The trapped region

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(joint work with Marc Mars, Walter Simon, Jan Metzger)
Outline

1. Marginally outer trapped surfaces
2. Curvature bounds
3. Existence of MOTS
4. Area bound
5. The trapped region
I will discuss some recent results on marginally outer trapped surfaces (MOTS), dynamical horizons and the trapped region.

- MOTS are black hole boundaries in GR
- They are analogues of minimal surfaces in Riemannian geometry
  - Curvature bounds
  - Barriers give existence (Proof uses Jang’s equation)
- Area bounds, “maximum principle” hold for outermost MOTS
  - Application: coalescence of black holes
- The trapped region can be characterized
- Exterior Cauchy problem
Notation

\((M, g, K) \subset L\) Cauchy hypersurface in a 3+1 dimensional Lorentz spacetime. \(\Sigma\) spacelike surface in \(M\).

Declare \(\ell^+\) to be outer

\[ A = \langle \nabla \nu, \cdot \rangle, \quad K^\Sigma = K|_{T\Sigma \times T\Sigma}, \]

\[ \chi^\pm = K^\Sigma \pm A, \]

\[ H = \Sigma \text{tr} A, \quad P = \Sigma \text{tr} K^\Sigma = \text{tr} K - K(\nu, \nu), \quad \theta^\pm = \Sigma \text{tr} \chi^\pm = P \pm H. \]

Definition

\(\Sigma\) is a marginally outer trapped surface (MOTS) if \(\theta^+ = 0\)
MOTS and singularities

- \( \theta^+ \) is the logarithmic variation of area: \( \theta^+ = \delta \ell + \mu \Sigma / \mu \Sigma \)
- \( \Sigma \) MOTS \( \leftrightarrow \) outgoing null rays marginally collapsing
- We call \( \Sigma \) (weakly) outer trapped if \( (\theta^+ \leq 0) \Rightarrow \theta^+ < 0 \).
- Recall NEC: \( G(\nu, \nu) \geq 0 \) for any null vector \( \nu \).

Theorem

**Suppose NEC holds. If a marginally outer trapped surface \( \Sigma \) separates and has noncompact exterior, then \( L \) is null geodesically complete.**

**⇒** MOTS can be viewed as black hole boundaries

The usual definition of trapped surface is \( \theta^+ < 0, \theta^- < 0 \).
A graph in $\mathbb{R}^3$ is minimal iff

$$H[f] := \sum_i D_i \left( \frac{D_i f}{\sqrt{1 + |Df|^2}} \right) = 0$$

but by Raychaudhuri

$$\delta_{\ell^+} \theta^+ = -Wf = -(|\chi^+|^2 + G(\ell^+, \ell^+)f)$$

so $|\Sigma|$ is not an elliptic functional w.r.t. null variations.

However, $Lf = \delta_{\ell^+} \theta^+$ is elliptic.
If there are barriers, there is a minimal $(H=0)$ surface between $H > 0$ and $H < 0$.

Proof by minimization
Persistence of MOTS

- Suppose we have analogue for MOTS of existence in the presence of barriers.
- By Rauchaudhuri, MOTS should persist if NEC holds.

\[ \theta^+ < 0 \quad \text{barrier} \]

Therefore expect MOTS are generically in a marginally outer trapped tube (MOTT) which is spacelike if NEC holds: Dynamical horizon
Trapped region

The trapped region $T$ is the union of all weakly outer trapped domains

$$T = \cup \{ \Omega \subset M : \partial \Omega \text{ is weakly outer trapped} \}$$

Theorem (Andersson & Metzger, 2007)

$\partial T$ is a MOTS, the unique outermost MOTS

Theorem (Galloway, 2008)

Suppose NEC holds. Then the outermost MOTS is an $S^2$
Stability operator

Let \( Lf = \delta f \nu \theta^+ \). Then

\[
Lf = -\Delta f + 2S(\nabla f) + f[\text{div } S - |S|^2 - \frac{1}{2} |\chi^+|^2 + \frac{1}{2} \Sigma \text{Sc} + (\mu - J(\nu))]
\]

where \( S(X) = K(X, \nu) \).

\( L \) is the analogue of the minimal surface stability operator.

Facts:

\begin{itemize}
  \item \( L \) is 2:nd order elliptic, non-self adjoint in general,
  \item \( \exists! \) principal eigenvalue \( \lambda \in \mathbb{R} \), with positive eigenfunction \( \phi \)
  \item \( \Sigma \) locally outermost \( \Rightarrow \lambda \geq 0 \)
  \item \( \lambda \geq 0 \iff \exists f \geq 0 : Lf \geq 0 \Rightarrow \text{if } \lambda \geq 0 \text{ max. principle holds.} \)
\end{itemize}

Definition

\( \Sigma \) is \textbf{stable} if \( \lambda \geq 0 \)

\begin{itemize}
  \item \( \Sigma \) locally outermost \( \Rightarrow \Sigma \) is stable.
\end{itemize}
Local existence of horizons

Theorem (Andersson, Mars, & Simon, 2005)

Suppose $\Sigma$ is strictly stable ($\lambda > 0$). Then $\exists$ MOTT $\mathcal{H}$ containing $\Sigma$. $\mathcal{H}$ is weakly spacelike if NEC holds.

Proof is an application of the implicit function theorem.

This is a **local** result. The outermost MOTS can **jump**.
Bifurcation

As a MOTS is created, the MOTT bifurcates in general.

\[ \theta^+ \leq 0 \]
\[ \theta^+ \geq 0 \]

**Theorem**

*The above picture holds assuming \( W \neq 0 \).*
Theorem (Andersson & Metzger, 2005)

Let $\Sigma$ be a stable MOTS. Then

$$|A| \leq C(|\text{Riem}|_{C^0}, |K|_{C^1}, \text{inj}(M))$$

Proof ingredients:

- Many ideas from (Schoen, Simon, & Yau, 1975).
- Symmetrized stability (Galloway & Schoen, 2006)
- Simons identity
- Kato inequality
- Hoffmann-Spruck Sobolev inequality
- Stampacchia iteration
- Local area bounds (Pogorelov)
Jang’s equation

- Consider $\mathbb{R} \times M$, with metric $ds^2 + g$
- Define $\overline{K}$ by pullback
  Let $\overline{M}$ be the graph of $f$.

- On $\overline{M}$ we have induced mean curvature $\mathcal{H}$ and $\mathcal{P} = \text{tr}_{\overline{M}} \overline{K}$.
- Jang’s equation is $\mathcal{J}[f] := \mathcal{H} - \mathcal{P} = 0$
- Analogue of the equation $\theta^+ = 0$
Jang’s equation

- Translation invariance: $J[f] = J[f + t] \Rightarrow$ stability operator has $L\phi = 0$, with $\phi = \langle \nu, e_4 \rangle$
- $L$ is the analogue of the minimal surface stability operator for $\overline{M}$ (Schoen & Yau, 1981):
  - local curvature bounds ($\Rightarrow$ compactness)
  - existence proof using capillarity deformation & Leray-Schauder
    - $\mathcal{H} - \sigma \mathcal{P} = \tau f_{\sigma, \tau}$
    - deform $\sigma$ from 0 to 1 $\rightarrow f_\tau$
    - let $\tau \downarrow 0$
  - $\rightarrow$ get convergence of subsequence of $f_\tau$ to solution $f$. 
Jang’s equation

Solution has blowups in general. Blowups project to MOTS.

\[ \nu \quad M \quad \theta^+ = 0 \]

Therefore Jang’s equation can be used to prove existence of MOTS. (Andersson & Metzger, 2007): Blowup surfaces are stable MOTS
Existence of MOTS

Theorem (Schoen, 2004; Andersson & Metzger, 2007)

Suppose $M$ is compact with barrier boundaries $\partial^\pm M$, such that

$$\theta^+[\partial^- M] < 0, \quad \theta^+[\partial^+ M] > 0$$

Then $M$ contains a MOTS $\Sigma$

This is the analogue of the barrier argument for existence of minimal surfaces.
Proof makes use of a Dirichlet problem for Jang’s equation.
(Eichmair, 2007) has studied the Plateau problem for MOTS using Picard’s method.
Existence of MOTS

\[ \partial^+ M \]
\[ \theta^+ > 0 \]

\[ \Sigma \]
\[ \theta^+ = 0 \]

\[ \partial^- M \]
\[ \theta^+ < 0 \]

The trapped region
Proof is by solving a sequence of Dirichlet problems for Jang's equation which forces a blowup solution.

\[ \mathcal{J}[f] = 0, \quad f \bigg|_{\partial^{\pm} M} = \mp Z \]

Let \( Z \to \infty \)

Converges to solution with blowups
Existence of MOTS

- Proof uses bending to get $H > 0$ at $\partial M$ (needed to have barriers for gradient control at the boundary)

- Solution must blow up somewhere $\Rightarrow \exists$ MOTS
- We have foliation by barriers near $\partial M$
- $\Rightarrow$ can show the MOTS constructed are in the undeformed region of $M$
- Can allow $\theta^+[\partial^- M] \leq 0$: deform data inside $\partial^- M$
Application: Persistence of MOTS

Theorem (Andersson, Mars, Metzger, & Simon, 2008)

Let $L$ be a spacetime which satisfies NEC. Let $\{M_t\}$, be a Cauchy foliation of $L$, and assume we have outer barriers. If $M_0$ contains a MOTS, then each $M_t$, $t \geq 0$ contains a MOTS.

It seems natural to view the collection of MOTS as the black hole boundary in $L$.

Further regularity and continuation results for MOTT, cf. (Andersson et al., 2008).
Area bound

Theorem (Andersson & Metzger, 2007)

Suppose $M$ has outer barrier. There is a constant $C = C(|\text{Riem}|_{C^0}, |K|_{C^1}, \text{inj}(M), \text{Vol}(M))$ such that for a bounding MOTS $\Sigma$ in $M$, either

$$|\Sigma| \leq C$$

or there is a MOTS $\Sigma'$ outside $\Sigma$.

If $|\Sigma|$ is very large, due to curvature bounds and the bounded $\text{Vol}(M)$, it must nearly meet itself from the outside $\Rightarrow$ outer injectivity radius $i^+(\Sigma)$ must be small.
Glue in a neck with $\theta^+ < 0$. $\theta^+$ heat flow: $\dot{x} = -\theta^+ \nu$ gives $\Sigma_s$, $s \geq 0$.

Maximum principle $\Rightarrow \Sigma_s$ outside $\Sigma$, with $\theta^+ [\Sigma_s] < 0$ for $s > 0$. 
Area bound

- $\Sigma_s$ is an inner barrier $\Rightarrow$ apply existence result
- $\Rightarrow \exists$ MOTS $\Sigma_{\text{new}}$ outside $\Sigma$
- Each step eats up at least $\delta_{\text{Vol}}$ of the outside volume:
  $\text{Vol}_{\text{outside}}(\Sigma_{\text{new}}) \leq \text{Vol}_{\text{outside}}(\Sigma) - \delta_{\text{Vol}}$
  $\Rightarrow$ only finitely many steps
- $\Rightarrow$ eventually get $\Sigma_0$ outside $\Sigma$ with outer injectivity radius $i^+(\Sigma_0) > \delta_*$.
- $\Sigma_0$ has the claimed area bound. To estimate the area, use the estimates on curvature and $i^+$ to estimate the volume of a tube around $\Sigma$ from below (divergence theorem) in terms of $|\Sigma|$.
- This tube must have volume bounded by $\text{Vol}(M)$. 

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Application: Coalescence of black holes

\[ \Sigma_1 \quad \Sigma_2 \]
Application: Coalescence of black holes

By the gluing result: if $\Sigma_1, \Sigma_2$ are sufficiently close, there is a MOTS $\Sigma$ surrounding them.

This gives a “maximum principle for MOTS”.

The usual maximum principle does not apply for MOTS which meet on the outside.
The trapped region

Definition
Let \((M, g, K)\) be an AF data set. The trapped region is

\[ \mathcal{T} = \bigcup_{\Omega \subset M} \{ \partial \Omega \text{ is weakly outer trapped} \} \]

Theorem (Andersson & Metzger, 2007)
If \( \exists \Omega \subset M \), with \( \partial \Omega \) weakly outer trapped, then \( \mathcal{T} \) has smooth boundary \( \partial \mathcal{T} \), with \( \theta^+ [\partial \mathcal{T}] = 0 \). In particular, \( \partial \mathcal{T} \) is the unique outermost MOTS in \( M \).
Replace $\mathbb{T}$ by

$$\mathcal{T} = \bigcup_\Omega \{ \theta^+ [\partial \Omega] \leq 0, \quad \text{and} \quad i^+ (\partial \Omega) \geq \delta_* \}$$

For the collection of subsets defining $\mathcal{T}$ we have compactness $\Rightarrow \partial \mathcal{T} = \Sigma$ is a MOTS.

**Lemma**

$\mathbb{T} \subset \mathcal{T}$. 
To see this, suppose $\exists$ WOT $\Omega \notin T$. We can argue that this means $\Omega \cap T \neq 0$.

Smoothing gives a barrier and hence there is a MOTS outside. This can be taken to be in $T$. So $T = T$.
MOTS and MITS

We may have weakly inner trapped $S$ in the region outside the outermost MOTS $\Sigma_{\text{out}}$. 

\[
\Sigma_{\text{out}} \quad \Omega_{\text{out}} \quad \theta^- \leq 0 \quad |x| = R
\]
MOTS and MITS

Then \( \exists \) outermost MITS \( S_{\text{out}} \) in \( \Omega_{\text{out}} \)

By (Galloway, 2008), \( \Sigma_{\text{out}}, S_{\text{out}} \sim S^2 \).

- There is a solution to Jang’s equation which blows up precisely at \( \Sigma_{\text{out}} \) and \( S_{\text{out}} \) and \( \to 0 \) at \( |x| = \infty \).
- This result can be used to complete the proof of the positive mass theorem (Schoen & Yau, 1981) without deforming the Cauchy data.
Exterior Cauchy problem

Suppose NEC holds + suitable matter equation.

![Diagram showing \( \Sigma = \partial \mathbb{T} \) and \( M_t \)]

- Expect: local well-posedness for the exterior IVP, due to curvature bounds for \( \partial \mathbb{T} \)
- Potentially interesting for black hole evolutions and stability of Kerr.
Concluding remarks

- Bray proposed Generalized Apparent Horizons: $H = |P|$ and a generalized Jang’s equation, as part of an approach to the general Penrose Inequality. (Eichmair, 2008) proved existence of outermost GAH. These are outer minimizing. However, not clear how they are related to black holes.

- Large families of GAH conditions can be treated using the techniques discussed here.

- Global properties of MOTT may be relevant for understanding the strong field Cauchy problem for the Einstein equations.

- The known conditions for existence of MOTS in a Cauchy data set (Schoen & Yau, 1983; Yau, 2001; Galloway & O’Murchadha, 2008) involve nonvacuum data.

- A better understanding of conditions for the existence of MOTS in vacuum, due to concentration of curvature terms of, say, curvature radii, conformal spectral gap, etc. is needed.
References I


