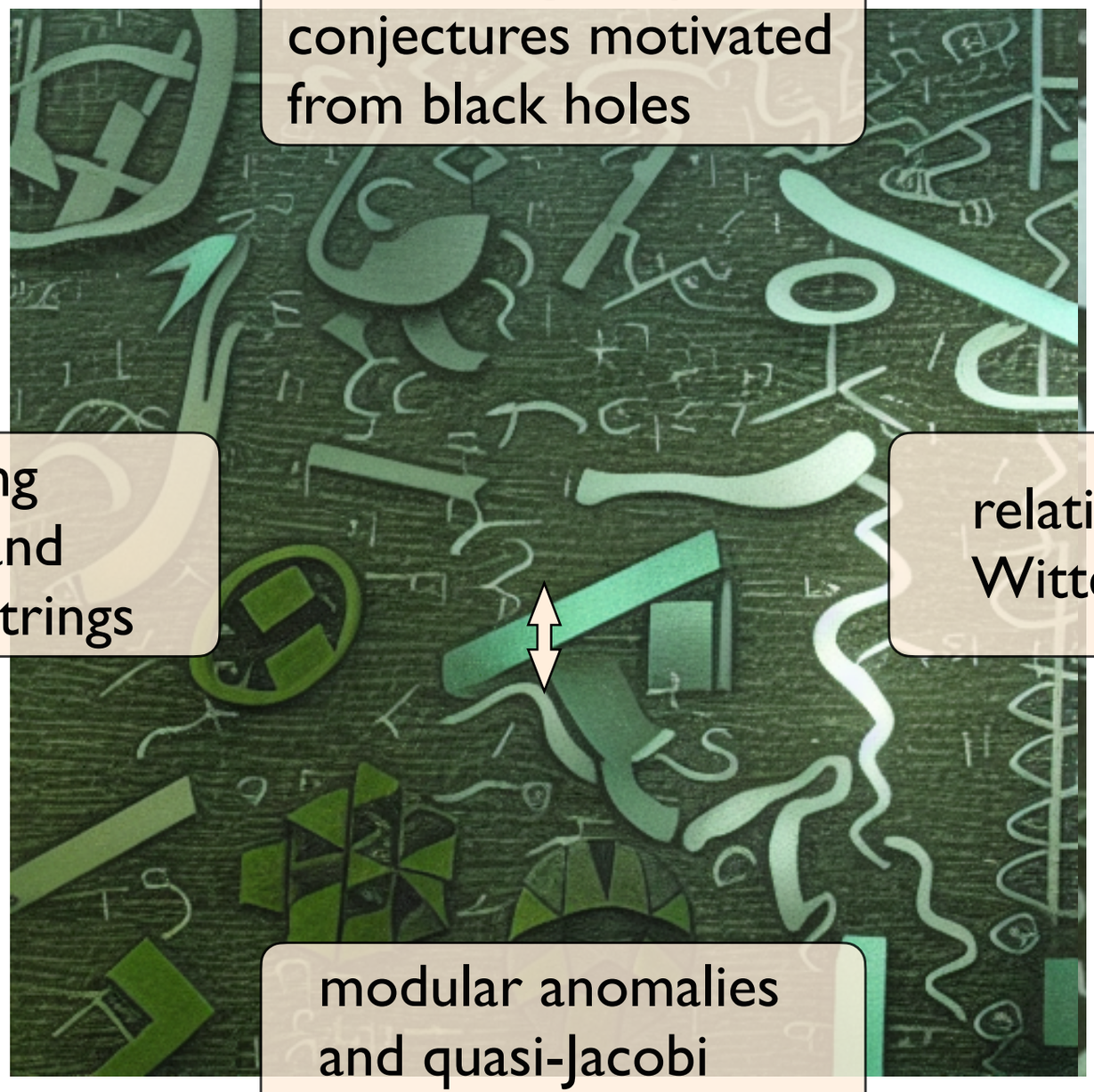
Some Math underlying the Stringy Landscape

W.Lerche, HermannFest, 9-2022

Quantum gravity conjectures motivated from black holes



degenerating geometry and emergent strings

relative Gromow-Witten invariants

modular anomalies and quasi-Jacobi Forms

DALL-E

Swampland Philosophy

V '05, OV '06,
AH-MNV '07+….

- The important role of math for string consistency has been recognized since long.

  However there seems a universal Uncertainty Principle at work:
  The cleaner and more precise a statement is, the less generally valid and useful it tends to be.

  One can turn this around and see how much one can gain by dropping rigour while widening the physical scope from "bottom up", by asking

- Which low energy theories that naively look perfectly alright can be consistently coupled to gravity? What can possibly go wrong?

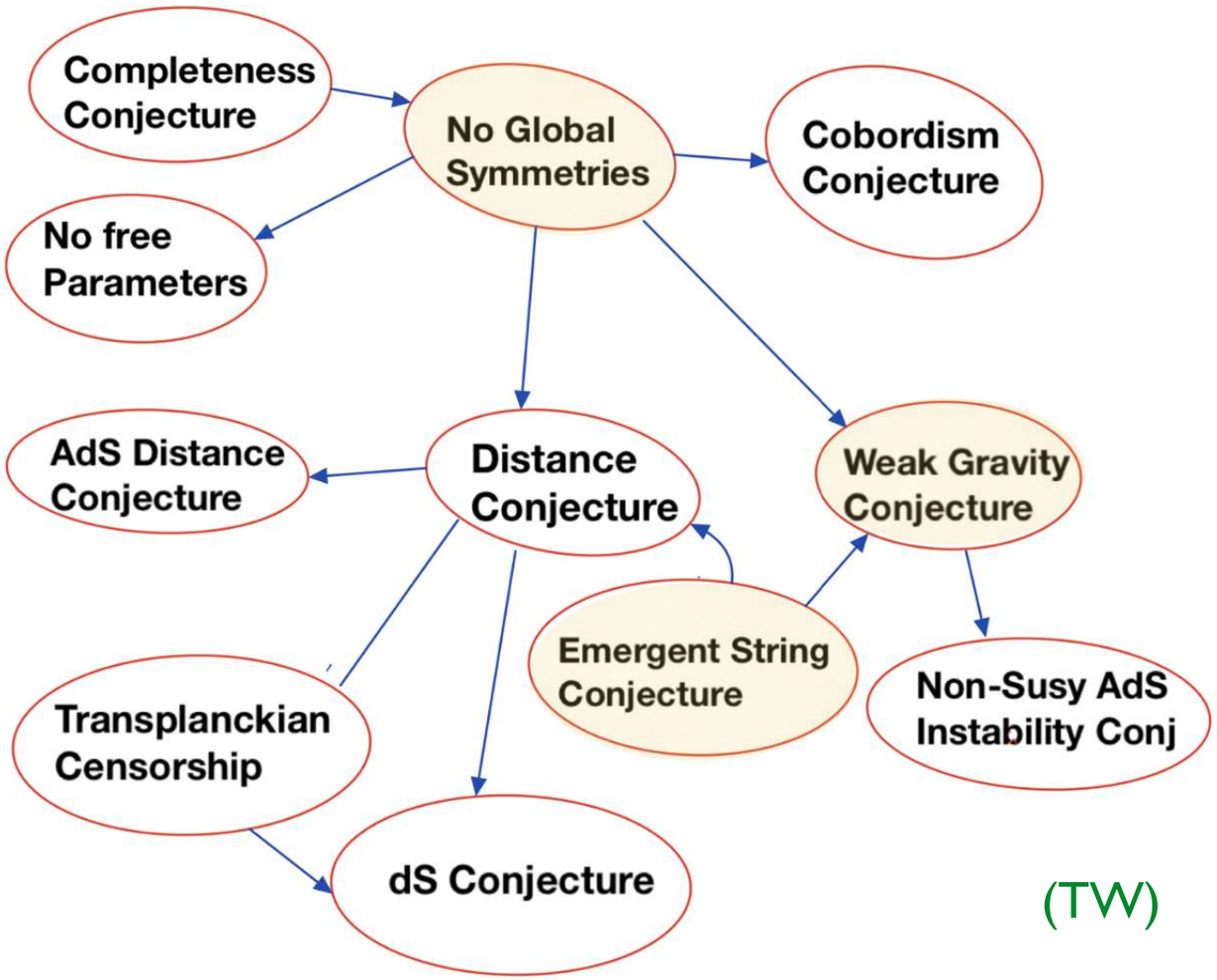
  Those theories define the "landscape", while the bad ones lie in the "Swampland".

  We don't mean standard field theoretical consistency here such as UV finiteness or anomalies, but rather heuristic principles based on subtle phenomena that are invisible in standard QFT, such as the decay of black holes.

- This has let to a web of mostly weak results or conjectures, which gains strength through their interplay.

Web of Conjectures: Sum is more than its Parts

Review: Palti '20



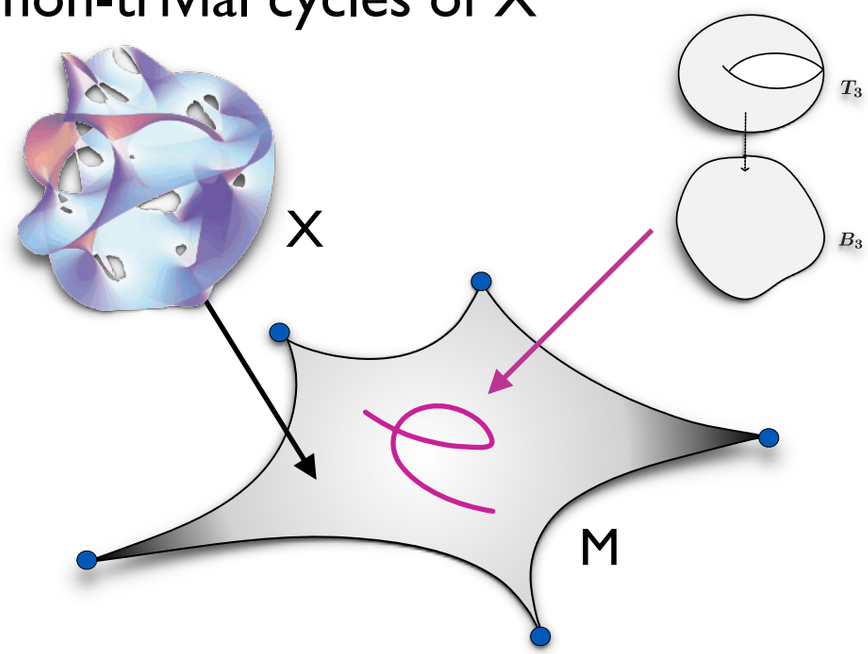
(TW)

Which underlying mathematical properties guarantee, if any, that the theories we consider do actually satisfy the various consistency requirements?

Consistency across parameter space M

- Consider string compactification on some Calabi-Yau space, X.
  Effective action depends on shape and size parameters, the moduli (=VEVs of scalar fields), t.

Physical states arise from strings and branes wrapped around non-trivial cycles of X



Massless particle states or tensionless strings are associated with singularities of M

Finite distance singularities: gravity decouples

Infinite distance (cusp) singularities: gravity stays

- There are various weak gravity conjectures (WGC) concerning the limiting behavior near large distance singularities, eg. for vanishing gauge couplings.

We focus on two Swampland Conjectures:

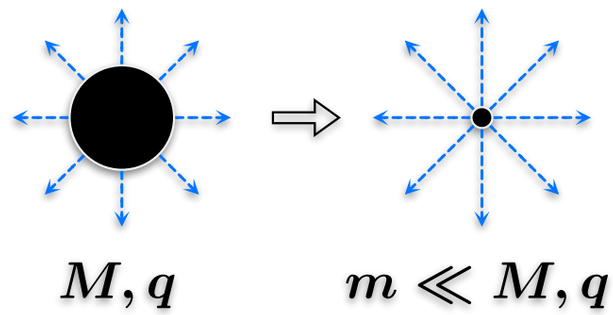
- No global symmetries in QG

  BD '88, BS '11,…, HO '18

  A black hole, highly charged under a global symmetry, cannot loose charge q under Hawking radiation:

  ⇒ either clashes with entropy formula $S = \frac{\text{Area}}{4G}$

  ⇒ or leads to highly charged remnants



What happens if we tune gauge couplings to zero? ⇒ degeneration geometry of CY manifolds

- Gravity as weakest force (WGC)

  Certain superextremal states must exist into which extremal black holes can decay

  AH-MNV '07

  ⇒ GW invariants, modular properties of elliptic genera, quasi-Jacobi forms

Geometry of global symmetries

- Dogma: there can't be global symmetries in a EFT coupled to quantum gravity

  ⇨ What happens if we dial appropriate parameters to go to vanishing gauge coupling while not decoupling gravity?

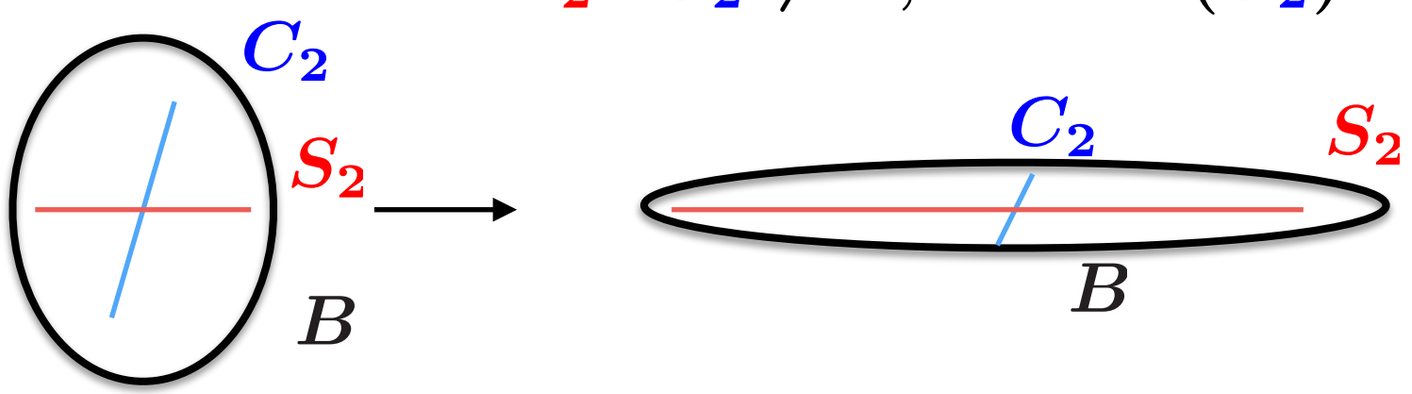
- Relevant piece of local geometry: 4d submanifold B

  want to keep gravity: $M_{pl}{}^4 \sim \mathrm{Vol}(B) = \mathrm{const}$

  weak coupling: $\dfrac{1}{g^2} \sim \mathrm{Vol}(S_2) = t \to \infty$

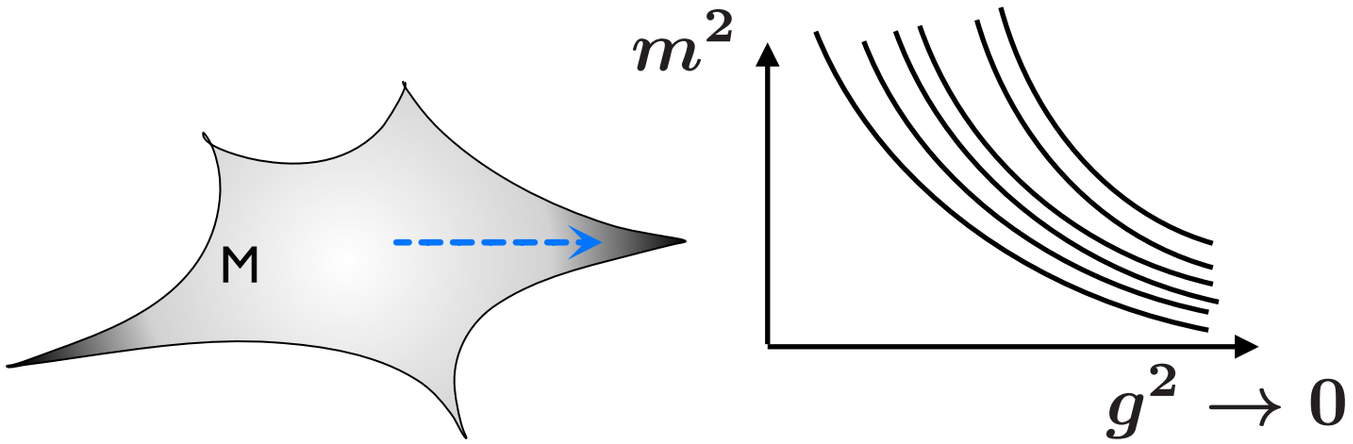
Theorem: there exists a unique rational curve C_2 with

$$S_2 \cdot C_2 \neq 0, \qquad \mathrm{Vol}(C_2) = \frac{1}{t} \to 0$$



Infinite Distance Limits

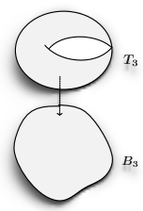
- Upshot: vanishing gauge coupling implies that a solitonic string becomes tensionless… so the effective field theory approximation breaks down!



Infinite tower of charged particles with exponentially vanishing mass gap, as posited by the Distance Conjecture

OV '06,
AH-MNV '07 + many

- While this picture seems naive, there are powerful mathematical theorems behind the possible large distance degenerations of Calabi-Yau spaces that guarantee this outcome: "theory protects itself".



Complex structure moduli: degeneration of Hodge structures
Kähler moduli: fiber structure theorems of Oguiso

GPV '18, CGV '18,
GRH '19, G,M…

LLW '19,

Emergent String Conjecture

LLW '19

- Can we possibly run into surprises?

  No: tensionless higher dimensional branes, or multiple strings, or potential so far unknown weakly coupled theories of quantum gravity do never appear as dominant d.o.f!

ESC: If a quantum gravity theory admits an infinite distance limit, then

- either it reduces to a weakly coupled (heterotic or Type II) string theory
  $\Rightarrow$ infinite tower of nearly massless string excitations
- or it decompactifies $\Rightarrow$ infinite tower of n.m. Kaluza-Klein excitations

Completeness of known consistent theories with gravity (@susy)

- Confirmed for many highly non-trivial (non-perturbative) toy models:

Existence and uniqueness of emergent critical string $\Longleftrightarrow$ Quantum geometry of string compactification

Weak Gravity Conjectures

AH-MNV '07 + many

- We have seen that with nearly vanishing gauge coupling, a tower of states becomes arbitrarily light.

  This is related to the

  - Magnetic WGC
    There exists a scale in the theory bounded from above:

$$\frac{\Lambda}{M_{\rm pl}} \lesssim g$$

  There is another variant, the

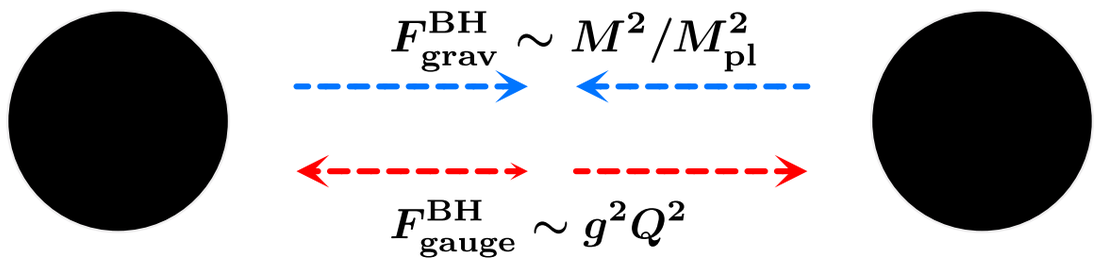
  - Electric WGC
    There exists a "light"particle in the theory for which:

$$\frac{m}{M_{\rm pl}} \leq \text{const.}\, g\, q$$

WGC: Gravity is Weakest Force

AH-MNV '07 + many

- Heuristic Argument:
  Charged black holes should be able to decay, at least for small g
  (otherwise infinitely many charged remnants with small mass)



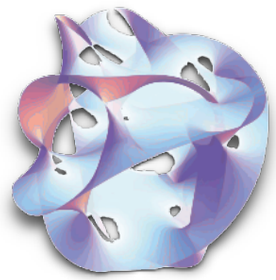
⇨ "Superextremal" states must exist into which extremal black holes can decay, reducing their charge/mass ratio

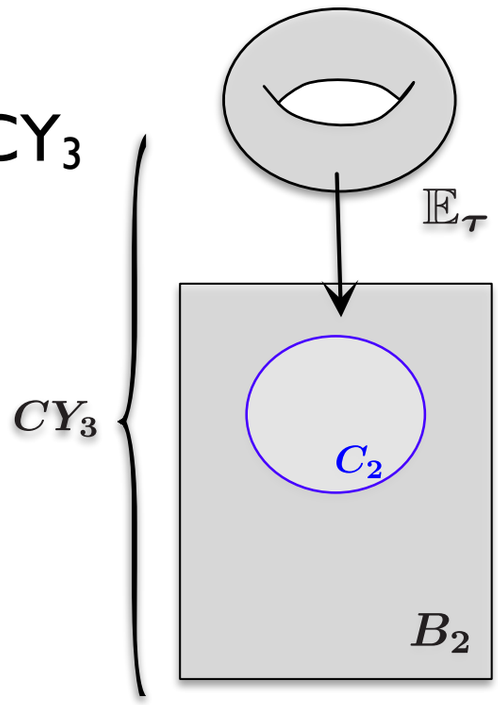
$$\left.\frac{q^2}{m^2}\right|_{\rm particle} \overset{!}{>} \left.\frac{Q^2}{M^2}\right|_{\substack{\rm extr\\ \rm BH}}$$

⇨ How to prove generic existence for a large class of string models ?
Stringy geometry

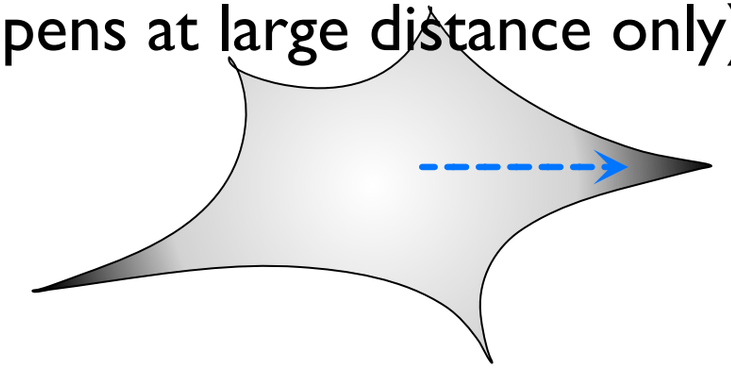
Non-perturbative Elliptic Genera from Geometry

F-theory
on elliptic CY_3





D3-brane wrapped around
2-cycle C_2 yields a solitonic
heterotic string in 6d, if $C_2 . C_2 = 0$
(happens at large distance only)



What is its charge/mass spectrum?

Quantum algebraic geometry
(mirror symmetry) allows to
count curves in CY_3

Relevant are "Gromow-Witten" invariants $N_{k,l}$
relative to C_{2s} with partition function

$$\mathcal{F}_{C_2}(q,\xi) = \sum N_{k,\ell} q^k \xi^\ell$$

Magic of string duality:
this yields the non-perturbative elliptic genus (=stringy index) of the
solitonic heterotic string $\mathcal{F}_{C_2} = \mathrm{Tr}_{het}(-1)^F q^H \xi^J$

HMVV '16, ….

Elliptic genus of heterotic strings

- Elliptic genus = loop space index = RR partition function SW '86, W '87

  Encodes protected, largely deformation invariant sub-sector of spectrum

- Consider 2-dim (0,2)-sigma model, refined by a single left-moving U(1) charge Q

$$Z^{ell}(\tau, z) = \mathrm{Tr}_{RR}\left[(-1)^{F_R} F_R^{1/2(d-2)} q^{L_0} \bar{q}^{\bar{L}_0} \xi^Q\right]$$

- Z^{ell} is naively a meromorphic function of $q \equiv e^{2\pi i \tau}, \quad \xi \equiv e^{2\pi i z}$ where τ is the complex structure of toroidal world-sheet, and z the U(1) field strength

  Generic expansion: $Z^{ell}(\tau, z) = -q^{E_0} \sum_{n \geq 0, r} N(n, r) q^n \xi^r$

  n= excitation level,
  r= U(1) charge ,
  N(n,r)= BPS degeneracies

Elliptic genus as Jacobi form

- Ell. genera as partition functions are expected to behave well under modular transf.

  When refined by an extra U(1), they should be "Jacobi forms" KYY '93, EZ'95, DMZ '12, … SW,W…

$$Z^{ell}(\tau, z) \;=\; \varphi_{w,m}(\tau, z)$$

Defining properties: modularity and double periodicity

$$\varphi_{w,m}\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = \qquad (c\tau+d)^w e^{2\pi i \frac{m\,c}{c\tau+d} z^2} \varphi_{w,m}(\tau, z)\,,$$

$$\varphi_{w,m}\left(\tau, z+\lambda\tau+\mu\right) = \quad e^{-2\pi i m(z^2\tau+2\lambda z))} \varphi_{w,m}(\tau, z) \quad \lambda, \mu \in \mathbb{Z}$$

Het. strings: modular weight $w = 1 - d/2$ and index(*) $m = 1/2 C_0 \cdot b \in \mathbb{N}$

- Ring of relevant Jacobi forms with given w and m is finitely generated:

$$\mathcal{R}^{Jac} \;=\; \mathbb{Q}\Big[E_4, E_6, \varphi_{-2,1}, \varphi_{-1,2}, \varphi_{0,1}\Big]$$ (standard defs; see lit.)

⇒

Need to determine just a finite number of parameters to find exact ell genus!

(*)b="height pairing": geom. definition of U(1)

Examples:

- F-theory on CY_3 on elliptic fibration over Hirzebruch surface F1 with extra U(1). Shrinking C_2 leads to emergent heterotic string on K3 surface (index m=2):

$$\mathcal{F}_{C_2}(\tau,z) = \frac{q}{\eta(\tau)^{24}}\left(-\frac{1}{72}E_4(\tau)^2E_6(\tau)\varphi_{-2,1}(\tau,z)^2+\frac{7}{432}E_4(\tau)^3\varphi_{-2,1}(\tau,z)\varphi_{0,1}(\tau,z)\right.$$
$$\left.+\frac{5}{432}E_6^2(\tau)\varphi_{-2,1}(\tau,z)\varphi_{0,1}(\tau,z)-\frac{1}{72}E_4(\tau)E_6(\tau)\varphi_{0,1}^2(\tau,z)\right)$$

- F-theory on CY_3 on elliptic fibration over the del Pezzo surface dP_2 with extra U(1).

  Again leads to an emergent dual heterotic string on K3 surface,
  however without a perturbative world-sheet description (NS5 brane defect):

$$\mathcal{F}_{C_2}(\tau,z) = \frac{q}{\eta^{24}}\Big(-\frac{23}{1728}E_4^2E_6\varphi_{-2,1}^2+\frac{1}{64}E_4^3\varphi_{-2,1}\varphi_{0,1}+\frac{19}{1728}E_6^2\varphi_{-2,1}\varphi_{0,1}-\frac{23}{1728}E_4E_6\varphi_{0,1}^2$$
$$+E_2\Big(-\frac{1}{1728}E_6^2\varphi_{-2,1}^2+\frac{1}{864}E_4E_6\varphi_{-2,1}\varphi_{0,1}-\frac{1}{1728}E_4^2\varphi_{0,1}^2\Big)\Big)$$

quasi-modular

Spectral flow property of Jacobi forms

- Ell genus is a Jacobi form…as such it has automatically a theta-expansion:

$$\varphi_{w,m}(\tau,z) = \sum_{\ell\in\mathbb{Z}/2m\mathbb{Z}} h_\ell(\tau)\,\theta_{m,\ell}(\tau,z) \qquad \text{EZ '95, DMZ'14}$$

$$= \sum_{n\geq 0}\sum_{r^2\leq 4mn} c(n,r) q^n \xi^r$$

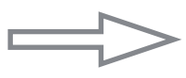
Theta-fct = partition function of a free 2d boson, with built-in relation between charge and excitation numbers:

$$\theta_{m,\ell}(\tau,z) = \sum_n q^{(\ell+2mn)^2/4m}\xi^{\ell+2mn}$$

States fall in "spectral flow" orbits characterized by discriminant: $\Delta = 4mn - r^2$

$$c(n,r) = C(\Delta, r)$$

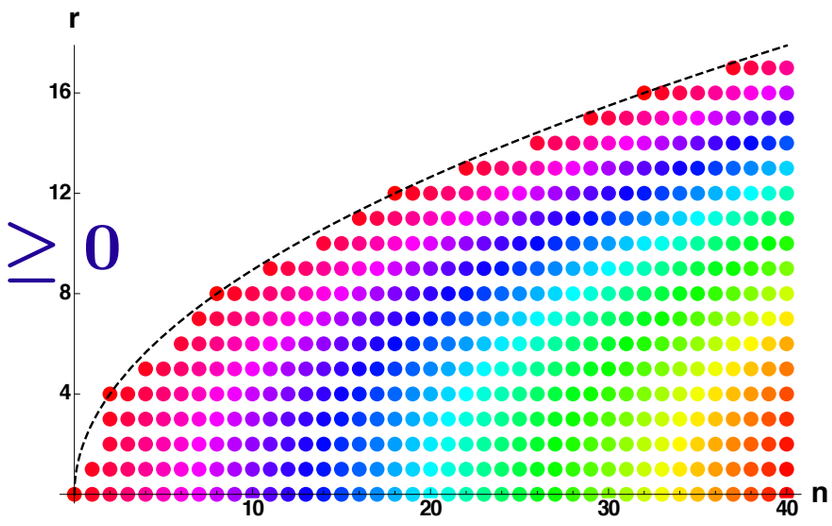
- Distinguish:



Holomorphic Jacobi forms: $c(n,r) = 0$ unless $\Delta \geq 0$

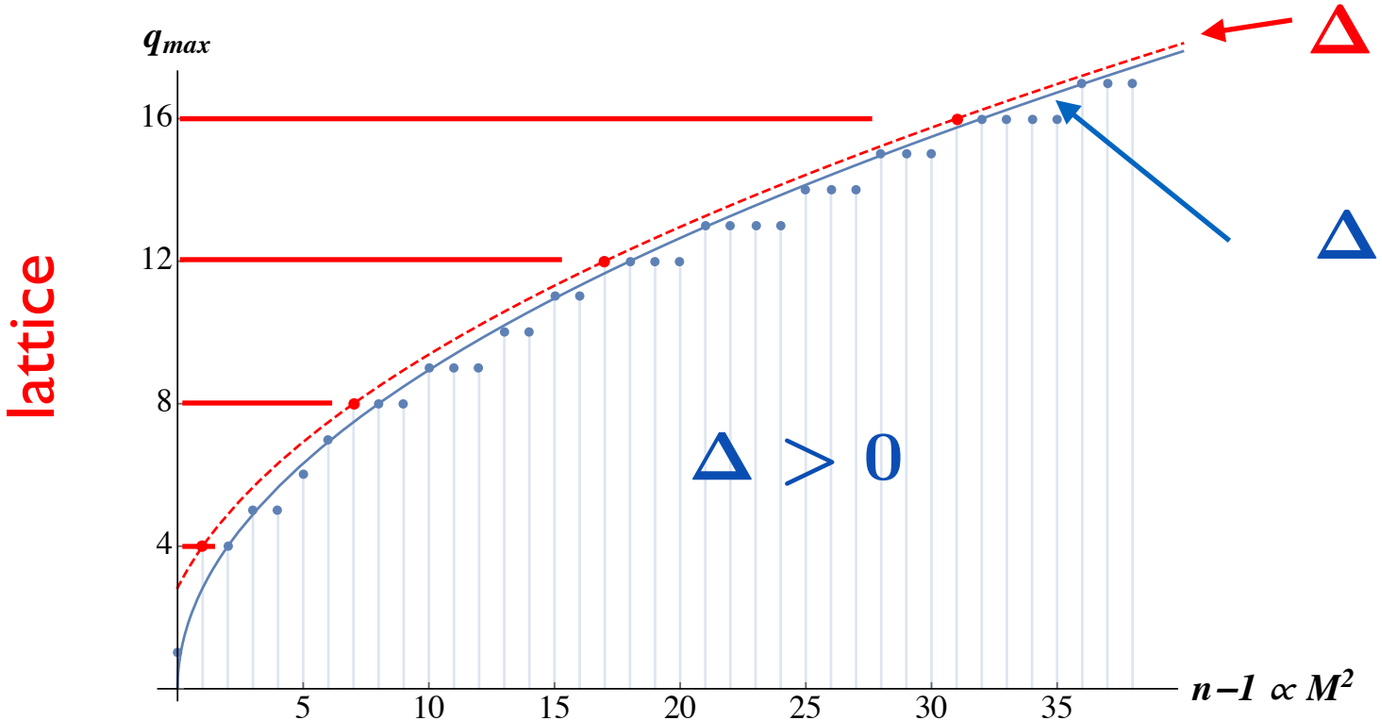
Weak Jacobi forms: $c(n,r) = 0$ unless $n \geq 0$

Jacobi cusp forms: $c(n,r) = 0$ unless $\Delta > 0$



Superextremality from Jacobi Forms

- The Jacobi property implies the following structure of the U(1) charge/mass spectrum encoded in the elliptic genus (here index m=2):



$\Delta = 0$ Maximal superextremal states, no smooth horizon BH

$\Delta = 4mn$ Extremal states, asymptote to black holes

$$\Delta \equiv 4mn - q^2$$

Thus not only superextremal states appear as required by the WGC, but also they lie on a charge lattice as postulated by the refined "lattice WGC".

It is the mathematics of elliptic fibrations and Jacobi forms which makes it work as expected from quantum gravity!

(but watch quantum corr)

LLW

Summary

- Heuristic consistency principles, while not rigorously defined, gain strength through their interrelationships and highly non-trivial verification within toy models.

- In this talk I have presented examples for mathematical structures underlying subtle physical principles, like avoiding pathologies such as highly charged remnants of decaying black holes.

- Most are string-inspired and the question about their general validity/significance remains (supersymmetry, existence of moduli spaces). At least they give confidence that string theory really is a fully consistent framework.

- The hope is that some principles are universally valid (eg no global symmetries in QG)