

\[
\frac{1}{2} \left( -3C^{12} - C^{21} + z \left( (z^2 - 3)C^{11} + 3z(C^{12} + C^{21}) + 6C^{22} \right) \right) = 0. \quad (4.194)
\]

As in the BPS case, the charges of the black hole are read off from the current $J$ using (4.119). We have checked that the attractor point is a non-supersymmetric critical point of the black hole potential $V_{BH} = |Z|^2 + |DZ|^2$:

\[
\partial V_{BH} = 0 \quad \text{and} \quad DZ \neq 0. \quad (4.195)
\]

It reproduces the results reported in the literature [145]. An example of the non-BPS attractor flow is shown in Figure 4.4, with $(p^0, p^1, q_1, q_0) = (5, 2, 7, 3)$ and attractor point $(x^*, y^*) = (-0.323385, 0.580375)$. Note that $J_4(5, 2, 7/3, 3) < 0$, so this is indeed a non-BPS black hole. Unlike the BPS attractor flow, all the non-BPS flows starting from different asymptotic moduli have the same tangent direction at the attractor point. The mass matrix of the black-hole potential at a BPS critical point has two identical eigenvalues, whereas the eigenvalues at a non-BPS critical point are different. The common tangent direction for the non-BPS flows corresponds to the eigenvector associated with the smaller mass.

**4D solution for given set of charges**

We now discuss how to construct the non-BPS black hole solution for a specific set of charges $(p', q_t)$. 
One major difference between the non-BPS case and the BPS case is that

\[ [k_{\text{NonBPS}}, g_{\text{NonBPS}}] = 0 \]  

(4.196)

automatically, since the forms of \((w_1, w_2)\) and \((m_1, m_2)\) guarantee that \(w_1 \cdot m_2 = w_2 \cdot m_1 = 0\). Thus the charge equation (4.169) becomes simply

\[ Q_{\text{NonBPS}} = S_0(k_{\text{NonBPS}})S_0. \]  

(4.197)

These five coupled equations determine the two twistors \((z, u)\) and \(C^{\alpha a}\) in terms of \((p^I, q_I)\). Similar to the BPS case, the four equations which determine the D-brane charge allow us to write \(C^{\alpha a}\) in terms of the charges \((p^0, p^1, q_1, q_0)\) and the twistor \(z\) via

\[
\begin{align*}
C^{11} &= \frac{(-2q_0 + 6(p^1 - q_0)z^2 + 4(p^0 + q_1)z^3 - 6p^1z^4)}{\sqrt{2}(1 + z^2)^3}, \\
C^{12} &= \frac{(p^0 + \frac{q_1}{3}) - 2(2p^1 - q_0)z - (p^0 + 5\frac{q_1}{3})z^2 + 2p^1z^3}{\sqrt{2}(1 + z^2)^2}, \\
C^{21} &= \frac{(p^0 - q_1) - 4q_1z^2 + 4(3p^1 - q_0)z^3 + (3p^0 + q_1)z^4}{\sqrt{2}(1 + z^2)^3}, \\
C^{22} &= \frac{2p^1 + (p^0 + 5\frac{q_1}{3})z - 2(2p^1 - q_0)z^2 - (p^0 + \frac{q_1}{3})z^3}{\sqrt{2}(1 + z^2)^2}.
\end{align*}
\]  

(4.198)

and \(u = \frac{C^{22}}{C^{11}}\). In contrast to the BPS case, the \(G^{\alpha a}\) do not enter the equations and therefore cannot be used to eliminate the twistor \(z\). Requiring the Taub-NUT charge to vanish gives the following degree-six equation for the \(z:\)

\[ p^0z^6 + 6p^1z^5 - (3p^0 + 4q_1)z^4 - 4(3p^1 - 2q_0)z^3 + (3p^0 + 4q_1)z^2 + 6p^1z - p^0 = 0. \]  

(4.199)

The three parameters in \(g_{\text{NonBPS}}\), namely, \(G^{\alpha a}\) with the constraint \(\frac{C^{22}}{C^{11}} = u\) are then fixed by the given asymptotic moduli \((x_0, y_0)\) and \(u_0 = 1\).
Similar to the BPS flow, the full non-BPS flow can be generated from the attractor value by replacing $C^{aa}$ with the harmonic function $H^{aa}(\tau) = C^{aa} \tau + G^{aa}$, while leaving $z$ unchanged as in (4.188). However, there are two major differences. First, the harmonic functions $H^{aa}$ have to satisfy the constraint

$$\frac{H^{22}(\tau)}{H^{12}(\tau)} = u = \frac{C^{22}}{C^{12}} = \frac{G^{22}}{G^{12}}.$$  \hspace{1cm} (4.200)

Note that this does not impose any constraint on the allowed asymptotic moduli since there are still three degrees of freedom in $G^{aa}$ to account for $(x_0, y_0, u_0)$. We will see later that it instead imposes a stringent constraint on the allowed D-brane charges in the multi-centered non-BPS solution.

Secondly, unlike the BPS flow, replacing $C^{aa}$ in the attractor moduli by the harmonic function $H^{aa}(\tau)$ is not equivalent to replacing the charges $Q$ with $H = Q \tau + h$ as in (4.189). The twistor $z$ here is no longer free, but is determined in terms of the charges as a root of the degree-six equation (4.199), so replacing $Q$ by $Q \tau + h$, for generic $Q$ and $h$, would not leave $z$ invariant. Therefore, the generic non-BPS flow cannot be given by the naive harmonic function procedure, as proposed by Kallosh et al [91]. Next, we will define the subset of the NonBPS single-centered flow that can be constructed by the harmonic function procedure.

When the attractor has only D4-D0 charges, namely, $Q_{40} = (0, p^1, 0, q_0)$, (4.199) has a root $z = 0$, which is independent of the value of charges. If the asymptotic moduli $h$ is also of the form of $h_{40} = (0, h^1, 0, h_0)$, replacing $Q_{40}$ by $Q_{40} \tau + h_{40}$ would leave the solution $z = 0$ invariant. Now we will use the duality symmetry to extend the subset to a generic charge system with restricted asymptotic moduli.

The one-modulus system can be considered as the STU attractor with the three
moduli \((S, T, U)\) identified. Since the STU model has \(SL(2, \mathbb{Z})^3\) duality symmetry at the level of the equations of motion, the one-modulus system has an \(SL(2, \mathbb{Z})\) duality symmetry coming from identifying the three \(SL(2, \mathbb{Z})\) symmetries of the STU model. That is, the one-modulus system is invariant under the following element of \(SL(2, \mathbb{Z})^3\)

\[
\Gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{with} \quad ad - bc = 1.
\] (4.201)

The modulus \(z = x + iy\) transforms as

\[
z \rightarrow \Gamma z = \frac{az + b}{cz + d},
\] (4.202)

and the transformation on the charges is given by [20]. A generic charge \((p^0, p^1, q_1, q_0)\) can be reached by applying the transformation \(\Gamma\) on a D4-D0 system.

Under the aforementioned transformation, a D4-D0 system transforms into \(\Gamma Q_{40}\)

\[
Q_{40} = \begin{pmatrix} 0 \\ p^1 \\ 0 \\ q_1 \end{pmatrix} \rightarrow \Gamma Q_{40} = \begin{pmatrix} -c(3d^2p^1 + c^2q_0) \\ d(2bc + ad)p^1 + ac^2q_0 \\ 3(b(bc + 2ad)p^1 + a^2cq_0) \\ a(3b^2p^1 + a^2q_0) \end{pmatrix}.
\]

The solution of the twistor \(z\) with the new charges \(\Gamma Q_{40}\) is

\[
z = \frac{a \pm \sqrt{a^2 + \frac{c^2}{c}}}{c},
\] (4.203)

independent of the D4-D0 charges we started with. Now given an arbitrary charge \(Q\), there exists a transformation \(\Gamma_Q\) such that \(Q = \Gamma_Q Q_{40}\) for some \(Q_{40}\). The twistor \(z\) remains invariant under \(Q \rightarrow Q + \Gamma_Q h_{40}\) for arbitrary \(h_{40}\). We conclude that the non-BPS single-centered black holes that can be constructed via the naive harmonic
function procedure are only those with \((Q, h)\) being the image of a single transformation \(\hat{\Gamma}\) on the \((Q_{40}, h_{40})\) from a D4-D0 system:

\[x_{NB}(\tau) = x^{\ast}_{NB}(\hat{\Gamma}Q_{40} \to \hat{\Gamma}Q_{40}\tau + \hat{\Gamma}h_{40}), \quad y_{NB}(\tau) = y^{\ast}_{NB}(\hat{\Gamma}Q_{40} \to \hat{\Gamma}Q_{40}\tau) + \hat{\Gamma}h_{40}.\]  

(4.204)

Since we are considering arbitrary charge system, the constraint is on the allowed values of \(h\).

### 4.7 Multi-centered Attractor Flows in \(G_{2(2)}/(SL(2, \mathbb{R}) \times SL(2, \mathbb{R}))\) Model

As proven in the pure gravity system, the multi-centered attractor solutions are given by exponentiating the matrix harmonic function \(K(\vec{x})\):

\[S(\vec{x}) = e^{K(\vec{x})}S_0\]  

(4.205)

with \(K(\vec{x})\) having the same properties as the generator \(k\):

\[K^3(\vec{x}) = 0 \quad \text{and} \quad K^2(\vec{x}) \text{ rank two.}\]  

(4.206)

We now describe how to formulate \(K(\vec{x})\) for multi-centered solutions in \(G_{2(2)}\).

The \(K(\vec{x})\) satisfying all the above constraints is constructed as

\[K(\vec{x}) = \sum_{a=1,2} [v_a W_a(\vec{x})^T + W_a(\vec{x}) v_a^T]S_0,\]  

(4.207)

with \(v_a\) being the same two constant null vectors in \(k\), and the multi-centered harmonic function

\[W_a(\vec{x}) = \sum_i \left(\frac{(w_a)_i}{||\vec{x} - \vec{x}_i||} + m_a\right)\]  

(4.208)
is everywhere orthogonal to $v_a$. The two 7-vectors $(m_1, m_2)$ contain the information of asymptotic moduli and has the same form as $(w_1, w_2)$. Write $K(\vec{x})$ as $K(\vec{x}) = \sum_i \frac{k_i}{|\vec{x} - \vec{x}_i|} + g$ where

$$k_i = \sum_{a=1,2} [v_a(w_a)_i^T + (w_a)_i v_a^T]S_0 \quad \text{and} \quad g = \sum_{a=1,2} [v_a m_a^T + m_a v_a^T]S_0 . \quad (4.209)$$

Since $v$ only depends on the twistor $(z, u)$, and $(w_1, w_2)$ are linear in $C^A$ or $C^{\alpha a}$, the above generating procedure is equivalent to replace $C^A$ or $C^{\alpha a}$ by the multi-centered harmonic functions while keeping the twistor $(z, u)$ fixed.

Next we discuss the properties of the BPS multi-centered attractor solution and non-BPS ones separately, since they are very different in character.

### 4.7.1 BPS multi-centered solutions

In contrast with the multi-centered solutions in pure gravity, now the second term of the current $J = \nabla K + \frac{1}{2}[\nabla K, K]$ does not vanish automatically since

$$[k_i^{BPS}, k_j^{BPS}] \neq 0 \quad \text{and} \quad [k_i^{BPS}, g^{BPS}] \neq 0 . \quad (4.210)$$

Therefore, the centers are no longer free, and we cannot simply read off the charges from $J$. Instead, we need to solve for $C_i^A$ and $C^A$ in a set of $5N$ coupled equations. The divergence of the current is

$$\nabla \cdot J = 4\pi \sum_i \delta(\vec{x} - \vec{x}_i)S_0(k_i + \frac{1}{2}[k_i, g] + \frac{1}{2} \sum_j \frac{[k_i, k_j]}{|\vec{x}_i - \vec{x}_j|})S_0 . \quad (4.211)$$

Using $Q_i$ to denote the charge matrix which relates to the D-brane charge $\{p^0, p^1, q_1, q_0\}$, as in (4.121), and with $Q_{32}$ as the vanishing NUT charge, we have $5N$ coupled equa-
tions from $Q_i = \frac{1}{4\pi} \int \nabla \cdot J$:

$$Q_i = S_0 (k_i + \frac{1}{2} [k_i, g] + \frac{1}{2} \sum_j \frac{[k_i, k_j]}{|\vec{x}_i - \vec{x}_j|}) S_0 .$$  \hspace{1cm} (4.212)

The generators of the multi-centered BPS attractor solution \{\(k_i\)} and \(g\) have \(4(N+1)+2\) parameters in total: the two twistors \((z,u)\) and \(\{C_i^A, G^A\}\). On the left hand side of (4.212), there are also \(3N-3\) degrees of freedom from the position of the centers \(\vec{x}_i\). On the other hand, a generic \(N\)-centered attractor solution has \(4N\) D-brane charges \((p^I, q_I)\), and three additional constraints from the asymptotic moduli \((x_0, y_0)\) and \(u_0 = 1\). As we will show, like the single-centered BPS solution, the three asymptotic moduli, together with the vanishing of the total Taub-NUT charge, determine the \(4G^A\) inside \(g\). Moreover, as in the single-centered case, we can solve \(C_i^A\) in terms of the 4D D-brane charges \(Q_i\) while leaving \((z,u)\) unfixed. The remaining \(N-1\) zero Taub-NUT charge conditions at each center will impose \(N-1\) constraints on the distances between the \(N\)-centers \(\vec{x}_i\).

First, integrating over the circle at the infinity, \(\sum_i Q_i = \frac{1}{4\pi} \int \nabla \cdot J\) gives the sum of the above \(N\) matrix equations:

$$Q_{tot} = \sum_i Q_i = S_0 \left( \sum_i k_i + \frac{1}{2} \left[ \sum_i k_i, g \right] \right) S_0 ,$$  \hspace{1cm} (4.213)

which is the same as the one for the single-center attractor with charge \(Q_{tot}\). This determines \(g\) to be \(g = g_h + \Lambda\), same as the one for single-centered attractor in (4.176).

As in the single centered case, \(h\) is fixed by the asymptotic moduli \((x_0, y_0)\) by

$$x_0 = x_{BPS}^*(Q \rightarrow h) , \quad y_0 = y_{BPS}^*(Q \rightarrow h) ,$$  \hspace{1cm} (4.214)

and the two constraints:

$$< Q_{tot}, h >= 0 , \quad J_4(h^0, h^1, \frac{h_1}{3}, h_0) = 1 .$$  \hspace{1cm} (4.215)
We have used the vanishing of the total Taub-NUT charge to determine $\Lambda$. Next, we will use the remaining coupled $5N - 1$ equations to solve for the $4N \{C^A_i\}$ and impose $N - 1$ constraints on the relative positions between the $N$ centers $\{\vec{x}_i\}$ where $i = 1, \ldots, N$.

The tentative solutions of $C^A_i$ are given by (4.181) with $(p^I, q_I)$ replaced by $(p^I_i, q_{I,i})$. The flow generator of each center $k_i$ is then $k_i = a_{\alpha A} z^\alpha C^A_i$. Substituting the solution of $k_i$ and $g = g_h + \Lambda$ into (4.212), and using

$$[k_i, g_h] = 2 < Q_i, h > \Theta , \quad [k_i, k_j] = 2 < Q_i, Q_j > \Theta ,$$

where all the $k_i$'s and $g_h$ have the same value for the twistor $z$, we get

$$Q_i = S_0 (k_i + < Q_i, h > \Theta + \frac{1}{2} [k_i, \Lambda] + \sum_j \frac{< Q_i, Q_j >}{|\vec{x}_i - \vec{x}_j|} \Theta) S_0 .$$

Just as in the single-centered case, the solution of $k_i$ and the form of $\Lambda$ guarantee that

$$Q_i = S_0 (k_i + \frac{1}{2} [k_i, \Lambda]) S_0 .$$

We see that as long as the following integrability condition is satisfied:

$$< Q_i, h > + \sum_j \frac{< Q_i, Q_j >}{|\vec{x}_i - \vec{x}_j|} = 0 ,$$

the $k_i$ and $g$ given above indeed produce the correct multi-centered attractor solution. Just like in the single-centered case, the multi-centered solution flows to the correct attractor moduli $(x^*_i, y^*_i)$ near each center, independent of the value of $z$. It also follows that the multi-centered solution can be generated by replacing the charges inside the attractor value by the multi-centered harmonic function:

$$x_{BPS}(\vec{x}) = x_{BPS}^*(Q \to \sum_i \frac{Q_i}{|\vec{x} - \vec{x}_i|} + h) , \quad y_{BPS}(\vec{x}) = y_{BPS}^*(Q \to \sum_i \frac{Q_i}{|\vec{x} - \vec{x}_i|} + h) .$$
The sum of the $N$ equations in the integrability condition (4.219) reproduces the constraint on $h$: $< Q_{\text{tot}}, h > = 0$. Thus the remaining $N - 1$ equations impose $N - 1$ constraints on the relative positions between the $N$ centers $\{ \vec{x}_i \}$ with $i = 1, \cdots , N$. From (A.5) and (4.94), we see that $\ast d\omega$ is given by $J_{23}$. Defining the angular momentum $\vec{J}$ by

$$\omega_i = 2\epsilon_{ijk} J^j \frac{x^k}{r^3} \quad \text{as } r \to \infty ,$$

we see that there exists a nonzero angular momentum given by

$$\vec{J} = \frac{1}{2} \sum_{i<j} \frac{\vec{x}_i - \vec{x}_j}{|\vec{x}_i - \vec{x}_j|} \langle Q_i, Q_j \rangle .$$

Thus we have shown that our multi-centered BPS attractor solution reproduces the one found in [18].

### 4.7.2 Non-BPS multi-centered solutions.

For given $(z, u)$ and $\{C^{\alpha a}_i, G^{\alpha a}\}$, the non-BPS multi-centered solution is the same as the single-centered one as in (4.191) with $H^{\alpha a}(\tau)$ replaced by the multi-centered harmonic function $H^{\alpha a}(\vec{x}) = \sum_i \frac{C^{\alpha a}_i}{|\vec{x} - \vec{x}_i|} + G^{\alpha a}$ satisfying the constraint

$$u = \frac{H^{22}_{i}(\vec{x})}{H^{12}_{i}(\vec{x})} = \frac{C^{22}_i}{C^{12}_i} = \frac{G^{22}}{G^{12}} .$$

Accordingly, the attractor values at each center is the same as (4.193) with the corresponding $C^{\alpha a}_s$. The asymptotic moduli are obtained by extraction from $S = e^g S_0$.

The equation of motion for the non-BPS multi-centered solution simplifies a great deal since

$$[\nabla K(\vec{x}), K(\vec{x})] = 0$$

(4.224)
automatically, following from the fact that for the non-BPS system:

\[(w_1)_i \cdot m_2 = (w_2)_i \cdot m_1 = 0, \quad (4.225)\]

which are guaranteed by the forms of NonBPS \((w_1, w_2)_i\) and \((m_1, m_2)\). Therefore, the 5N equations (4.212) decouple into \(N\) sets of 5 coupled equations:

\[Q_i = S_0(k_i)S_0. \quad (4.226)\]

Equation (4.226) differs greatly from the BPS counterpart (4.212). Firstly, the generators of the multi-centered non-BPS attractor solution \(\{k_i\}\) and \(g\) have \(3(N + 1) + 2\) parameters: the two twistors \((z, u)\) and \(\{C^{\alpha a}_i, G^{\alpha a}\}\) with the constraint (4.223). In contrast to the BPS case, \(g\) does not enter the equation. Thus we can simply use the three asymptotic moduli, without invoking the zero Taub-NUT condition, to determine the 3 \(G^{\alpha a}\) inside \(g\). Secondly, unlike the BPS multi-centered solution, the position of the centers \(\vec{x}_i\) do not appear in the equation, therefore there will be no constraint imposed on them: the centers are free. Last but not least, the remaining \(3N + 2\) parameters in \((z, u)\) and \(C^{\alpha a}\) are not enough to parameterize a generic \(N\)-centered attractor solution, which has \(4N\) D-brane charges \((p_I^i, q_I,i)\). Accordingly, the multi-centered non-BPS attractor generated by this ansatz will not have arbitrary charges. Combining with the fact that \(\vec{x}_i\) do not appear in the R.H.S of the equation, we find that all the \(N\) vanishing Taub-NUT charge conditions can only act on the charges on the L.H.S. We conclude that, in total, there will be \(2N - 2\) constraints on the allowed charges.

Now we will show in detail the derivation of the constraints. First, like in the single-centered NonBPS solution, the absence of the Taub-NUT charge at infinity
fixes $z$ via

$$\sum_i Q_i = S_0 \left( \sum_i k_i \right) S_0$$ \hspace{1cm} (4.227)

The solution is the same as the solution to (4.199) with the charges replaced by the total charges of $N$ centers: $z = z(Q \rightarrow \sum_i Q_i)$. Since all the $N$ centers share the same twistor $z$, the absence of the NUT charge at each center imposes $N - 1$ constraints on the allowed charges $Q_i$: all $z(Q_i)$ have to be equal.

The remaining $4N$ equations in (4.226) determine $C^{\alpha a}$ in terms of $z$ and $Q_i$. Since the $N$-centers decouple, (4.226) for each center is the same as the single-center one (4.197). Thus the solution of $C^{\alpha a}_i$ is given by (4.198) with $(p^I, q_I)$ replaced by $(p^I_i, q_{I,i})$. Again, since all the centers share the same twistor $u$, the condition (4.223) imposes another $N - 1$ constraints on the allowed charges. Solving these $2N - 2$ constraints, we see all the charges $\{Q_i\}$ are the image of a single transformation $\hat{\Gamma}$ on a multi-centered D4-D0 system $Q_{40,i}$:

$$Q_i = \hat{\Gamma}Q_{40,i} \hspace{1cm} (4.228)$$

The charges at different centers are all mutually local

$$\langle Q_i, Q_j \rangle = 0 \hspace{1cm} (4.229)$$

Except for the constraint on the charges, the $N$ centers are independent, and there is no constraint on the position of the centers. A related fact is that the angular momentum is zero.

Like the non-BPS single-centered case, though the multi-centered solution can be generated from the attractor value by replacing $C^{\alpha a}$ with the multi-centered harmonic function $H^{\alpha a}(\vec{x}) = \sum_i \frac{C^{\alpha a}_i}{|\vec{x} - \vec{x}_i|} + G^{\alpha a}$ under the constraint (4.223), while leaving $z$
unchanged as in (4.188), the generic solution cannot be generated via the harmonic function procedure used in the BPS case, namely, by replacing the charges inside the attractor value by the corresponding multi-centered harmonic functions. The reason is again due to the fact that the twistor $z$, being a function of charges, does not remain invariant under this substitution of charges by harmonic functions. The multi-centered non-BPS solutions that can be generated by the harmonic function procedure are those with $\{Q_i, h\}$ being the image of a single $\hat{\Gamma}$ on the $\{Q_{40,i}, h_{40}\}$ of a pure D4-D0 system:

$$
x_{NB}(\vec{x}) = x_{NB}^*(\hat{\Gamma}Q_{40} \rightarrow \sum_i \frac{\hat{\Gamma}Q_{40,i}}{|\vec{x} - \vec{x}_i|} + \hat{\Gamma}h_{40}) ,
$$

$$
y_{NB}(\vec{x}) = y_{NB}^*(\hat{\Gamma}Q_{40} \rightarrow \sum_i \frac{\hat{\Gamma}Q_{40,i}}{|\vec{x} - \vec{x}_i|} + \hat{\Gamma}h_{40}) .
$$

(4.230)

It appears that the existence of a simple linear ansatz for “superimposing” single center solutions exists in general only for mutually local extremal black holes, and only in the supersymmetric case does it extend to mutually non-local centers.

To summarize, the non-BPS multi-centered solution is different from the BPS case because it imposes no constraints on the position of the centers, but instead on the allowed charges $Q_i$: the choice of charges at each center are restricted to a three-dimensional subspace, and they are mutually local. The result is that the centers can move freely, and there is no angular momentum in the system. It does not have interesting moduli spaces of centers with mutually non-local charges, so it is as “boring” as the pure gravity case.
4.8 Conclusion and Discussion

In this chapter, we find exact single-centered and multi-centered black hole solutions in theories of gravity which have a symmetric 3D moduli space. The BPS and extremal non-BPS single-centered solutions correspond to certain geodesics in the moduli space. We construct these geodesics by exponentiating different types of nilpotent elements in the coset algebra. Using the Jordan form of these nilpotent elements, we are able to write them down in closed explicit form. Furthermore, we can use a symmetric matrix parametrization to recover the metric and full flow of the scalars in four dimensions.

We have also generalized the geodesics to find solutions for non-BPS and BPS multi-centered black holes. The BPS multi-centered solution reproduces the known solution of Bates and Denef. Given our assumption that the 3D spatial slice is flat, we find that a non-BPS multi-centered black hole is very different from its BPS counterpart. It is constrained to have mutually local charges at all of its centers and therefore carries no intrinsic angular momentum. It is possible that if we dropped this assumption, we could find more general non-BPS multi-centered solutions. Such configurations would probably be amenable to exact analysis only in the axially symmetric case, using inverse scattering methods.

There are many other avenues for future work. One could explore nilpotent elements in other symmetric spaces, and see whether non-BPS bound states with nonlocal charges exist. In particular, it would be interesting to study $E_{8(8)}/SO^*(16)$, which is the 3D moduli space for $d = 4, \mathcal{N} = 8$ supergravity. We would also like to find a way to modify our method so that we can apply it to non-symmetric homogeneous
spaces, and eventually to generic moduli spaces. We could then study the much larger class of non-BPS extremal black holes in generic $\mathcal{N} = 2$ supergravities.
Figure 4.3: Sample BPS flow. The attractor point is labeled $(x^*, y^*)$. The initial points of each flow are given by $(x_1 = 1.5, y_1 = 0.5), (x_2 = 2, y_2 = 4), (x_3 = -0.2, y_3 = 0.1), (x_4 = -1, y_4 = 3)$
Figure 4.4: Sample non-BPS flow. The attractor point is labeled \((x^*, y^*)\). The initial points of each flow are given by: \((x_1 = 0.539624, y_1 = 5.461135), (x_2 = 1.67984, y_2 = 0.518725), (x_3 = -0.432811, y_3 = 0.289493), (x_4 = 1.28447, y_4 = 1.49815), (x_5 = -0.499491, y_5 = 0.181744)\)
Chapter 5

Chiral Gravity in Three Dimensions

5.1 Introduction

In three dimensions, the Riemann and Ricci tensor both have the same number (six) of independent components. Hence Einstein’s equation, with or without a cosmological constant $\Lambda$, completely constrains the geometry and there are no local propagating degrees of freedom. At first sight this makes the theory sound too trivial to be interesting. However in the case of a negative cosmological constant, there are asymptotically $AdS_3$ black hole solutions [16] as well as massless gravitons which can be viewed as propagating on the boundary. These black holes obey the laws of black hole thermodynamics and have an entropy given by one-quarter the horizon area. This raises the interesting question: what is the microscopic origin of the black hole entropy in these “trivial” theories?
In order to address this question one must quantize the theory. One proposal [147] is to recast it as an $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$ Chern-Simons gauge theory with $k_L = k_R$. Despite some effort this approach has not given a clear accounting of the black hole entropy (see however [64] for an interesting attempt). So the situation remains unsatisfactory.

More might be learned by deforming the theory with the addition of the gravitational Chern-Simons term with coefficient $\frac{1}{\mu}$ [51, 50]. The resulting theory is known as the topologically massive gravity (TMG) and contains a local, massive propagating degree of freedom, as well as black holes and massless boundary gravitons. The addition of the Chern-Simons term leads to more degrees of freedom because it contains three, rather than just two, derivatives of the metric. It is the purpose of this chapter to study this theory for negative $\Lambda = -1/\ell^2$. We will argue that the theory is unstable/inconsistent for generic $\mu$: either the massive gravitons or BTZ black holes have negative energy. The exception occurs when the parameters obey $\mu\ell = 1$, at which point several interesting phenomena simultaneously arise:

(i) The central charges of the dual boundary CFT become $c_L = 0$, $c_R = 3\ell/G$.

(ii) The conformal weights as well as the wave function of the massive graviton, generically $\frac{1}{2}(3 + \mu\ell, -1 + \mu\ell)$, degenerate with those of the left-moving weight $(2, 0)$ massless boundary graviton. They are both pure gauge, but the gauge transformation parameter does not vanish at infinity.

(iii) BTZ black holes and all gravitons have non-negative masses. Further the angular momentum is fixed in terms of the mass to be $J = M\ell$.

This suggests the possibility of a stable, consistent theory at $\mu\ell = 1$ which is
dual to a holomorphic boundary CFT (i.e. containing only right-moving degrees of freedom) with \( c_R = 3\ell/G \). The hope — which remains to be investigated — is that for a suitable choice of boundary conditions the zero-energy left-moving excitations can be discarded as pure gauge. We will refer to this theory as 3D chiral gravity. As we will review herein, if such a dual CFT exists, and is unitary, an application of the Cardy formula gives a microscopic derivation of the black hole entropy [139, 95, 96].

Related recent work [148, 106, 72, 67, 14, 150, 149, 105] has considered an alternative deformation of pure 3D gravity, locally described by the \( SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R \) Chern-Simons gauge theory with \( k_L \neq k_R \). This is a purely topological theory with no local degrees of freedom and is not equivalent to TMG. It contains all the subset of solutions of TMG which are Einstein metrics but not the massive gravitons. It is nevertheless possible that the arguments given in [148] (adapted to the case \( k_L = 0 \) as in [106]) which are quite general apply to the chiral gravity discussed herein. Indeed discrepancies with the semiclassical analysis mentioned in [148, 106, 72, 67, 14, 150, 149, 105] disappear for the special case \( k_L = 0 \).\(^1\) Moreover the main assumption of [148] — holomorphic factorization of the partition function — is simply a consistency requirement for chiral gravity because there are only right movers.

This chapter is organized as follows. Section 5.2 gives a brief review of the cosmological TMG and its \( AdS_3 \) vacuum solution, and shows that the theory is purely chiral at the special value of \( \mu = 1/\ell \). Section 5.3 describes the linearized gravitational excitations around \( AdS_3 \). Section 5.4 shows how the positivity of energy imposes a

\(^1\)Although we do not study the Euclidean theory herein the relation \( M\ell = J \) for Lorentzian BTZ black holes in chiral gravity suggests that the saddle point action will be holomorphic.
stringent constraint on the allowed value of $\mu$. We end with a short summary and discussion of future directions.

5.2 Topologically Massive Gravity

5.2.1 Action

The action for topologically massive gravity (TMG) with a negative cosmological constant $\Lambda$ is

$$I = \frac{1}{16\pi G} \int d^3x \sqrt{-g}(R - 2\Lambda) + \frac{1}{16\pi G\mu}I_{CS}$$

(5.1)

where $I_{cs}$ is the Chern-Simons term

$$I_{cs} = -\frac{1}{2} \int d^3x \sqrt{-g} \epsilon^\lambda{}_{\mu\nu} \Gamma_\lambda^\sigma [\partial_\mu \Gamma_\nu^\sigma + \frac{2}{3} \Gamma_\mu^\sigma \Gamma_\nu^\tau].$$

(5.2)

In this work, we will focus the theory with negative cosmological constant $\Lambda$.

We have chosen the sign in front of the Einstein-Hilbert action so that BTZ black holes have positive energy for large $\mu$, while the massive gravitons will turn out to have negative energy for this choice. This contrasts with most of the literature which chooses the opposite sign in order that massive gravitons have positive energy (for large $\mu$). Note that had we chosen the opposite sign in front of the Einstein action, the BTZ black holes would have negative masses, which signifies a quantum instability. Namely, the vacuum would be unstable against decaying into the infinite degeneracy of zero energy solutions corresponding to negative mass black holes surrounded by positive energy gravitons.
The equation of motion of the theory is

\[ G_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} = 0 \quad \text{(5.3)} \]

where \( G_{\mu\nu} \) is the cosmological-constant-modified Einstein tensor:

\[ G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} \quad \text{(5.4)} \]

and \( C_{\mu\nu} \) is the Cotton tensor

\[ C_{\mu\nu} \equiv \epsilon^\alpha_{\mu\nu} \nabla_\alpha (R_{\beta\nu} - \frac{1}{4} g_{\beta\nu} R) \quad \text{(5.5)} \]

where \( \epsilon \) is Levi-Civita tensor.

Cotton tensor is the three-dimensional analogue of Weyl tensor, which vanishes identically in 3D. From its definition, it is symmetric, manifestly traceless, and obeys Bianchi identity. Since \( C_{\mu\nu} \) is manifestly traceless, its presence does not modify the value of Ricci scalar solved from the equation of motion. That \( C_{\mu\nu} \) obeys Bianchi identity shows that although the presence of the gravitational Chern-Simons term manifestly breaks the diffeomorphism invariance of TMG, the equation of motion of the bulk theory still obeys the diffeomorphism invariance. Finally, the metric compatibility ensures that Einstein metrics which have \( G_{\mu\nu} = 0 \) are a subset of the general solutions of (5.3).

\(^2\)The Cotton tensor is symmetric thanks to the Bianchi identity of \( G_{\mu\nu} \):

\[ \epsilon^{\alpha\mu\nu} C_{\mu\nu} = \nabla_\beta G^\alpha_{\beta} = 0 \quad \text{.} \]
5.2.2 AdS\(_3\) vacuum solution

TMG has an AdS\(_3\) solution:

\[
    ds^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = \ell^2 \left(- \cosh^2 \rho d\tau^2 + d\rho^2 + \sinh^2 \rho d\phi^2\right) \tag{5.6}
\]

where the radius is related to \(\Lambda\) by

\[
    \Lambda = -\frac{1}{\ell^2}. \tag{5.7}
\]

The Riemann tensor, Ricci tensor and Ricci scalar of the AdS\(_3\) are:

\[
    \bar{R}_{\mu\alpha\nu\beta} = \Lambda (\bar{g}_{\mu\nu} \bar{g}_{\alpha\beta} - \bar{g}_{\mu\beta} \bar{g}_{\alpha\nu}) ,
    \bar{R}_{\mu\nu} = 2\Lambda \bar{g}_{\mu\nu} ,
    \bar{R} = 6\Lambda , \tag{5.8}
\]

The metric (5.6) has isometry group \(SL(2, \mathbb{R})_L \times SL(2, \mathbb{R})_R\). The \(SL(2, \mathbb{R})_L\) generators are

\[
    L_0 = i\partial_u ,
    L_{-1} = ie^{-iu} \left[\frac{\cosh 2\rho}{\sinh 2\rho} \partial_u - \frac{1}{\sinh 2\rho} \partial_v + \frac{i}{2} \partial_\rho\right] ,
    L_1 = ie^{iu} \left[\frac{\cosh 2\rho}{\sinh 2\rho} \partial_u - \frac{1}{\sinh 2\rho} \partial_v - \frac{i}{2} \partial_\rho\right] , \tag{5.9-11}
\]

where \(u \equiv \tau + \phi, v \equiv \tau - \phi\). The \(SL(2, \mathbb{R})_R\) generators \(\{L_0, L_{\pm 1}\}\) are given by the above expressions with \(u \leftrightarrow v\). The normalization of the \(SL(2, \mathbb{R})\) algebra is

\[
    [L_0, L_{\pm 1}] = \mp L_{\pm 1} ,
    [L_1, L_{-1}] = 2L_0 . \tag{5.12}
\]

The quadratic Casimir of \(SL(2, \mathbb{R})_L\) is \(L^2 = \frac{1}{2}(L_1 L_{-1} + L_{-1} L_1) - L_0^2\). When acting on scalars, \(L^2 + \bar{L}^2 = -\frac{\ell^2}{2} \nabla^2\) [103].
5.2.3 Chiral gravity

As shown by Brown and Henneaux [31], quantum gravity on asymptotically $\text{AdS}_3$ spacetimes with appropriate boundary conditions is described by a 2D CFT which lives on the boundary. They computed the total central charge of the CFT and found

$$c_L + c_R = 3\ell/G .$$

(5.13)

Simplified calculations were given in [85, 44, 15]. The difference $c_L - c_R$ corresponds to the diffeomorphism anomaly. In reference [96, 94] it was shown that

$$c_L - c_R = -\frac{3}{\mu G} .$$

(5.14)

Combining (5.13) and (5.14), we have

$$c_L = \frac{3\ell}{2G}(1 - \frac{1}{\mu \ell}) , \quad c_R = \frac{3\ell}{2G}(1 + \frac{1}{\mu \ell}) .$$

(5.15)

In order that both central charges are non-negative, we must have, as was also noticed in [138],

$$\mu \ell \geq 1 .$$

(5.16)

An interesting special case is

$$\mu \ell = 1 ,$$

(5.17)

which implies

$$c_L = 0 , \quad c_R = \frac{3\ell}{G} .$$

(5.18)

We will refer to this theory as chiral gravity. If the chiral gravity is unitary it can have only right-moving excitations.
5.3 Gravitons in $AdS_3$

5.3.1 Equation of motion for gravitons

In this section we describe the linearized excitations around background $AdS_3$ metric $\bar{g}_{\mu\nu}$. Expanding

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

with $h_{\mu\nu}$ small, the linearized Ricci tensor and Ricci scalar are:

$$R^{(1)}_{\mu\nu} = \frac{1}{2}(-\nabla^2 h_{\mu\nu} - \nabla_\mu \nabla_\nu h + \nabla^\sigma \nabla_\nu h_{\sigma\mu} + \nabla^\sigma \nabla_\mu h_{\sigma\nu}),$$

$$R^{(1)} = (R_{\mu\nu} g^{\mu\nu})^{(1)} = -\nabla^2 h + \nabla_\mu \nabla_\nu h^{\mu\nu} - 2\Lambda h.$$  

The leading terms in $G$ and $C$ are

$$G^{(1)}_{\mu\nu} = R^{(1)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu} R^{(1)} - 2\Lambda h_{\mu\nu},$$

$$C^{(1)}_{\mu\nu} = \epsilon^\alpha_{\mu\beta} \nabla_\alpha \left( R^{(1)}_{\beta\nu} - \frac{1}{4}\bar{g}_{\beta\nu} R^{(1)} - 2\Lambda h_{\beta\nu} \right).$$

We note $\text{Tr} C^{(1)} = 0$ and the Bianchi identity implies $\nabla^\mu G^{(1)}_{\mu\nu} = \nabla^\mu C^{(1)}_{\mu\nu} = 0$. The linearized equations of motion are then

$$G^{(1)}_{\mu\nu} + \frac{1}{\mu} C^{(1)}_{\mu\nu} = 0.$$  

Tracing this equation yields $\text{Tr} G^{(1)} = -\frac{1}{2} R^{(1)} = 0$. So the equation of motion becomes

$$G^{(1)}_{\mu\nu} + \frac{1}{\mu} \epsilon^\alpha_{\mu\beta} \nabla_\alpha G^{(1)}_{\beta\nu} = 0,$$

where in terms of $h_{\mu\nu}$

$$G^{(1)}_{\mu\nu} = \frac{1}{2}(-\nabla^2 h_{\mu\nu} - \nabla_\mu \nabla_\nu h + \nabla^\sigma \nabla_\nu h_{\sigma\mu} + \nabla^\sigma \nabla_\mu h_{\sigma\nu}) - 2\Lambda h_{\mu\nu}.$$
Now we fix the gauge. We define $\tilde{h}_{\mu\nu} \equiv h_{\mu\nu} - \bar{g}_{\mu\nu} h$, which gives $\tilde{h} = -2h$. Plugging $h_{\mu\nu} = \tilde{h}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \tilde{h}$ into (5.20) and setting it to zero gives

$$\nabla_{\mu} \nabla_{\nu} \tilde{h}^{\mu\nu} = -\Lambda \tilde{h}.$$  \hspace{1cm} (5.27)

Thus, the gauge

$$\bar{\nabla}_{\mu} \tilde{h}^{\mu\nu} = 0$$ \hspace{1cm} (5.28)

together with the linearized equation of motion implies tracelessness of $h_{\mu\nu}$: $\tilde{h} = -2h = 0$. This gauge is equivalent to the harmonic plus traceless gauge $\nabla_{\mu} h^{\mu\nu} = h = 0$. Noting that

$$[\bar{\nabla}_{\sigma}, \nabla_{\mu}] h^{\sigma}_{\nu} = \bar{R}^{\sigma}_{\lambda \sigma \mu} h^{\lambda}_{\nu} - \bar{R}^{\lambda}_{\nu \sigma \mu} h^{\sigma}_{\lambda} = 3\Lambda h_{\mu\nu} - \Lambda g_{\mu\nu},$$ \hspace{1cm} (5.29)

and imposing the gauge condition, (5.26) is just

$$\mathcal{G}^{(1)}_{\mu\nu} = \frac{1}{2}(-\nabla^{2} h_{\mu\nu} + 2\Lambda h_{\mu\nu}).$$ \hspace{1cm} (5.30)

The equation of motion (5.25) thus becomes

$$\left(\nabla^{2} + \frac{2}{\ell^{2}}\right)(h_{\mu\nu} + \frac{1}{\mu} \epsilon^{\alpha\beta} \nabla_{\alpha} h_{\beta\nu}) = 0.$$ \hspace{1cm} (5.31)

### 5.3.2 Massless and massive gravitons

Define three mutually commuting operators ($\mathcal{D}^{L}, \mathcal{D}^{R}, \mathcal{D}^{M}$):

$$(\mathcal{D}^{L/R})_{\mu}^{\beta} \equiv \delta_{\mu}^{\beta} \pm \epsilon^{\alpha\beta} \nabla_{\alpha}, \quad \text{and} \quad (\mathcal{D}^{M})_{\mu}^{\beta} \equiv \delta_{\mu}^{\beta} + \frac{1}{\mu} \epsilon^{\alpha\beta} \nabla_{\alpha},$$ \hspace{1cm} (5.32)

where the meaning of superscripts will become clear presently. The equations of motion (5.25) can then be written as

$$(\mathcal{D}^{L} \mathcal{D}^{R} \mathcal{D}^{M} h)_{\mu\nu} = 0,$$ \hspace{1cm} (5.33)
where we have used the linearized Bianchi identity. Since the three operators commute

equation (5.33) has three branches of solutions. First, the massive gravitons \( h^M_{\mu\nu} \) given by

\[
(D^M h^M)_{\mu\nu} = h^M_{\mu\nu} + \frac{1}{\mu} \epsilon^\alpha_\mu \nabla^\alpha h^M_{\beta\nu} = 0
\]  

(5.34)

are solutions special for TMG. The other two branches are massless gravitons which are also solutions of Einstein gravity: \( G^{(1)}_{\mu\nu} = 0 \). The left-mover \( h^L_{\mu\nu} \) and right-mover \( h^R_{\mu\nu} \) have different first order equations of motion:

\[
(D^L h^L)_{\mu\nu} = h^L_{\mu\nu} + \epsilon^\alpha_\mu \nabla^\alpha h^L_{\beta\nu} = 0 ,
\]

\[
(D^R h^R)_{\mu\nu} = h^R_{\mu\nu} - \epsilon^\alpha_\mu \nabla^\alpha h^R_{\beta\nu} = 0 .
\]

(5.35)

Note that the components of (5.34) and (5.35) tangent to the AdS\(_3\) boundary relate those components of \( h_{\mu\nu} \) to their falloff at infinity. This could be used to directly infer their conformal weights but we will instead just find the full solutions.

### 5.3.3 Mass of massive gravitons

At this stage, we can infer the mass of the massive graviton without solving its full equations of motion. It requires knowing its second order equations of motion. Define linear operator

\[
(\tilde{D}^M)_{\mu}^\beta \equiv \delta^\beta_\mu - \frac{1}{\mu} \epsilon^\alpha_\mu \nabla^\alpha ,
\]

(5.36)

which commutes with \( D^M \) defined earlier. Applying \( \tilde{D}^M \) on (5.25), we obtain the second order equation of motion,

\[
(\tilde{D}^M D^M G^{(1)})_{\mu\nu} = (D^M \tilde{D}^M G^{(1)})_{\mu\nu} = -\frac{1}{\mu^2}[\nabla^2 - (\mu^2 + 3\Lambda)]G^{(1)}_{\mu\nu} = 0
\]

(5.37)

where we have used the linearized Bianchi identity. Note that \( \tilde{D}^M G = 0 \) is just the linearized gravitational wave equation if we exchange \( \mu \) for \( -\mu \). So solutions of TMG
with both signs of $\mu$ are solutions of (5.37). It can conversely be shown that all solutions of (5.37) are solutions of TMG for one sign of $\mu$ or the other.

The mass of the massive graviton can be inferred by comparing the second-order equation of motion of massive gravitons:

$$\[\nabla^2 + \frac{2}{\ell^2} - (\mu^2 - \frac{1}{\ell^2})]h^{M}_{\mu\nu} = 0 \quad (5.38)$$

with that of massless gravitons:

$$\[\nabla^2 + \frac{2}{\ell^2}]h^{L/R}_{\mu\nu} = 0 \quad (5.39)$$

The mass of massive gravitons is thus

$$m = \sqrt{\mu^2 - \frac{1}{\ell^2}} \quad (5.40)$$

At $\mu\ell = 1$, the massive graviton becomes massless. The mass-squared saturates the Breitenlohner-Freedman bound $m^2 = -\frac{1}{\ell^2}$ [28, 27] at zero $\mu$.

### 5.3.4 Solutions of massless and massive gravitons

Next we solve for the three branches of solutions. Rewrite the Laplacian acting on rank-two tensors in terms of the sum of two $SL(2, \mathbb{R})$ Casimirs:

$$\nabla^2 h_{\mu\nu} = -\left[\frac{2}{\ell^2}(L^2 + \bar{L}^2) + \frac{6}{\ell^2}\right]h_{\mu\nu} \quad (5.41)$$

$G^{(1)}_{\mu\nu}$ can be written as

$$G^{(1)}_{\mu\nu} = \left[\frac{1}{\ell^2}(L^2 + \bar{L}^2) + \frac{2}{\ell^2}\right]h_{\mu\nu} \quad (5.42)$$

Thus (5.37) becomes

$$\left[-\frac{2}{\ell^2}(L^2 + \bar{L}^2) - \frac{3}{\ell^2} - \mu^2\right]h_{\mu\nu} = 0 \quad (5.43)$$
This allows us to use the $SL(2, \mathbb{R})_L \times SL(2, \mathbb{R})_R$ algebra to classify the solutions of (5.37). Consider states with weight $(h, \bar{h})$:

$$L_0 |\psi_{\mu\nu}\rangle = h |\psi_{\mu\nu}\rangle, \quad \bar{L}_0 |\psi_{\mu\nu}\rangle = \bar{h} |\psi_{\mu\nu}\rangle. \quad (5.44)$$

From the explicit form of the generators, we see

$$\psi_{\mu\nu} = e^{-ih\rho - i\bar{h}\rho} F_{\mu\nu}(\rho). \quad (5.45)$$

Now let’s specialize to primary states $|\psi_{\mu\nu}\rangle$ which obey $L_1 |\psi_{\mu\nu}\rangle = \bar{L}_1 |\psi_{\mu\nu}\rangle = 0$. These conditions plus the gauge conditions give $h - \bar{h} = \pm 2$ and

$$F_{\mu\nu}(\rho) = f(\rho) \begin{pmatrix} 1 & \frac{h-\bar{h}}{2} & \frac{i}{\sinh(\rho)\cosh(\rho)} \\ \frac{h-\bar{h}}{2} & 1 & \frac{i(h-\bar{h})}{2\sinh(\rho)\cosh(\rho)} \\ \frac{i}{\sinh(\rho)\cosh(\rho)} & \frac{i(h-\bar{h})}{2\sinh(\rho)\cosh(\rho)} & -\frac{1}{\sinh^2(\rho)\cosh^2(\rho)} \end{pmatrix} \quad (5.46)$$

where

$$\partial_\rho f(\rho) + \frac{(h + \bar{h}) \sinh^2 \rho - 2 \cosh^2 \rho}{\sinh \rho \cosh \rho} f(\rho) = 0 \quad (5.47)$$

for the primary states. The solution is

$$f(\rho) = (\cosh \rho)^{-(h+\bar{h})} \sinh^2 \rho. \quad (5.48)$$

Note that the solution for massive graviton and left and right massless gravitons have the same form (5.48), with the only difference being their different conformal weights $(h, \bar{h})$.

Now we solve for the conformal weights $(h, \bar{h})$ for the massless and massive gravitons. Using

$$L^2 |\psi_{\mu\nu}\rangle = -h(h - 1) |\psi_{\mu\nu}\rangle \quad (5.49)$$
for primaries, the weights \((h, \bar{h})\) of primary states obey:

\[
[2h(h - 1) + 2\bar{h}(\bar{h} - 1) - 3\mu^2\ell^2][h(h - 1) + \bar{h}(\bar{h} - 1) - 2] = 0, \quad h - \bar{h} = \pm 2. \tag{5.50}
\]

There are two branches of solutions. The first branch has

\[
h(h - 1) + \bar{h}(\bar{h} - 1) - 2 = 0, \tag{5.51}
\]

which gives:

\[
h = \frac{3 + 1}{2}, \quad \bar{h} = \frac{-1 \pm 1}{2} \quad \text{or} \quad h = \frac{-1 \pm 1}{2}, \quad \bar{h} = \frac{3 \pm 1}{2}. \tag{5.52}
\]

These are the solutions for left and right massless gravitons, which already appear in Einstein gravity. From (5.48), we see that the solutions with the lower sign will blow up at the infinity, so we will only keep the upper ones corresponding to weights \((2, 0)\) and \((0, 2)\). We will refer to these as left-moving and right-moving massless gravitons.

The second branch has

\[
2h(h - 1) + 2\bar{h}(\bar{h} - 1) - 3\mu^2\ell^2 = 0, \tag{5.53}
\]

which gives:

\[
h = \frac{3}{2} \pm \frac{\mu\ell}{2}, \quad \bar{h} = \frac{-1 \pm \mu\ell}{2}, \tag{5.54}
\]

\[
or \quad h = \frac{-1}{2} \pm \frac{\mu\ell}{2}, \quad \bar{h} = \frac{3}{2} \pm \frac{\mu\ell}{2}. \tag{5.55}
\]

The solution (5.54) are the solutions of (5.25), which is the TMG under consideration. The solution (5.55) are the solutions of (5.25) with \(\mu\) replaced by \(-\mu\), namely, the theory with opposite chirality. Moreover, for positive \(\mu\), the solutions with negative signs in (5.54) will blow up at the infinity. Thus we only need to consider the solution
with positive signs in (5.54). Hence the relevant solutions corresponding to massive gravitons are

\[ h = \frac{3}{2} + \frac{\mu \ell}{2}, \quad \bar{h} = -\frac{1}{2} + \frac{\mu \ell}{2}. \]  

(5.56)

Descendants are obtained by simply applying \( L_{-1} \) and \( \bar{L}_{-1} \) on the primary \( |\psi_{\mu\nu}\rangle \).

One immediately sees that the positivity of conformal weights \((h, \bar{h})\) also requires \( \mu \ell \geq 1 \).

(5.57)

Note that for chiral gravity at \( \mu \ell = 1 \), with \( c_L = 0 \), \( c_R = \frac{3\mu}{G} \), the conformal weights of the massive graviton (5.56) become those of left-moving massless gravitons:

\[ (h = \frac{3}{2} + \frac{\mu \ell}{2}, \bar{h} = -\frac{1}{2} + \frac{\mu \ell}{2}) \rightarrow (h = 2, \bar{h} = 0). \]  

(5.58)

Recall that the solutions for massive graviton and massless gravitons differ only in their conformal weights, so the wave function of the massive graviton becomes identical to that of the left-moving massless graviton at \( \mu \ell = 1 \).

The 3D massless graviton are pure gauge in the bulk. In fact, we can explicitly eliminate the \((2,0)\) modes with the residual gauge transformation

\[ \epsilon_t = e^{-2iu} \frac{i\sinh^4(\rho)}{6 \cosh^2(\rho)}, \]  

(5.59)

\[ \epsilon_\phi = e^{-2iu} \frac{-i\sinh^2(\rho) (2 + \cosh^2(\rho))}{6 \cosh^2(\rho)}, \]  

(5.60)

\[ \epsilon_\rho = e^{-2iu} \frac{\sinh(\rho) (1 + 2 \cosh^2(\rho))}{6 \cosh^3(\rho)}, \]  

(5.61)

which satisfies

\[ \bar{\nabla}_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu + \psi^{(2,0)}_{\mu\nu} = 0. \]  

(5.62)
That is, in the chiral gravity with $\mu \ell = 1$, the 3D massive graviton becomes pure gauge in the bulk. Note that the above gauge transformation diverges at the boundary, so whether or not the $(2,0)$ solution should be regarded as gauge equivalent to the vacuum depends on the so-far-unspecified boundary conditions.

5.4 Positivity of Energy

5.4.1 BTZ black holes

**BTZ black holes in Einstein gravity**

BTZ black holes are the only black hole solutions in pure 3D Einstein gravity. Its metric is

$$ds^2 = -N(r)^2 dt^2 + \frac{dr^2}{N(r)^2} + r^2 (d\phi + N^\phi(r) dt)^2,$$

(5.63)

where $N(r)$ is the lapse function and $N^\phi(t)$ the angular shift:

$$N(r)^2 = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{\ell^2 r^2},$$

(5.64)

$$N^\phi(r) = \pm \frac{r_+ r_-}{\ell r^2},$$

(5.65)

and $r_\pm$ are the outer and inner horizon. It is locally $AdS_3$ but has different global topology from $AdS_3$. In fact, BTZ black hole can be constructed as the quotient of $AdS_3$ space: identifying points in $AdS_3$ along Killing vector $\xi$ at intervals of Killing parameter $2\pi n$:

$$P \rightarrow e^{2\pi n \xi} P,$$

(5.66)

where Killing vector $\xi$ mixes between left and right $SL(2,\mathbb{R})$’s and has positive norm.
In pure Einstein gravity, the mass and angular momentum of BTZ black holes are computed simply as conserved ADM charges:

\[ m = \frac{1}{8G} \left( \frac{r_+^2 + r_-^2}{\ell^2} \right), \quad (5.67) \]

\[ j = \pm \frac{1}{8G} \frac{2r_+r_-}{\ell}. \quad (5.68) \]

Inversely, the outer and inner horizons are related to the conserved charges by:

\[ r_{\pm} = \ell (4Gm(1 \pm \sqrt{1 - \left( \frac{j}{\ell m} \right)^2}))^{1/2}. \quad (5.69) \]

One immediately sees that there is an upper bound on the angular momentum:

\[ |j| \leq \ell m. \quad (5.70) \]

The “=” sign gives the extremal BTZ black hole, for which the inner and outer horizons coincide.

The macroscopic entropy of the BTZ black hole in Einstein gravity can be calculated via Area Law since there is no higher-derivative term in the action:

\[ S_{BH} = \frac{\pi}{2G} r_+ \quad (5.71) \]

It satisfies the first law of black hole thermodynamics:

\[ dm = T_+ dS + \Omega_+ dj, \quad (5.72) \]

where \( T_+ \) is the Hawking temperature on the outer horizon and \( \Omega_+ \) is the angular velocity of the spacetime at the outer horizon:\(^3\)

\[ T_+ = \frac{1}{2\pi \ell^2} \frac{r_+^2 - r_-^2}{2r_+} \quad (5.73) \]

\(^3\) The Hawking temperature can be calculated from the surface gravity:

\[ T_\pm = \frac{\kappa}{2\pi} |_{r=r_\pm} \]
and

\[ \Omega_+ = N^\phi (r)|_{r=r_+} = \frac{r_-}{\ell r_+}. \]  \hspace{1cm} (5.74)

One can further check that it obeys the microscopic Cardy Formula

\[ S_{BH} = \frac{\pi^2 c_L(\mu)T_L \ell}{3} + \frac{\pi^2 c_R(\mu)T_R \ell}{3} \]  \hspace{1cm} (5.75)

with the left and right temperatures of the black hole given by

\[ T_{L/R} = \frac{r_+ \mp r_-}{2\pi \ell^2}. \]  \hspace{1cm} (5.76)

**BTZ black holes in topologically massive gravity**

As all solutions of Einstein gravity are also solutions of TMG, one can also study the BTZ black holes living in TMG. The metric was the same as given in eq (5.63). Thus the value of geometric quantities, e.g. inner and outer horizons, surface gravity, hence the Hawking temperature are unchanged.

However, the values of conserved charges are modified by the addition of the gravitational Chern-Simons term. The reason is that the addition of the Chern-Simons term to the bulk action requires additional surface terms, which in turn modify the definition of conserved charges such as energy and angular momentum [50]. These surface terms are non-vanishing in general even for those solutions which satisfy the Einstein’s equation — namely BTZ black holes and massless gravitons. That is, where the surface gravity is given by

\[ \kappa = \frac{1}{2} \frac{d(N^2(r))}{dr}. \]

The temperature can also be determined from the periodicity of the Euclidean time in the Euclidean action.
same solutions in general have different conserved charges when measured in TMG compared to the values measured in Einstein gravity.

For Einstein metrics, the mass $M(\mu)$ and angular momentum $J(\mu)$ at general coupling $\mu$ are related to their values at $\mu = \infty$ by [109, 96]:

$$\ell M(\mu) = \ell M(\infty) + \frac{J(\infty)}{\mu \ell},$$

$$J(\mu) = J(\infty) + \frac{M(\infty)}{\mu}.$$  (5.77)

(5.78)

Here $(M(\infty), J(\infty))$ are just $(m, j)$ computed previously. Note that our conventions differ from [109] in the orientation, and from [96] in the coefficient before the Chern-Simons term, which is related by $\frac{1}{\mu} = -32\pi \beta G_3$.

For $\mu \ell > 1$, the bound

$$\ell M(\infty) \geq |J(\infty)|,$$  (5.79)

ensures positivity of energy for Einstein metrics when $\mu \ell > 1$. For $\mu \ell < 1$, the energy of BTZ black holes becomes negative for sufficiently large rotation (yet within bound).\(^4\) Thus the stability of BTZ black holes in TMG also requires $\mu \ell \geq 1$.

Note that for chiral gravity at $\mu \ell = 1$ we have

$$\ell M\left(\frac{1}{\ell}\right) = J\left(\frac{1}{\ell}\right).$$  (5.80)

This can be interpreted as the statement that all Einstein geometries are right-moving.

Now let’s compute the macroscopic entropy of BTZ black holes in TMG. The Area Law no longer applies here due to the presence of the gravitational Chern-Simons term. Usually, in theories with higher-derivative terms which preserve the

\(^4\)There are some discussion in the $\mu \ell < 1$ region in [124, 125, 121, 122, 123], where other signs of instability also appear.
diffeomorphism invariance, the entropy can be computed via Wald’s formula. The validity of Wald’s formula requires diffeomorphism invariance in order to compute the entropy as the conserved surface charge associated with the horizon Killing field. However, because the gravitational Chern-Simons term breaks the manifest diffeomorphism invariance, Wald’s formula in its original form does not apply in TMG either. Fortunately, since the equation of motion is still diffeomorphism invariant, it is possible to generalize the Wald’s formula to include the Chern-Simons term, as done in [142]. One can also reduce to two dimensions and apply Wald’s formula there [130]. Finally, assuming validity of the first law of black hole thermodynamics, one can compute the entropy after knowing their conserved mass and angular momentum.

The inner and outer horizons are at

\[ r_\pm = \sqrt{2G\ell(\ell M(\infty) + J(\infty))} \pm \sqrt{2G\ell(\ell M(\infty) - J(\infty))} \] (5.81)

and do not depend on \( \mu \). In terms of these, the macroscopic formula for the entropy, including a contribution from the Chern-Simons term, is [138, 130, 125, 142].

\[ S_{BH}(\mu) = \pi \frac{2}{G} \left( r_+ \mp \frac{1}{\mu \ell} r_- \right). \] (5.82)

The left and right temperatures of the black hole are determined by periodicities and also do not depend on \( \mu \), thus they are unchanged from their Einstein values (5.76). Assuming the existence of a unitary dual CFT, the microscopic Cardy formula for the entropy is

\[ S_{BH}(\mu) = \frac{\pi^2 c_L(\mu) T_L \ell}{3} + \frac{\pi^2 c_R(\mu) T_R \ell}{3}. \] (5.83)

Using formula (5.15) for the central charges one readily finds that this agrees with the macroscopic result (5.82).
5.4.2 Gravitons

For $\mu \ell > 1$ the weights $(h, \bar{h})$ of the massive gravitons are positive. The energy of the massive gravitons is proportional to these weights but with a possible minus sign. In particular, if the overall sign of the action is changed, so is the energy, but the equations of motion and hence the weights are unaffected. In [51, 50, 52], it was shown that with the sign taken herein but no cosmological constant massive gravitons have negative energy. In this section we redo this analysis for the case of negative cosmological constant, by constructing the Hamiltonian [32].

The fluctuation $h_{\mu \nu}$ can be decomposed as

$$h_{\mu \nu} = h^M_{\mu \nu} + h^L_{\mu \nu} + h^R_{\mu \nu},$$  \hspace{1cm} (5.84)

where we use the subscript $M$ to represent the $(\frac{3+\mu \ell}{2}, \frac{-1+\mu \ell}{2})$ primary and their descendants, $L$ to represent the $(2, 0)$ primary and their descendants, and $R$ to represent the $(0, 2)$ primary and their descendants. We will call them massive modes, left-moving modes and right-moving modes hereafter.

Up to total derivatives, the quadratic action of $h_{\mu \nu}$ is

$$S_2 = -\frac{1}{32\pi G} \int d^3 x \sqrt{-g} h^{\mu \nu} (g^{(1)}_{\mu \nu} + \frac{1}{\mu} C^{(1)}_{\mu \nu})$$

$$= \frac{1}{64\pi G} \int d^3 x \sqrt{-g} \left( -\bar{\nabla}^\lambda h^{\mu \nu} \bar{\nabla}_\lambda h_{\mu \nu} + \frac{2}{\ell^2} h^{\mu \nu} h_{\mu \nu} - \frac{1}{\mu} \bar{\nabla}_\alpha h^{\mu \nu} \epsilon_{\mu \alpha}^\beta (\bar{\nabla}^2 + \frac{2}{\ell^2}) h_{\beta \nu} \right).$$ \hspace{1cm} (5.85)

The momentum conjugate to $h_{\mu \nu}$ is

$$\Pi^{(1)}_{\mu \nu} = -\frac{\sqrt{-g}}{64\pi G} \left( \nabla^0 (2h^{\mu \nu} + \frac{1}{\mu} \epsilon^{\mu \alpha}_{\beta} \bar{\nabla}_\alpha h_{\beta \nu}) - \frac{1}{\mu} \epsilon_{\beta \nu}^{\mu} (\bar{\nabla}^2 + \frac{2}{\ell^2}) h_{\beta \nu} \right).$$ \hspace{1cm} (5.86)
Using the equations of motion we find,

\[\Pi_{(1)}^{\mu\nu} = \frac{\sqrt{-g}}{64\pi G} \left(-\bar{\nabla}^0 h^{\mu\nu} + \frac{1}{\mu}(\mu^2 - \frac{1}{\ell^2})\epsilon_\beta^{\quad 0\mu} h^{\beta\nu}_M\right),\]

\[\Pi_{(1)}^{(1)}_{\mu\nu} = -\frac{\sqrt{-g}}{64\pi G} (2 - \frac{1}{\mu\ell}) \bar{\nabla}^0 h^{\mu\nu}_L,\]

\[\Pi_{(1)}^{(1)}_{\mu\nu} = -\frac{\sqrt{-g}}{64\pi G} (2 + \frac{1}{\mu\ell}) \bar{\nabla}^0 h^{\mu\nu}_R.\]

Because we have up to three time derivatives in the Lagrangian, using the Ostrogradskii method we should also introduce \(K_{\mu\nu} \equiv \bar{\nabla}^0 h_{\mu\nu}\) as a canonical variable, whose conjugate momentum is

\[\Pi_{(2)}^{\mu\nu} = \frac{-\sqrt{-g}g^{00}}{64\pi G\mu} \epsilon_\beta^{\quad 0\mu} \bar{\nabla}_\lambda h^{\beta\nu},\]

again using equations of motion we have

\[\Pi_{(2)}^{\mu\nu} = \frac{-\sqrt{-g}g^{00}}{64\pi G h_{\mu\nu}^M},\]

\[\Pi_{(2)}^{\mu\nu} = \frac{-\sqrt{-g}g^{00}}{64\pi G\mu\ell h_{\mu\nu}^L},\]

\[\Pi_{(2)}^{\mu\nu} = \frac{\sqrt{-g}g^{00}}{64\pi G\mu\ell h_{\mu\nu}^R}.\]

The Hamiltonian is then

\[H = \int d^2x \left(\dot{h}_{\mu\nu} \Pi_{(1)}^{\mu\nu} + \dot{K}_{ij} \Pi_{(2)}^{ij} - \mathcal{L}\right).\]

Specializing to linearized gravitons, and using their equations of motion, we then have the energies

\[E_M = \frac{1}{\mu}(\mu^2 - \frac{1}{\ell^2}) \int d^2x \frac{\sqrt{-g}}{64\pi G} \epsilon_\beta^{\quad 0\mu} h^{\beta\nu}_M h_{\mu\nu},\]

\[E_L = (-1 + \frac{1}{\mu\ell}) \int d^2x \frac{\sqrt{-g}}{32\pi G} \bar{\nabla}^0 h^{\mu\nu}_L h_{\mu\nu},\]

\[E_R = (-1 - \frac{1}{\mu\ell}) \int d^2x \frac{\sqrt{-g}}{32\pi G} \bar{\nabla}^0 h^{\mu\nu}_R h_{\mu\nu}.\]
All the integrals above are negative, as can be shown by plugging in the solutions (5.45), (5.46), and (5.48) for primaries and by using the $SL(2,\mathbb{R})$ algebra for descendants. For the massive mode, the energy is negative when $\mu\ell > 1$ (it becomes infinitely negative at $\mu = \infty$) and positive when $\mu\ell < 1$. The energy of left-moving modes is positive when $\mu\ell > 1$, and negative when $\mu\ell < 1$. The energy of right-moving modes is always positive. $\mu\ell = 1$ is a critical point, where the energies of both massive modes and the left-moving modes become zero. Note $E_L$ and $E_R$ are consistent with (5.77).

Recall that positivity of central charges requires that $\mu\ell \geq 1$, so the only possibility for avoiding negative energy is to take the chiral gravity theory with $\mu\ell = 1$. In that case, we see that massive and left-moving gravitons carry no energy, and might perhaps be regarded as pure gauge.

### 5.5 Conclusion and Discussion

In this note, we investigated the cosmological TMG, and found that the theory can be at most sensible at $\mu\ell = 1$. At this special point, the theory is completely chiral. In order to show that the chiral theory is classically sensible asymptotic boundary conditions which consistently eliminate the infinite degeneracy of zero-energy excitations must be specified. One must also prove a non-linear positive energy theorem. Renormalizability must be addressed to define the quantum theory.

At the classical level the chiral structure might enable an exact solution of the theory. Some exact non-Einstein solutions of TMG with arbitrary $\mu$ have been found in [126, 83, 117, 6, 7, 49, 24]. Should there turn out to also be a consistent quantum
theory it would be interesting to find the chiral CFT dual. Towards this end the approach of [148] may prove useful.
Chapter 6

Falling D0-Branes in 2D Superstring Theory

6.1 Introduction

In both bosonic and supersymmetric Liouville Field Theory (LFT), there exist static D0 and D1-branes. In particular, the static D0-branes — the so-called ZZ branes — sit in the strong coupling region $\phi \to +\infty$ [60]. In the bosonic system with Euclidean time, Lukyanov, Vitchev, and Zamolodchikov, showed the existence of a time-dependent boundary state, the paperclip brane, that breaks into two hairpin-shaped branes in the UV region [100]. They derived the wave function of the boundary state from the classical shape of the brane in the spacetime. Under the Wick-rotation from Euclidean time into Minkowski time, the hairpin brane is reinterpreted as the falling D0-brane.

The falling D0-brane in the supersymmetric system was first considered by Ku-
tasov [97]. He studied the classical dynamics of the falling D0-brane in the vicinity of a stack of NS5-branes that produce a linear dilaton background. In his treatment, the radial position (along the Liouville direction) of the D0-brane is a dynamical field living on its worldvolume, thus the corresponding DBI action gives the classical trajectory of the D-brane in this background:

\[ e^{-\frac{Q\phi}{2}} = \frac{\tau_p}{E} \cosh \frac{Qt}{2}, \]  

(6.1)

where \( Q \) is the background charge of the linear dilaton, and \( \tau_p \) and \( E \) are the tension and energy of the D0-brane, respectively.

In [100], they considered free bosonic string theory with a linear dilaton and boundary conditions on the bosonic fields. As was noted in [112], although [100] considered no Liouville potential, they required their boundary state to carry the \( \mathcal{W} \)-symmetry, which is defined as the operators commuting with two screening charges. These screening charges are essentially just Liouville potentials. In fact, in [101] it was shown that the linear dilaton theory with the particular boundary conditions considered in [100] is dual to a linear dilaton theory with a boundary Liouville potential. In \( \mathcal{N} = 2 \) SLFT, which has Euclidean time, a type of time-dependent boundary state solution was derived in [58]; the Wick-rotation was then carried out in [112] to study the wave function of the falling D0-brane in the \( \mathcal{N} = 2 \) SLFT system and was found to reproduce the trajectory (6.1) in the classical limit.

In the \( \mathcal{N} = 2 \) case, the \( \mathcal{W} \)-algebra is then replaced by the \( \mathcal{N} = 2 \) SCA, leading to the suggestion that this is the supersymmeterized version of the hairpin brane [112]. While the hairpin construction was shown to live in a theory with a boundary Liouville potential [101], their theory contained no bulk Liouville potential. Since
the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ theories typically contain both bulk and boundary Liouville potentials, to claim a supersymmeterized version of the hairpin brane we will make a comparison (at the end of section 6.3.3) in the limit that the bulk cosmological constant is turned off.

We show that in $\mathcal{N} = 1$, 2D superstring theory with a linear dilaton background — which we will use interchangeably with $\hat{c}_m = 1 \mathcal{N} = 1$ SLFT — there exists a similar, time-dependent boundary state corresponding to the falling D0-brane. The naive argument for the existence of the falling D0-brane is as follows. As is well known, the mass of the D0-brane is inversely related to the string coupling as

$$m = \frac{1}{g_s} = e^{-\phi},$$

(6.2)

so the mass of the D0-brane decreases as it runs along the Liouville direction from the weak coupling region ($\phi \to -\infty$) to the strong coupling region ($\phi \to +\infty$). Thus, if we set a D0-brane free at the weak coupling region, it will roll along the Liouville direction towards the strong coupling region until it is reflected back by the boundary Liouville potential (this point will be expanded upon at the end of section 6.3.3). This is the falling D0-brane solution which can be described by a time-dependent closed string boundary state of the $\mathcal{N} = 1$, 2D superstring.

In the bosonic case, the hairpin brane satisfies symmetries in addition to those of the action (conformal symmetry). The additional symmetry is known as the $\mathcal{W}$-symmetry and is generated by higher spin currents [100]. The hairpin brane is then constructed from the integral equations that are defined by the $\mathcal{W}$-symmetry. In the $\mathcal{N} = 1$, 2D superstring, it should be possible to use the supersymmetrized version of the $\mathcal{W}$-symmetry to go through a similar construction and find a falling D0-brane.
However, we will argue that it can also be obtained by adapting the falling D0-brane solution in $\mathcal{N} = 2$ SLFT [112], [58], to the $\mathcal{N} = 1$, 2D superstring.

In section 6.2, we briefly review properties of $\mathcal{N} = 1$ SLFT, including the construction of boundary states corresponding to ZZ-branes and FZZT-branes using the modular bootstrap approach. In section 6.3, we review properties of $\mathcal{N} = 2$ SLFT as well as the construction of the boundary state corresponding to the falling D0-brane. Finally, in section 6.4 we argue that we may slightly modify the $\mathcal{N} = 2$ SLFT falling D0-brane boundary state to obtain the solution in $\mathcal{N} = 1$, 2D superstring theory, and we discuss the number of falling D0-branes in the Type 0A and 0B projections. It would be interesting to understand these falling D0-branes in the context of matrix models, but this is beyond the scope of this paper.

6.2 $\mathcal{N} = 1$, 2D Superstring Theory and its Boundary States

6.2.1 $\hat{c}_m = 1 \mathcal{N} = 1$ SLFT

The $\mathcal{N} = 1$ super Liouville theory can be obtained by the quantization of a two dimensional supergravity theory [129]. After eliminating the auxiliary field by its equation of motion, adding $\hat{c}_m = 1$ matter, and setting $\alpha' = 2$, the free part of the action is

$$S_0 = \frac{1}{2\pi} \int d^2z \left[ \delta_{\mu\nu} \left( \partial X^\mu \bar{\partial} X^\nu + \bar{\psi}^\mu \bar{\partial} \psi^\nu + \bar{\tilde{\psi}}^\mu \partial \tilde{\psi}^\nu \right) + \frac{Q}{4} R X^1 \right],$$

(6.3)
where $\mu, \nu = 1, 2$. Since we are considering 2D superstring theory below, we will write $\phi = X^1$ and $Y = X^2$ as is common in the literature. The $\mathcal{N} = 1$ SLFT also includes a potential term

$$S_{\text{int}}^{\mathcal{N}=1} = 2i\mu b^2 \int d^2 z : e^{b\phi} : \left( \psi \psi^\dagger + 2\pi \mu e^{b\phi} \right) : , \quad (6.4)$$

where we must have $Q = b + \frac{1}{b}$ for conformal invariance (note that the normal ordering is crucial for this result and comes from the elimination of the auxiliary field). In the case of Neumann boundary conditions (FZZT brane) we can also have a boundary term preserving the superconformal invariance

$$S_B = \int_{\partial \Sigma} \left[ \frac{QK}{4\pi} \phi + \mu_B b\gamma \psi \psi^\dagger e^{b\phi/2} \right] , \quad (6.5)$$

where $K$ is the boundary curvature scalar, $\mu_B$ is the boundary cosmological constant, and $\gamma$ is a “boundary fermion” normalized so that $\gamma^2 = 1$ [111].

The stress energy tensor and superconformal current are

$$T = -\frac{1}{2} \partial Y \partial Y - \frac{1}{2} \partial \phi \partial \phi + \frac{Q}{2} \partial^2 \phi - \frac{1}{2} \delta_{\mu\nu} \psi^\mu \partial \psi^\nu , \quad G = i(\psi \partial \phi + \psi^2 \partial Y - Q \partial \psi^1) , \quad (6.6)$$

which produce the $\mathcal{N} = 1$ superconformal algebra (SCA)

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12} (m^3 - m)\delta_{m,-n} ,$$

$$\{G_r, G_s\} = 2L_{r+s} + \frac{c}{12} (4r^2 - 1)\delta_{r,-s} ,$$

$$[L_m, G_r] = \frac{m - 2r}{2} G_{m+r} , \quad (6.7)$$

where $c = \frac{3}{2} \hat{c} = \frac{3}{2} (1 + 2Q^2 + 1)$, and $r$ and $s$ take integer (half-integer) values in the R (NS) sector. For a critical string theory, we must set $Q = 2$, corresponding to $b = 1$. 
The primary fields in the NS sector are \( N_{p,\omega} =: e^{(Q^2 + ip)\phi + i\omega Y} \) : with weights \( h_{NS}^{p,\omega} = \frac{1}{2}(Q^2 + p^2 + \omega^2) \), while in the R sector they are \( R_{p,\omega}^\pm = \sigma^\pm N_{p,\omega} \) with weights \( h_{R}^{p,\omega} = h_{NS}^{p,\omega} + \frac{1}{16} \). (If we bosonize the complex fermion \( \psi^\pm = \frac{1}{\sqrt{2}}(\psi^2 \pm i\psi^1) =: e^{\pm iH} \), then \( \sigma^\pm \) is given by \( e^{\pm iH/2} \).) The open string character is

\[
\chi_{\sigma,\pm}^{p,\omega}(\tau) = \text{Tr}_{e^{p,\omega}}[q^{L_0 - c/24}(\pm 1)^F], \tag{6.8}
\]

where \( q = e^{2\pi i\tau} \) and we trace over the descendants of \( N_{p,\omega} \) or \( R_{p,\omega} \) for \( \sigma = \text{NS, R} \), respectively. For non-degenerate representations, the open string characters (which result from a trace over a corresponding primary state and its descendants) are \([3, 111],\)

\[
\chi_{\text{NS}}^{p,\omega, +}(\tau) = q^{\frac{1}{2}(p^2 + \omega^2)} \frac{\theta_{00}(\tau, 0)}{\eta(\tau)^3},
\]

\[
\chi_{\text{NS}}^{p,\omega, -}(\tau) = q^{\frac{1}{2}(p^2 + \omega^2)} \frac{\theta_{01}(\tau, 0)}{\eta(\tau)^3},
\]

\[
\chi_{\text{R}}^{p,\omega, +}(\tau) = q^{\frac{1}{2}(p^2 + \omega^2)} \frac{\theta_{10}(\tau, 0)}{\eta(\tau)^3},
\]

\[
\chi_{\text{R}}^{p,\omega, -}(\tau) = 0. \tag{6.9}
\]

### 6.2.2 Open/Closed duality: boundary states

As is well known, we can realize boundary conditions for an open string as constraints on states in the closed string spectrum \([56, 55, 66, 115]\). For example, Neumann and Dirichlet boundary conditions in the open string are realized classically as \( \partial X(y) \mp \bar{\partial} X(y) = 0 \) and \( \psi(y) \mp \eta \bar{\psi}(y) = 0 \), where \( \eta = \pm 1, y \in \mathbb{R} \), and we have taken the boundary to lie along the real axis (the upper sign is for Neumann boundary conditions and the lower for Dirichlet). To transform from the open channel to the closed channel, we must perform the coordinate transformation \( z \to z \) and \( \bar{z} \to \bar{z}^{-1} \),
resulting in the conditions \( \partial X(y) \pm y^{-2} \partial X(y^{-1}) = 0 \) and \( \psi(y) \pm i\eta y^{-1} \tilde{\psi}(y^{-1}) = 0 \).

When we quantize the theory, these become constraints on closed string boundary states:

\[
\text{Neumann: } (\alpha_m + \bar{\alpha}_{-m}) |B, \eta\rangle = (\psi_r + i\eta \tilde{\psi}_r) |B, \eta\rangle = 0, \\
\text{Dirichlet: } (\alpha_m - \bar{\alpha}_{-m}) |B, \eta\rangle = (\psi_r - i\eta \tilde{\psi}_r) |B, \eta\rangle = 0. \tag{6.10}
\]

In general, for a state in the closed string Hilbert space to be a boundary state, it must satisfy two conditions. First, the state must satisfy constraints coming from the requirement that the corresponding boundary vertex operator preserve the symmetries of the original theory. In the case of a simple bosonic theory, this amounts to requiring conformal invariance which, in the language of boundary states, translates to the constraints

\[
(L_m - \bar{L}_{-m}) |B\rangle = 0. \tag{6.11}
\]

Second, the state must satisfy constraints coming from the open/closed duality of a cylinder diagram. If an open string satisfies some boundary conditions \( \alpha \) and \( \beta \) on its left and right ends, respectively, then the corresponding closed string boundary states must satisfy:

\[
\langle B, \alpha |q_c^\frac{1}{2} H_c|B, \beta\rangle = \text{Tr}_{\mathcal{H}_{\alpha\beta}} [q_o^{H_o}], \tag{6.12}
\]

where \( H_c \) and \( H_o \) are the closed and open string Hamiltonians, \( q_c = e^{2\pi i\tau_c} \) and \( q_o = e^{2\pi i\tau_o} \), and the trace on the right is taken over the open string spectrum that satisfies the specified boundary conditions, \( \mathcal{H}_{\alpha\beta} \). The open and closed string moduli are related through worldsheet duality by a modular transformation, \( \tau_c = -\frac{1}{\tau_o} \).
6.2.3 Ishibashi and Cardy states: the modular bootstrap

In $\hat{c}_m = 1 \mathcal{N} = 1$ SLFT, a closed string boundary state must satisfy [3, 111],

\[
(L_m - \tilde{L}_{-m})|B, \alpha; \eta, \sigma\rangle = 0,
\]
\[
(G_r - i\eta \tilde{G}_{-r})|B, \alpha; \eta, \sigma\rangle = 0,
\]
(6.13)

where $\alpha$ labels an open string conformal family, $\sigma = \text{NS, R}$ (really NS-NS and R-R, but we will commonly use this short-hand), and $\eta = \pm$ gives the spin structure of the boundary states. Additionally, the boundary states must satisfy the open/closed duality requirement (6.12). These states are commonly referred to as the Cardy states.

To find the Cardy states, it is convenient to use an orthonormal basis of states satisfying (6.13). The so-called Ishibashi states $|i; \eta, \sigma\rangle$ of the theory form such a basis [89] and are defined to satisfy the additional constraints

\[
\langle\langle i; \eta, \sigma|q^{H^c_{H_c}}|j; \eta', \sigma'\rangle\rangle = \delta_{ij}\delta_{\sigma\sigma'}\chi_i^{\sigma,\eta\eta'}(\tau_c) \equiv \delta_{ij}\delta_{\sigma\sigma'}\text{Tr}_{\mathcal{H}_c^i}[q^{H^c_{H_c}}(\eta\eta')^F],
\]
(6.14)

where $\mathcal{H}_c^i$ is spanned by the conformal family corresponding to the ‘$i$’ representation of the constraint algebra, and the $\chi_i$ are the characters. As a point of clarification, note that in an Ishibashi state $i$, $\eta$, and $\sigma$, denote the representation of a closed string conformal family, the boundary condition on a closed string state, and the closed string sector (NS-NS or R-R), respectively. In a Cardy state, $\eta$ and $\sigma$ have the same meaning while $\alpha$ labels an open string conformal family. This statement will become more clear in section 6.2.4.
These constraints imply that the Ishibashi states are constructed as \([111, 65]\),

\[
|i; \eta, \text{NS}\rangle = |i; \text{NS}\rangle_L|i; \text{NS}\rangle_R + \text{descendants}
\]

\[
|i; \eta, \text{R}\rangle = a|i; \text{R}^+\rangle_L|i; \text{R}^+\rangle_R - i\eta a|i; \text{R}^+\rangle_L|i; \text{R}^-\rangle_R + b|i; \text{R}^+\rangle_L|i; \text{R}^+\rangle_R - i\eta b|i; \text{R}^+\rangle_L|i; \text{R}^-\rangle_R + \text{descendants}, \quad (6.15)
\]

where the coefficients \(a\) and \(b\) are determined by the constraint equations (note that this implies the coefficients of descendants in both sectors will have some \(\eta\) dependence). In fact, when we take the Type 0A projection we will have \(a = 0\) and when we take the Type 0B projection we will have \(b = 0\). Now we may use these Ishibashi states to represent the Cardy states schematically as

\[
|B, \alpha; \eta, \sigma\rangle = \sum_i \Psi_\alpha(i; \eta, \sigma)|i; \eta, \sigma\rangle, \quad (6.16)
\]

where \(\alpha\) and \(i\) may range over some combination of a continuous and discrete spectrum. Cardy showed \([33]\) that the trace in (6.12) may also be represented as

\[
\text{Tr}_{\mathcal{H}_\alpha^\sigma}[q_\alpha^{H_\alpha}(\pm1)^F] = \sum_i n^{i,z}_\alpha \chi_i^{\sigma,\pm}(q_\alpha), \quad (6.17)
\]

where the \(n^{i,z}_\alpha\) are non-negative integers representing the multiplicity of \(\mathcal{H}_i^\sigma\) in \(\mathcal{H}_\alpha^\sigma\) (Cardy’s condition). Using (6.12), (6.16), (6.17), and the modular transformations of the open string characters, we may determine the ‘wave functions’ \(\Psi_\alpha(i; \eta, \sigma)\) (actually, there is an extra freedom that is fixed by noting that these wave functions are one-point functions on the disk and have specific transformation properties under reflection \([65]\)). This is what is known as the modular bootstrap construction.
6.2.4 ZZ and FZZT boundary states

As an example, let us demonstrate how the modular bootstrap is applied to determine the boundary states corresponding to the ZZ brane and the FZZT brane. Since the stress tensor and the superconformal current of the $\hat{c}_m = 1 \mathcal{N} = 1$ SLFT are simply a sum of the corresponding currents of the $\hat{c}_m = 1$ theory and the $\mathcal{N} = 1$ SLFT, the tensor product of an Ishibashi state from each theory will be an Ishibashi state of the combined theory. The ZZ and FZZT boundary states have been constructed for the $\mathcal{N} = 1$ SLFT \cite{3, 65}, and so a trivial modification allows us to write them in our theory.

The open string spectrum corresponding to the descendants of the vacuum state is given by an open string stretching between two vacuum Cardy states, while the spectrum corresponding to an excited state is given by an open string stretching between the corresponding excited Cardy state and a vacuum Cardy state:

\[
\chi^{\tilde{\sigma}, \tilde{\eta}\eta'}_{\text{vac}}(\tau_o) = \langle \text{vac}; \eta, \sigma | q^c H_{\text{c}}^{1/2} | \text{vac}; \eta', \sigma \rangle,
\]
\[
\chi^{\tilde{\sigma}, \tilde{\eta}\eta'}_{p, \omega}(\tau_o) = \langle \text{vac}; \eta, \sigma | q^c H_{\text{c}}^{1/2} | p, \omega; \eta', \sigma \rangle, \quad (6.18)
\]

where $\nu(\tilde{\sigma}) = \frac{1}{2}|\eta - \eta'|$; $\nu(\sigma) = 0, 1$, corresponds to NS and R, respectively; and $\tilde{\eta}\eta' = e^{i\pi \nu(\sigma)}$.

Our goal is to determine the wave function of the Cardy states expressed as a linear combination of the Ishibashi states, which we denote by

\[
|B, p, \omega; \eta, \sigma\rangle = \int_{-\infty}^{\infty} dp' \ d\omega' \ \Psi_{p, \omega}(p', \omega'; \eta, \sigma) |p', \omega'; \eta, \sigma\rangle. \quad (6.19)
\]

To apply this to the vacuum, note that the vacuum state has zero weight and so, in the NS sector, has momentum $p = -\frac{i}{2} (\frac{1}{b} + b)$, $\omega = 0$. Recall that this corresponds to
a (1,1) degenerate representation since every conformal family built from a primary of momentum \( p = -\frac{i}{2}(\frac{m}{b} + nb) \) is degenerate at level \( mn \). This means that the open string character corresponding to the vacuum representation is

\[
\chi^{\text{NS},+}(\tau) = [q^{-(\frac{1}{2}+b)^2/8} - q^{-(\frac{1}{2}-b)^2/8}] \frac{\theta_{00}(\tau,0)}{\eta(\tau)^3},
\]

(6.20)

and the others are obtained similarly. Then by inserting the expansion of the Cardy states into (6.18) and using the normalization of the Ishibashi states (6.14), the above open/closed duality equations are rewritten as

\[
\begin{align*}
\chi^{\tilde{c},\eta\eta'}_{\text{vac}}(\tau_o) &= \int_{-\infty}^{\infty} dp' d\omega' \Psi^{(\sigma)}_{\text{vac}}(p', \omega'; \eta) \Psi^{(\sigma)}_{\text{vac}}(p', \omega'; \eta') \chi^{\tilde{c},\eta\eta'}_{\omega',\omega'}(\tau_c), \\
\chi^{\tilde{c},\eta\eta'}_{p,\omega}(\tau_o) &= \int_{-\infty}^{\infty} dp' d\omega' \Psi^{(\sigma)}_{\text{vac}}(p', \omega'; \eta) \Psi^{(\sigma)}_{p,\omega}(p', \omega'; \eta') \chi^{\tilde{c},\eta}\eta'(\tau_c).
\end{align*}
\]

(6.21)

The wave functions of the vacuum boundary states are simply one-point functions on the disk which must transform in specific ways under reflection [65]. The transformation properties under reflection, combined with the modular transformations of the open string characters (given by the S-transformation matrix) determine the wave functions of the vacuum boundary states to be

\[
\begin{align*}
\Psi^{(\text{NS})}_{\text{vac}}(p, \omega; \eta) &= \frac{\pi(\mu \pi \gamma(\frac{bQ}{2}))^{-ip/b}}{ip \Gamma(-ipb) \Gamma(-ipb)} \Psi^{(\text{NS}),\tilde{c}=1}_{\omega'=0}(\omega; \eta), \\
\Psi^{(\text{R})}_{\text{vac}}(p, \omega; +) &= \frac{\pi(\mu \pi \gamma(\frac{bQ}{2}))^{-ip/b}}{\Gamma(\frac{1}{2} - ipb) \Gamma(\frac{1}{2} - ipb)} \Psi^{(\text{R}),\tilde{c}=1}_{\omega'=0}(\omega; +),
\end{align*}
\]

(6.22)

where \( \Psi^{(\sigma),\tilde{c}=1}_{\omega'} \) denotes the wave function for the \( \tilde{c}_m = 1 \) matter boundary state whose properties do not concern us here. (Note that there is no R-sector (1,1) boundary state with \( \eta = - \) for the same reason that the open string character in this sector is zero [65, 57].) These vacuum boundary states, the ZZ branes, correspond to static Euclidean D0-branes that sit in the strong coupling region, \( \phi \to +\infty \).
The behavior of the Fourier transform of (6.22) is most transparent by utilizing the product representation of the gamma functions. If this product is cutoff after \( N \) terms, one will be taking the Fourier transform of a function of the form \( P_N(p) e^{i a_N(p)} \), where \( P_N(p) \) is a polynomial in \( p \) and \( a_N \) tends to infinity as \( N \) tends to infinity. Such a Fourier transform will give a sum of delta functions and derivatives of delta functions located at \( a_N \). Thus, the Fourier transform of the wave functions in (21) will be localized at infinity as claimed.

All the excited states are non-degenerate representations with continuous momentum \( p' \). The modular transformation of the open string character of the continuous representation together with the solution of the ZZ boundary state gives the excited boundary state (FZZT brane)

\[
\Psi^{(NS)}_{p',\omega'}(p, \omega; \eta) = -\frac{\cos(2\pi pp')}{2\pi} \left( \mu \pi \gamma \left( \frac{bQ}{2} \right) \right)^{-ip/b} i p \Gamma(ipb) \Gamma \left( i \frac{b}{2} \right) \Psi^{(NS),\hat{c}_m=1}_{\omega'}(\omega; \eta),
\]

\[
\Psi^{(R)}_{p',\omega'}(p, \omega; +) = \frac{\cos(2\pi pp')}{2\pi} \left( \mu \pi \gamma \left( \frac{bQ}{2} \right) \right)^{-ip/b} \Gamma \left( \frac{1}{2} + ipb \right) \Gamma \left( \frac{1}{2} + i \frac{b}{2} \right) \Psi^{(R),\hat{c}_m=1}_{\omega'}(\omega; +).
\]

Their pole structures show that they are Euclidean D1-branes, extended in the Liouville direction.

### 6.2.5 An argument for additional symmetry

As we saw above, the characters for \( \hat{c}_m = 1 \mathcal{N} = 1 \) SLFT are simply a product of the individual characters for \( \hat{c}_m = 1 \) matter and \( \mathcal{N} = 1 \) SLFT. This led us to a trivial modification of the ZZ and FZZT boundary states of \( \mathcal{N} = 1 \) SLFT that resulted in static branes, as we saw in section 6.2.4. In fact, we were destined to realize this result because we restricted ourselves to the subset of \( \hat{c}_m = 1 \mathcal{N} = 1 \) Ishibashi states that were simply a tensor product of the Ishibashi states of the separate theories.
We thus implicitly required our boundary states to satisfy an additional symmetry (namely, that they separately satisfy $\mathcal{N} = 1$ boundary conditions for each direction). It was because of this additional symmetry we imposed, combined with the fact the characters decouple, that the wave functions did not mix the two directions.

If we want to find a falling D0-brane, we clearly cannot impose the restrictions mentioned above. Naturally, the time direction should satisfy ‘Neumann-like’ boundary conditions while the Liouville direction should satisfy ‘Dirichlet-like’ conditions. However, what conditions to impose are not obvious. In the bosonic case of the hairpin brane [100], the authors had to impose the $\mathcal{W}$-symmetry to get the time dependence necessary to obtain a falling D0-brane with the desired trajectory.

It turns out that in the $\mathcal{N} = 2$ SLFT, which has the same matter content as $\hat{c}_m = 1 \mathcal{N} = 1$ SLFT, there are additional symmetries that naturally follow from the action. Applying these additional constraints to $\mathcal{N} = 1$ Ishibashi states, one finds boundary states with trajectories that match that of the falling D0-brane (6.1). We will show that in $\mathcal{N} = 1$, 2D superstring theory with linear dilaton background (which is equivalent to $\hat{c}_m = 1 \mathcal{N} = 1$ SLFT), there exists a type of falling D0-brane that has the additional $\mathcal{N} = 2$ SCA symmetry and can be obtained by a slight modification of the falling D0-brane solution of the $\mathcal{N} = 2$ SLFT.
6.3 $\mathcal{N} = 2$ SLFT and its Boundary States

6.3.1 $\mathcal{N} = 2$ SLFT

The $\mathcal{N} = 2$ SLFT theory has the same free action as $\hat{c}_m = 1 \mathcal{N} = 1$ SLFT given by \((6.3)\), but has different interaction terms. In fact, there are two types of interaction terms that are consistent with the $\mathcal{N} = 2$ superconformal symmetry. The chiral interaction terms are

$$S^{\mathcal{N}=2}_{c} = 2\mu b^2 \int d^2 z \left( \frac{\pi}{2} \mu : e^{b(\phi+iY)} : e^{b(\phi-iY)} : + (\psi^1 \bar{\psi}^1 - \psi^2 \bar{\psi}^2) e^{b\phi} \cos bY ight. \\
\left. - (\psi^2 \bar{\psi}^1 + \psi^1 \bar{\psi}^2) e^{b\phi} \sin bY \right), \quad (6.24)$$

while the non-chiral interaction terms are

$$S^{\mathcal{N}=2}_{nc} = \mu' \int d^2 z \left( \partial \phi - i \partial Y + \frac{i}{b} \psi^1 \bar{\psi}^2 \right) \left( \bar{\partial} \phi + i \bar{\partial} Y + \frac{i}{b} \bar{\psi}^1 \bar{\psi}^2 \right) e^{\frac{1}{b} \phi}. \quad (6.25)$$

In this theory, the background charge does not get renormalized, so we have $Q = \frac{1}{b}$ instead of $Q = \frac{1}{b} + b$ as we had in the $\mathcal{N} = 1$ SLFT and the bosonic LFT. Only the non-chiral interaction preserves the $\mathcal{N} = 2$ supersymmetry after a Wick rotation of the Euclidean time, $Y$. Additionally, there is a boundary action (as in the $\mathcal{N} = 1$ case) with Liouville potentials multiplied by a boundary cosmological constant $\mu_B$ (the exact form is not particularly illuminating, the interested reader is referred to [2]).

Since the free part of the action is the same as for $\hat{c}_m = 1 \mathcal{N} = 1$ SLFT, $T$ and $G$ — which we will call $G^1$ for this section — are the same as before. However, since this is an $\mathcal{N} = 2$ theory, we have a current $G^2$ corresponding to the second supercharge. Both the chiral and non-chiral interaction terms are invariant under a
combined shift in Euclidean time and rotation between the two fermions, leaving us with an additional $U(1)$ current, $J$:

\[
T = -\frac{1}{2} \partial \phi \partial \phi - \frac{1}{2} \partial Y \partial Y - \frac{1}{2} \delta_{\mu \nu} \psi^\mu \partial \psi^\nu + \frac{Q}{2} \partial^2 \phi,
\]

\[
G^1 = i \left( \psi^1 \partial \phi + \psi^2 \partial Y - Q \partial \psi^1 \right),
\]

\[
G^2 = -i \left( \psi^2 \partial \phi - \psi^1 \partial Y - Q \partial \psi^2 \right),
\]

\[
J = i \left( \psi^1 \psi^2 + Q \partial Y \right).
\] (6.26)

It will be convenient to define $G^\pm \equiv \frac{1}{\sqrt{2}} (G^1 \pm iG^2)$, which allows us to write the $\mathcal{N} = 2$ SCA as

\[
[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} (m^3 - m) \delta_{m,-n},
\]

\[
[L_m, G^\pm_r] = \left( \frac{m}{2} - r \right) G^\pm_{m+r}, \quad [J_m, G^\pm_r] = \pm G^\pm_{m+r},
\]

\[
\{G^+_r, G^-_s\} = 2L_{r+s} + (r-s)J_{r+s} + \frac{c}{12} (4r^2 - 1) \delta_{r,-s}, \quad \{G^+_r, G^+_s\} = 0,
\]

\[
[L_m, J_n] = -nJ_{m+n}, \quad [J_m, J_n] = \frac{c}{3} m \delta_{m,-n},
\] (6.27)

with central charge $c = 3\hat{c} = 3(1 + Q^2)$, so again $Q = 2$ corresponds to a critical string theory, but in this case we must have $b = \frac{1}{2}$. The primary fields are the same as in section 6.2.1, with corresponding $U(1)$ charges

\[
\hat{j}^{\text{NS}}_p,\omega = Q \omega,
\]

\[
\hat{j}^{\text{R},\pm}_p,\omega = \hat{j}^{\text{NS}}_p,\omega \pm \frac{1}{2}.
\] (6.28)

The open string character is

\[
\chi_{\xi}^{\sigma,\pm}(\tau, \nu) = \text{Tr}_\xi \left[ q^{L_0-c/24} y^{J_0} (\pm 1)^F \right],
\] (6.29)
where \( q = e^{2\pi i \tau} \), \( y = e^{2\pi i \nu} \), and \( \xi \) denotes the ‘\( \xi \)’ representation of the constraint algebra. The characters of \( \mathcal{N} = 2 \) SCA representations split into three classes using the fermionic operator \( G^{\pm}_{-\frac{1}{2}} \) [58, 111]:

- **Class 1 (Graviton):** The graviton representation is defined by the open string primaries satisfying \( G^{\pm}_{-\frac{1}{2}} |\text{graviton}\rangle = 0 \), which implies that \( L_{-1}|\text{graviton}\rangle = 0 \). This constrains the momentum to \( p = -\frac{i}{2b} \), \( \omega = 0 \), which implies \( h = 0 \). Thus, the graviton representation corresponds to the unique vacuum state (the identity operator). After eliminating this state, the character becomes

\[
\chi^{\text{NS,+}}_{\text{vac}}(\tau, \nu) = q^{-\frac{1}{12}} \frac{1 - q}{(1 + y\sqrt{q})(1 + y^{-1}\sqrt{q})} \frac{\theta_{00}(\tau, \nu)}{\eta(\tau)^3}.
\]  

(Note that in all three classes, \( \chi^{\text{NS,-}}_{\xi} \) is obtained by replacing \( \theta_{00} \) with \( \theta_{01} \), \( \chi^{\text{R,+}}_{\xi} \) by replacing both \( \theta_{00} \) with \( \theta_{10} \) and \( j = Q\omega \) with \( j = Q\omega \pm \frac{1}{2} \) (chiral/anti-chiral), and \( \chi^{\text{R,-}}_{\xi} = 0 \).)

- **Class 2 (Massive):** The massive representation is defined by \( G^{\pm}_{-\frac{1}{2}} |\text{massive}\rangle \neq 0 \). For generic \( p \) and \( \omega \), this representation is non-degenerate. The NS character is obtained by summing over all the descendants while using \( j = Q\omega \):

\[
\chi^{\text{NS,+}}_{[p,\omega]}(\tau, \nu) = q^{\frac{1}{2}(p^2 + \omega^2)} y^{Q\omega} \frac{\theta_{00}(\tau, \nu)}{\eta(\tau)^3}.
\]  

- **Class 3 (Massless):** The massless representation is defined by \( G^{+}_{-\frac{1}{2}} |\text{chiral}\rangle = 0 \) or \( G^{-}_{-\frac{1}{2}} |\text{anti-chiral}\rangle = 0 \) for the chiral or anti-chiral representations, respectively. This implies that the momentum must satisfy \( \frac{Q}{2} + ip = \pm \omega \), respectively. The character for the chiral representation is obtained by eliminating the contribu-
tion from the $G^{+}_{\frac{1}{2}}$ mode:

$$\chi_{\omega}^{NS,+}(\tau, \nu) = q^{-\frac{1}{8^2}} \frac{(y \sqrt{q})^{Q\omega}}{1 + y \sqrt{q}} \frac{\theta_{00}(\tau, \nu)}{\eta(\tau)^{\frac{3}{8}}}.$$  \hspace{1cm} (6.32)

### 6.3.2 $\mathcal{N} = 2$ Ishibashi States and Cardy States

An $\mathcal{N} = 2$ boundary state is constructed as in the $\mathcal{N} = 1$ system. Naturally, an $\mathcal{N} = 2$ boundary state must satisfy the $\mathcal{N} = 1$ conditions

$$(L_m - \tilde{L}_m)|B; \eta, \sigma\rangle = (G^1_r - i\eta \tilde{G}^1_r)|B; \eta, \sigma\rangle = 0 .$$  \hspace{1cm} (6.33)

Additionally, an $\mathcal{N} = 2$ boundary state will satisfy one of two different conditions. An A-Type boundary state will satisfy

$$(J_m - \tilde{J}_m)|B; \eta, \sigma\rangle = (G^\pm_r - i\eta \tilde{G}^\pm_r)|B; \eta, \sigma\rangle = 0 ,$$  \hspace{1cm} (6.34)

while a B-Type state will satisfy

$$(J_m + \tilde{J}_m)|B; \eta, \sigma\rangle = (G^\pm_r - i\eta \tilde{G}^\pm_r)|B; \eta, \sigma\rangle = 0 .$$  \hspace{1cm} (6.35)

B-Type conditions correspond to 'Neumann-like' boundary conditions on Euclidean time while A-Type conditions correspond to ‘Dirichlet-like’ boundary conditions. Since we are interested in studying D0-branes, we will focus on the B-Type states for the rest of this paper.

If we denote the R-sector primary states as $|h, j; R^\pm\rangle_L$, where $j = Q\omega$ as in (6.28) and $\pm$ denotes the spin structure, then $J_0|h, j; R^\pm\rangle_L = (j \pm \frac{1}{2})|h, j; R^\pm\rangle_L$ (and similarly in the right-moving sector). We can check from (6.15) that the B-Type, R-R sector Ishibashi states can be constructed schematically as

$$|h, j; \eta, R\rangle \propto |h, j; R^-\rangle_L |h, -j; R^+\rangle_R - i\eta |h, j; R^+\rangle_L |h, -j; R^-\rangle_R + \text{descendants} ,$$  \hspace{1cm} (6.36)
Since $\psi_0^+|h, j; R^-\rangle_L = |h, j; R^+\rangle_L$, it is clear that $(-1)^{F+\tilde{F}} = -1$ on the primary states in (6.36). Furthermore, from the commutation relations $\{J_0, G_0^\pm\} = \pm G_0^\pm$, we can see that the constraint $(J_0 + \tilde{J}_0) = 0$ implies that all descendants in (6.36) must have an equal number of fermionic raising operators on the left-moving and right-moving sides, modulo 2. Therefore, B-Type, R-R sector Ishibashi states will be projected out by the Type 0B GSO projection and so are only present in Type 0A — note that a similar argument implies that A-Type, R-R sector Ishibashi states are only present in Type 0B. Thus, B-Type states will yield stable D0-branes in Type 0A and unstable D0-branes in Type 0B.

B-Type Ishibashi states are constructed to form an orthonormal basis for states satisfying (6.33) and (6.35), and must also satisfy

\begin{align}
\text{Class 1} & \quad \langle \langle \text{vac}; \eta, \sigma | q_{\xi}^\frac{1}{2} H_{c} \ y_{\xi}^\frac{1}{2} (J_0 - \tilde{J}_0) | \text{vac}; \eta, \sigma \rangle \rangle = \chi_{\text{vac}}^{\sigma, \eta \eta'}(\tau_c, \nu_c), \\
\text{Class 2} & \quad \langle \langle p, \omega; \eta, \sigma | q_{\xi}^\frac{1}{2} H_{c} \ y_{\xi}^\frac{1}{2} (J_0 - \tilde{J}_0) | p', \omega'; \eta, \sigma \rangle \rangle = \delta(p - p') \delta(\omega - \omega') \chi_{[p, \omega]}^{\sigma, \eta \eta'}(\tau_c, \nu_c), \\
\text{Class 3} & \quad \langle \langle \omega; \eta, \sigma | q_{\xi}^\frac{1}{2} H_{c} \ y_{\xi}^\frac{1}{2} (J_0 - \tilde{J}_0) | \omega'; \eta, \sigma \rangle \rangle = \delta(\omega - \omega') \chi_{\omega}^{\sigma, \eta \eta'}(\tau_c, \nu_c),
\end{align}

while all other correlators between Ishibashi states vanish. The open and closed parameters are related by the modular transformation $\tau_o = -\frac{1}{\tau_c}$ and $\nu_o = \frac{\nu_c}{\tau_c}$. The B-Type Cardy states are then constructed as a linear combination of the B-Type Ishibashi states such that the Cardy states satisfy

\begin{align}
\langle B, O; \eta, \sigma | q_{\xi}^\frac{1}{2} H_{c} \ y_{\xi}^\frac{1}{2} (J_0 - \tilde{J}_0) | B, \xi; \eta' \rangle \rangle &= e^{i \pi c^2 \tau_o} \chi_{\xi}^{\sigma, \eta \eta'}(\tau_o, \nu_o), \\
\langle B, O; \eta, \sigma | q_{\xi}^\frac{1}{2} H_{c} \ y_{\xi}^\frac{1}{2} (J_0 - \tilde{J}_0) | B, O; \eta' \rangle \rangle &= e^{i \pi c^2 \tau_o} \chi_{\text{vac}}^{\sigma, \eta \eta'}(\tau_o, \nu_o),
\end{align}

where $\chi_{\xi}^{\sigma, \pm}(\tau, \nu)$ is the open string character of the $\xi$ representation of the constraint algebra, $O$ represents the graviton state, and $\tilde{\sigma}$ and $\tilde{\eta} \tilde{\eta}'$ are defined as in equation
6.3.3 Falling Euclidean D0-brane in $\mathcal{N} = 2$ SLFT

As in $\hat{c}_m = 1$ $\mathcal{N} = 1$ SLFT, the modular transformation of the Class 1 (Graviton) representation (identity operator) gives the wave function of the vacuum boundary state. Then the modular transformation of the Class 2 (Massive) non-degenerate representation of the open string produces the wave function of the excited boundary state which corresponds to the FZZT brane (Falling Euclidean D0-brane) solution [58, 112, 4, 5]:

$$
\Psi[p', \omega'](p, \omega; \eta, \sigma) = \sqrt{2Q\tilde{\mu}e^{-2\pi i \omega'p'}\cos(2\pi pp')} \frac{\Gamma(-iQp)\Gamma(1 - i\frac{2\omega}{Q})}{\Gamma\left(\frac{1}{2} - i\frac{2}{Q} + \frac{\nu(\sigma)}{2}\right)\Gamma\left(\frac{1}{2} - i\frac{2}{Q} - \frac{\omega}{Q} + \frac{\nu(\sigma)}{2}\right)},
$$

where $\tilde{\mu}$ is the renormalized bulk cosmological constant (it is, in fact, proportional to $\mu$ [111] and we will henceforth drop the distinction between the two as it just corresponds to a finite, constant shift of the dilaton in the position space picture).

Note also that this wave function has no dependence on $\eta$.

The Fourier transform of the momentum space wave function into the position space wave function is

$$
\bar{\Psi}^{(NS)}_{[p', \omega']}(\phi, Y) \equiv \int_{-\infty}^{\infty} dpd\omega \frac{d}{(2\pi)^2} e^{-ip(\phi + Q \ln \mu)} e^{-i\omega Y} \Psi^{(NS)}[p', \omega'](p, \omega).
$$

We can construct solutions where $p'$ and $\omega'$ are nonzero from the solution in which they are both zero [112]

$$
\bar{\Psi}^{(NS)}_{[0,0]}(\phi, Y) = \frac{\sqrt{2}}{\mu \pi Q (2 \cos \frac{QY}{2})^{\frac{2}{Q} + 1}} \cdot \exp \left[ -\frac{\phi}{Q} - \frac{e^{-\frac{\phi}{2}}}{\mu (2 \cos \frac{Q Y}{2})^{\frac{2}{Q}}} \right].
$$
Then it is simple to see that

\[
\tilde{\Psi}^{(NS)}_{p', \omega'}(\phi, Y) = \frac{1}{2} \tilde{\Psi}^{(NS)}_{[0,0]}(\phi - 2\pi p', Y + 2\pi \omega') + \frac{1}{2} \tilde{\Psi}^{(NS)}_{[0,0]}(\phi + 2\pi p', Y + 2\pi \omega'). \tag{6.42}
\]

Actually, \(p'\) is not an independent variable but is instead related to the bulk and boundary cosmological constants through [4]

\[
\left( \mu_B^2 + \frac{\mu^2}{4Q^4 \mu_B^2} \right) = \frac{\mu}{32\pi Q} \cosh \left( \frac{2\pi p'}{Q} \right) \tag{6.43}
\]

(at least for \(\omega' = 0\)). Since [100] and [101] contain a theory with no bulk cosmological constant, we should take the limit \(\mu \to 0\) to make a comparison with the hairpin brane. In this limit, \(p' \to \pm \infty\), which is irrelevant since (6.42) contains a sum of both signs of \(p'\), so let us take \(p' \to +\infty\) in which case

\[
e^{2\pi p'/Q} \to \frac{64\pi Q \mu_B^2}{\mu}. \tag{6.44}
\]

One then finds

\[
\lim_{\mu \to 0} \tilde{\Psi}_{p,0}(\phi, Y) = \frac{\sqrt{2}}{64\pi^2 Q^2 \mu_B^2 (2 \cos \frac{QY}{2})^2 + 1} \cdot \exp \left[ -\frac{\phi}{Q} - \frac{e^{-\phi}}{64\pi Q \mu_B^2 (2 \cos \frac{QY}{2})^2 + 1} \right]. \tag{6.45}
\]

The classical shape of this falling Euclidean D0-brane is given by the peak of its wave function

\[
e^{-\frac{Q\phi}{2}} = 128\pi Q \mu_B^2 \cos \frac{QY}{2}, \tag{6.46}
\]

reproducing the trajectory of the hairpin brane [100] for appropriate shift of the Liouville direction. This supports the suggestion [97, 112] that this is the supersymmetric extension of the hairpin brane. Note that if we had instead considered the limit \(\mu_B \to 0\), we would have found that the wave function vanishes. Thus, the boundary Liouville potential is necessary for the existence of these boundary states.
The wave function $\tilde{\Psi}_{[0,0]}(\phi, Y)$ is also peaked along the trajectory
\[ e^{-\frac{Q\phi}{2}} = 2 \cos \frac{QY}{2} \] (6.47)
and we will simply refer to this wave function for the rest of our discussions. Since we are only interested in bulk one-point functions, limits can always be taken if one wishes to look at the case with vanishing bulk cosmological constant.

### 6.3.4 Falling D0-brane in $\mathcal{N} = 2$ SLFT

For the wave function $\tilde{\Psi}_{[0,0]}(\phi, Y)$, the Wick-rotation from the Euclidean time $Y$ into the Minkowski time $t$, together with a shift in the Liouville direction $\phi \rightarrow \phi - \frac{2}{Q} \ln \tilde{r} - Q \ln \mu$, produces the classical trajectory of the falling D0-brane in $\mathcal{N} = 2$ SLFT [112]:
\[ \tilde{r} e^{-\frac{Q\phi}{2}} = 2 \cosh \frac{Qt}{2}, \] (6.48)
which matches with (6.1) once we set $\tilde{r} = \frac{2E}{r_p}$.

Therefore, the falling D0-brane wave function in position space is [112]
\[
\tilde{\Psi}_{[0,0]}^{(\text{NS})}(\phi, t) = \frac{\sqrt{2}}{\mu \pi Q(2 \cosh \frac{Qt}{2})^{\frac{3}{2}} + 1} \cdot \exp \left[ -\phi - \frac{2}{Q} \ln \tilde{r} - \frac{e^{-\phi - \frac{2}{Q} \ln \tilde{r}}}{\mu (2 \cosh \frac{Qt}{2})^{\frac{3}{2}}} \right].
\] (6.49)

Then the Fourier transform to momentum space yields
\[
\Psi_{[0,0]}^{(\text{NS})}(p, q) = -i \sqrt{2} Q e^{-iQp} e^{i \frac{2}{Q} \ln \tilde{r}} \sinh \left( \frac{2\pi p}{Q} \right) \cosh \left( \frac{2\pi q}{Q} \right) \Gamma(-iQp) \Gamma(1 - i \frac{2p}{Q}) \frac{\Gamma(\frac{1}{2} - i \frac{p}{Q} + i \frac{q}{Q}) \Gamma(\frac{1}{2} - i \frac{p}{Q} - i \frac{q}{Q})}{\Gamma(\frac{1}{2} - i \frac{p}{Q} + i \frac{q}{Q}) \Gamma(\frac{1}{2} - i \frac{p}{Q} - i \frac{q}{Q})}. \] (6.50)

and a half spectral flow gives the R-sector wave function
\[
\Psi_{[0,0]}^{(R)}(p, q) = -i \sqrt{2} Q e^{-iQp} e^{i \frac{2}{Q} \ln \tilde{r}} \sinh \left( \frac{2\pi p}{Q} \right) \cosh \left( \frac{2\pi q}{Q} \right) \Gamma(-iQp) \Gamma(1 - i \frac{2p}{Q}) \frac{\Gamma(\frac{1}{2} - i \frac{p}{Q} + i \frac{q}{Q}) \Gamma(\frac{1}{2} - i \frac{p}{Q} - i \frac{q}{Q})}{\Gamma(\frac{1}{2} - i \frac{p}{Q} + i \frac{q}{Q}) \Gamma(\frac{1}{2} - i \frac{p}{Q} - i \frac{q}{Q})}. \] (6.51)
6.4 Falling D0-brane in $\mathcal{N} = 1$, 2D Superstring Theory

6.4.1 Using $\mathcal{N} = 2$ SLFT to study boundary states in 2D superstring

We propose that the $\mathcal{N} = 2$ SLFT boundary states may be used to study falling D0-branes in the $\mathcal{N} = 1$, 2D superstring with linear dilaton background (which is equivalent to $\hat{c}_m = 1 \mathcal{N} = 1$ SLFT theory). Notice that the field content of both theories is the same, as are the stress tensor and the first supercharge. Additionally, as is apparent from the constraints on an $\mathcal{N} = 2$ boundary state, any $\mathcal{N} = 2$ boundary state also satisfies the $\mathcal{N} = 1$ constraints (6.13). So an $\mathcal{N} = 2$ SLFT boundary state is also a boundary state of the $\mathcal{N} = 1$, 2D superstring, with the $\mathcal{N} = 2$ boundary interaction term.

However, this alone is not enough; if we want to use the $\mathcal{N} = 2$ Ishibashi states, we must also be able to construct a Cardy state from them that will generate the $\hat{c}_m = 1 \mathcal{N} = 1$ open string character. In fact, we can do this. In (6.31), $\nu$ is the ‘modulus’ of the $U(1)$ charge and appears nontrivially in the functional form of the character. But if we set $\nu = 0$ ($y = 1$), the $U(1)$ charge acts trivially on the $\mathcal{N} = 2$ states and we can see that the character of the $\mathcal{N} = 2$ Class 2 (Massive) representation (6.31) is equivalent to the $\hat{c}_m = 1 \mathcal{N} = 1$ character (6.8). Such a state was found in [58, 112, 4, 5], and presented in sections 6.3.3 and 6.3.4. The momentum space wave
functions are the same as in (6.50) and (6.51):

\[
\Psi_{[0,0]}^{(\text{NS})}(p, q) = \Psi_{[0,0]}^{(\text{R})}(p, q) = \frac{-i\sqrt{2}Q\mu e^{i\frac{2\pi}{Q} \ln \tilde{r}}}{\cosh(\frac{2\pi p}{Q}) + \cosh(\frac{2\pi q}{Q})} \cdot \frac{\Gamma(-iQp)\Gamma(1 - i\frac{2p}{Q})}{\Gamma(\frac{1}{2} - i\frac{2p}{Q}) + i\frac{2p}{Q} \Gamma(\frac{1}{2} - i\frac{2p}{Q} - i\frac{2q}{Q})} \cdot \frac{\Gamma(-iQp)\Gamma(1 - i\frac{2p}{Q})}{\Gamma(1 - i\frac{2p}{Q} + i\frac{2q}{Q}) \Gamma(-i\frac{2p}{Q} - i\frac{2q}{Q})} \quad (6.52)
\]

This is not a surprising result. Recall that in section 6.2.5, we argued that in \( \hat{c}_m = 1 \mathcal{N} = 1 \) SLFT we could find a falling boundary state with the Liouville and time directions coupled nontrivially by assuming additional symmetries that coupled the two directions. This additional symmetry is a symmetry only of the boundary state and not of the theory as a whole. The coupling of the Liouville and time directions is then achieved by imposing this additional symmetry on the Hilbert space of the original \( \hat{c}_m = 1 \mathcal{N} = 1 \) SLFT boundary states.

Note that it is also possible to derive this falling D0-brane in the \( \mathcal{N} = 1, 2 \)D superstring by directly solving the constraint equations satisfied by the boundary states, similar to the derivation of the bosonic hairpin brane [100]. The equations are constrained by the \( W \)-symmetry in the bosonic case, while by the \( \mathcal{N} = 2 \) SCA in the \( \mathcal{N} = 1, 2 \)D superstring.

### 6.4.2 Number of D0-branes after GSO projection

In \( \mathcal{N} = 1, 2 \)D superstring theory, there are two distinct types of boundary states in each of the NS-NS and R-R sectors, corresponding to the different boundary conditions for world sheet fermions (\( \eta = \pm \)). Therefore, the Type 0, non-chiral GSO projection produces four types of stable D0-branes (two branes and two anti-branes) in the Type 0A theory, and two unstable D0-branes in the Type 0B theory. In the
Type 0A theory, the $D0^\pm$-branes are sourced by two different R-R gauge fields, $C_1^{(\pm)}$ [144].

The D-brane boundary states in the Type 0A theories are given by the non-chiral GSO projection $\frac{1\pm(-1)^{F+\tilde{F}}}{2}$, with the upper sign for the NS-NS sector and the lower sign for the R-R sector. As explained in section 6.3.2, the D0-branes of our theory correspond to the B-Type boundary states. Since the B-Type, R-R Ishibashi states survive the Type 0A GSO projection, the D0-branes in Type 0A will be stable. We can represent them schematically as

$$|D0; +\rangle = |B0; +, NS\rangle + |B0; +, R\rangle,$$

$$|D0; -\rangle = |B0; -, NS\rangle + |B0; -, R\rangle,$$

$$|\overline{D}0; +\rangle = |B0; +, NS\rangle - |B0; +, R\rangle,$$

$$|\overline{D}0; -\rangle = |B0; -, NS\rangle - |B0; -, R\rangle.$$ (6.53)

On the other hand, the D-brane boundary states in the Type 0B theories are given by the non-chiral GSO projection $\frac{1+(-1)^{F+\tilde{F}}}{2}$ for both the NS-NS and R-R sectors. In this case, the B-Type, R-R Ishibashi states are projected out by the Type 0B projection. This leaves us with two unstable D0-branes represented schematically as

$$|\widehat{D}0; +\rangle = |B0; +, NS\rangle,$$

$$|\widehat{D}0; -\rangle = |B0; -, NS\rangle.$$ (6.54)

Note that the sign, $\eta = \pm$, really does denote different D0-branes since, by Cardy’s condition (6.12) and (6.18), these states yield different spectra. It would be interesting to see how states with different values of $\eta$ distinguish themselves from each other in the context of matrix models.
6.5 Discussion and Summary

Recall that in the static case, the D0-brane (ZZ brane) and the D1-brane (FZZT brane) boundary states were derived from the degenerate and the non-degenerate representation of the open string character, respectively. So it may seem a little puzzling that the falling D0-brane is constructed from the non-degenerate representation instead of the degenerate one.

In the static case, the difference between the D0-brane and the D1-brane is that the D0-brane is localized at the same point in the Liouville direction for all time, while the D1-brane is extended. The open strings ending on this D0-brane can only take on a fixed (imaginary) value of Liouville momentum, while the open strings ending on the D1-brane can take on any value of Liouville momentum.

In the case of the falling D0-brane, if we partition the two-dimensional spacetime into spacelike hypersurfaces, the falling D0-brane is again localized in the Liouville direction along each hypersurface. However, the Liouville position from one hypersurface to the next is not the same (the falling D0-brane is, not surprisingly, moving), and so open strings ending on the falling D0-brane can take on any value of Liouville momentum. This is the reason that we must use the non-degenerate representation of the open string characters to define the falling D0-brane boundary state.

To summarize, we have shown that a falling D0-brane boundary state in $\mathcal{N} = 1$, 2D superstring theory can be obtained by adapting the falling D0-brane boundary state solution in $\mathcal{N} = 2$ SLFT [112]. In particular, there exist four types of stable, falling D0-branes (two branes and two anti-branes) in Type 0A theory and two types of unstable, falling D0-branes in Type 0B theory. As is well known, Type 0, $\mathcal{N} = 1$,
2D superstring theory has a dual description in the language of matrix models. An interesting question would be to understand these falling D0-branes in the context of the dual matrix model.
Appendix A

Derivation of the Moduli Space

\( \mathcal{M}_{3D} \)

Here we briefly review the derivation of the 3d moduli space \( \mathcal{M}_{3D} \) from the c*-map of the 4D supergravity coupled to \( n_V \) vector-multiplets \[46 [35] [127].

The bosonic part of the action for the \( \mathcal{N} = 2 \) supergravity coupled to \( n_V \) vector-multiplets is:

\[
S = -\frac{1}{16\pi} \int d^4x \sqrt{g^{(4)}} \left[ R - 2g_{ij}dz^i \wedge *_4dz^j - F^I \wedge G_I \right]
\]  
(A.1)

where the ranges of the indices are \( i, j = 1, \ldots, n_V \) and \( I = 0, 1, \ldots, n_V \), and \( G_I = (\text{Re}\mathcal{N})_{IJ}F^J + (\text{Im}\mathcal{N})_{IJ}F^J \). The complex symmetric matrix \( \mathcal{N}_{IJ} \) is defined by

\[
F_I = \mathcal{N}_{IJ}X^J \hspace{1cm} D_iF_I = \overline{\mathcal{N}}_{IJ}D_iX^J.
\]  
(A.2)

For model endowed with a prepotential \( F(X) \),

\[
\mathcal{N}_{IJ} = F_{IJ} + 2i(\text{Im}F \cdot X)_I(\text{Im}F \cdot X)_J X \cdot \text{Im}F \cdot X
\]  
(A.3)

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where $F_{iJ} = \partial_i \partial_J F(X)$.

After reduction on the time-like isometry, the action is $S = -\frac{1}{8\pi} \int dt \int d^3x \mathcal{L}$.

The 3D lagrangian $\mathcal{L}$ has three parts: $\mathcal{L} = \mathcal{L}_{\text{gravity}} + \mathcal{L}_{\text{moduli}} + \mathcal{L}_{\text{e.m.}}$ where

\[
\mathcal{L}_{\text{gravity}} = -\frac{1}{2} \sqrt{g} \ R + dU \wedge \star dU - \frac{1}{4} e^{2U} d\omega \wedge \star d\omega
\]

\[
\mathcal{L}_{\text{moduli}} = g_{i\bar{j}} dz^i \wedge \star dar{z}^j
\]

\[
\mathcal{L}_{\text{e.m.}} = \frac{1}{2} e^{-2U} (ImN)_{IJ} dA_I^0 \wedge \star dA_J^0 + \frac{1}{2} e^{2U} (ImN)_{IJ} (dA^I + A^I_0 d\omega) \wedge \star (dA^J + A^J_0 d\omega) + (ReN)_{IJ} dA_I^0 \wedge (dA^J + A^J_0 d\omega)
\]

The dual scalars for $\omega$ and $A^I$ are defined by:

\[
e^{2U} (ImN)_{IJ} \star (dA^J + A^J_0 d\omega) + (ReN)_{IJ} dA^J_0 = -d\phi_{A^I},
\]

\[
e^{4U} \star d\omega + (A^I_0 d\phi_A - \phi_{A^I} dA^I_0) = -d\phi_{\omega}.
\]

After renaming the variables $\phi_{\omega} \rightarrow \sigma, A^I_0 \rightarrow A^I, \phi_{A^I} \rightarrow B_I$, we obtain the 3d lagrangian in terms of scalars only:

\[
\mathcal{L} = -\frac{1}{2} \sqrt{g} \ R + dU \wedge \star dU + \frac{1}{4} e^{-4U} (d\sigma + A^I dB_I - B_I dA^I) \wedge \star (d\sigma + A^I dB_I - B_I dA^I) + g_{i\bar{j}}(z, \bar{z}) dz^i \wedge \star d\bar{z}^j + \frac{1}{2} e^{-2U} (ImN)_{IJ} dA^I \wedge \star dA^J + \frac{1}{2} e^{-2U} (ImN^{-1})_{IJ} dB_I + (ReN)_{IK} dA^K) \wedge \star (dB_J + (ReN)_{JL} dA^L)
\]

\[
= -\frac{1}{2} \sqrt{g} \ R + g_{mn} \partial_a \phi^m \partial^a \phi^n,
\]

where, as before, $\phi^n$ are the $4(n_V + 1)$ moduli fields: $\phi^n = \{U, z^i, \bar{z}^i, \sigma, A^I, B_I\}$, and $g_{ab}$ is the space time metric, $g_{mn}$ is the moduli space metric. Therefore, the moduli
space $\mathcal{M}_{3D}$ has metric:

$$
\begin{align*}
\text{Appendix A: Derivation of the Moduli Space } \mathcal{M}_{3D} \\
\end{align*}
$$

$$
\begin{align*}
ds^2 &= dU \cdot dU + \frac{1}{4} e^{-4U} (d\sigma + A^I dB_I - B_I dA^I) \cdot (d\sigma + A^I dB_I - B_I dA^I) \\
&\quad + g_{ij}(z, \bar{z}) dz^i \cdot d\bar{z}^j \\
&\quad + \frac{1}{2} e^{-2U} [(\text{Im}N^{-1})^{IJ} (dB_I + N_{IK} dA^K) \cdot (dB_J + \bar{N}_{KL} dA^L)].
\end{align*}
$$

It is a para-quaternionic-Kähler manifold. Since the holonomy is reduced from $SO(4n_V + 4)$ to $Sp(2, \mathbb{R}) \times Sp(2n_V + 2, \mathbb{R})$, the vielbein has two indices $(\alpha, A)$ transforming under $Sp(2, \mathbb{R})$ and $Sp(2n_V + 2, \mathbb{R})$, respectively. The para-quaternionic vielbein is the analytical continuation of the quaternionic vielbein computed in [61]:

$$
V^{\alpha A} = 
\begin{pmatrix}
iu & v \\
e^a & iE^a \\
-\bar{E}^a & \bar{e}^a \\
-\bar{v} & i\bar{u}
\end{pmatrix}.
$$

The 1-forms are defined as

$$
\begin{align*}
u &\equiv e^{K/2-U} X^I (dB_I + N_{IJ} dA^J), \\
e^a &\equiv e_i^a dz^i, \\
E^a &\equiv e_i^a g^{ij} e^{K/2} \bar{D}^j X^I (dB_I + N_{IJ} dA^J), \\
v &\equiv -dU + \frac{i}{2} e^{-2U} (da + A^I dB_I - B_I dA^I),
\end{align*}
$$

where $e_i^a$ is the vielbein of the 4D moduli space, and the bar denotes complex conjugate. The line element is related to the vielbein by

$$
\begin{align*}
ds^2 &= -u \cdot \bar{u} + g_{ab} e^a \cdot \bar{e}^b - g_{ab} E^a \cdot \bar{E}^b + v \cdot \bar{v} = \epsilon_{\alpha\beta} \epsilon_{AB} V^{\alpha A} \otimes V^\beta B
\end{align*}
$$
where \( \epsilon_{\alpha\beta} \) and \( \epsilon_{AB} \) are the anti-symmetric tensors invariant under \( Sp(2, \mathbb{R}) \cong SL(2, \mathbb{R}) \) and \( Sp(2n_v + 2, \mathbb{R}) \).

The isometries of the \( \mathcal{M}^*_3 \) descends from the symmetry of the 4D system. The gauge symmetries in 4D gives the shifting isometries of \( \mathcal{M}^*_3 \):

\[
\begin{align*}
A^I & \rightarrow A^I + \Delta A^I, \\
B_I & \rightarrow B_I + \Delta B_I, \\
\sigma & \rightarrow \sigma + \Delta \sigma + \Delta B_I A^I - \Delta A^I B_I.
\end{align*}
\] (A.10)

The conserved currents and charges are given by (4.93) and the discussion thereafter.
Appendix B

Energy-Momentum Pseudotensor

In this appendix we show that the energy defined from the energy momentum pseudotensor for massive gravitons can be negative. For simplicity we specialize to $\Lambda = 0$: for $\mu \ell \gg 1$ the Compton wavelength of the gravitons is much shorter than the $AdS_3$ radius so the latter can be locally ignored.

Let us first review how the energy-momentum pseudo tensor at quadratic order is defined without the Chern-Simons term or cosmological constant term. The full Einstein equation in the presence of matter is

$$ G_{\mu \nu} = 16\pi G T^M_{\mu \nu}. \tag{B.1} $$

Expanding the metric $g_{\mu \nu} = \eta_{\mu \nu} + h_{\mu \nu}$, we have through quadratic order

$$ G_{\mu \nu}^{(1)} = -G_{\mu \nu}^{(2)} + 16\pi G T^M_{\mu \nu}, \tag{B.2} $$

where $G_{\mu \nu}^{(1)}$ and $G_{\mu \nu}^{(2)}$ are terms linear and quadratic in $h_{\mu \nu}$. The energy momentum pseudo tensor defined as

$$ t_{\mu \nu} = -\frac{1}{16\pi G} G_{\mu \nu}^{(2)}, \quad \text{with} \quad \partial^\mu t_{\mu \nu} = 0 \tag{B.3} $$
serves the Newtonian part of the gravitational potential in the same way that the
matter stress tensor does. When adding the Chern-Simons term, the energy momen-
tum pseudo tensor is similarly defined as

\[ t_{\mu\nu} = -\frac{1}{16\pi G} (G_{(2)}^{\mu\nu} + \frac{1}{\mu} C_{(2)}^{\mu\nu}) \]  

In flat background, the linearized equation of motion \((5.25)\) becomes

\[ \partial_2 h_{\mu\nu} + \frac{1}{\mu} \epsilon_{\mu}^{\alpha\beta} \partial_\alpha \partial_2 h_{\beta\nu} = 0 \]  

under the harmonic plus traceless gauge. For a plane wave solution in the form of

\[ h_{\mu\nu}(k) = \frac{1}{\sqrt{2k_0}} e^{-ik\cdot x} e_{\mu\nu}(k) \], the gauge conditions and the equations of motion are

\[
\begin{align*}
k^\mu e_{\mu\nu}(k) &= 0 \, , \quad e^\mu_{\mu}(k) = 0 \, , \\
k^2 [e_{\mu\nu}(k) - \frac{i}{{\mu}} \epsilon_{\mu}^{\alpha\beta} e_{\beta\nu}(k)] &= 0 .
\end{align*}
\]  

The \(k^2 = 0\) solution is pure gauge. So \(e_{\mu\nu}(k) - \frac{i}{{\mu}} \epsilon_{\mu}^{\alpha\beta} e_{\beta\nu}(k) = 0\), which implies

\((k^2 + \mu^2) e_{\mu\nu}(k) = 0\). When \(k_{\mu} = (\mu, 0, 0)\), the positive energy solution is

\[ h_{\mu\nu} = \frac{e^{-i\mu x}}{\sqrt{2\mu}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & i \\ 0 & i & -1 \end{pmatrix} \]  

It is convenient to define \(e_{\mu}(0) = (0, 1, i)\), such that

\[ e_{\mu\nu}(0) = e_{\mu}(0) e_{\nu}(0) \]  

and

\[ e_{\mu\nu}(\vec{k}) = e_{\mu}(\vec{k}) e_{\nu}(\vec{k}) \]  

where \(e_{\mu}(\vec{k})\) is obtained from \(e_{\mu}(0)\) by a boost. The metric fluctuation can be ex-
panded in Fourier modes,

\[ h_{\mu\nu} = \int d\vec{k}^2 \frac{1}{\sqrt{2k_0}} \{ \alpha(\vec{k}) e_{\mu\nu}(\vec{k}) e^{-ik\cdot x} + \alpha^\dagger(\vec{k}) e_{\mu\nu}^*(\vec{k}) e^{ik\cdot x} \} . \]
To calculate the energy momentum pseudo tensor, we will need the following,

\[ t_{\mu \nu} = -\frac{1}{16\pi G} (G_{\mu \nu}^{(2)} + \frac{1}{\mu} C_{\mu \nu}^{(2)}) , \quad (B.12) \]

\[ G_{\mu \nu}^{(2)} = R_{\mu \nu}^{(2)} - \frac{1}{2} \eta_{\mu \nu} R^{(2)} , \quad (B.13) \]

\[ R_{\mu \nu}^{(2)} = \frac{1}{2} h^{\rho \sigma} \partial_\mu h_{\rho \sigma} - h^{\rho \sigma} \partial_\rho h_{\mu \nu} + \frac{1}{4} (\partial_\mu h_{\rho \sigma}) \partial_\nu h^{\rho \sigma} \]
\[ + (\partial^\rho h^{\sigma \nu}) \partial_\rho h_{\mu \nu} + \frac{1}{2} \partial_\sigma (h^{\rho \sigma} \partial_\rho h_{\mu \nu}) , \quad (B.14) \]

\[ C_{\mu \nu}^{(2)} = \epsilon_\mu^{\alpha \beta} \partial_\alpha (R_{\beta \nu}^{(2)}) - \frac{1}{4} \eta_{\beta \nu} R^{(2)} + h_{\mu \lambda} \epsilon^{\lambda \alpha \beta} \partial_\alpha h_{\beta \nu} - \epsilon_\mu^{\alpha \beta} \Gamma_{\alpha \nu}^{(1)} R_{\beta \lambda}^{(1)} , \quad (B.16) \]

\[ \Gamma_{\nu \alpha}^{(1)} = \frac{1}{2} \eta^{\lambda \rho} (\partial_\alpha h_{\rho \nu} + \partial_\nu h_{\rho \alpha} - \partial_\rho h_{\alpha \nu} ) , \quad (B.17) \]

\[ R_{\mu \nu}^{(1)} = -\frac{1}{2} \partial^2 h_{\mu \nu} . \quad (B.18) \]

Using the equation of motion \((5.34)\), it simplifies to

\[ G_{\mu \nu}^{(2)} = -\frac{1}{4} (\partial_\mu h_{\rho \sigma}) \partial_\nu h^{\rho \sigma} - \frac{\mu^2}{2} h^{\rho \mu} h_{\rho \nu} - \frac{\mu^2}{8} \eta_{\mu \nu} h^{\rho \sigma} h_{\rho \sigma} , \quad (B.19) \]

\[ C_{\mu \nu}^{(2)} = \frac{\mu^2}{4} (3 h^{\mu \nu} h_{\rho \nu} + \epsilon_\mu^{\alpha \beta} h^{\lambda \alpha \beta} h_{\lambda \nu}) \quad (B.20) \]

up to total derivatives. So the energy is

\[ E = \int d^2 \vec{x} t_{00} \]
\[ = \int d^2 \vec{k} \frac{n(\vec{k})}{16\pi G k_0} \left\{ (k_0^2 - \frac{\mu^2}{2} - k_0 \mu) - \frac{1}{2} \mu^2 e_0(\vec{k}) e_0(\vec{k}) \right\} . \quad (B.21) \]

For vanishing spatial momentum \(n(\vec{k}) \propto \delta^2(\vec{k})\), the energy is just \(-\mu\) times a positive normalization factor.
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