AEI '22

Nicolai rules in supersymmetry – a legacy from the 1980s

- based on 2005.12324, 2104.00012, 2104.09654, 2109.00346, 2111.13223, 2204.02094, 2207.09471, 2208.06420

  and works from the early 1980s (!)

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- Content

  ① A question raised in 1980 and answered until 1984 (?)

  ② A more universal answer in 2021

  ③ The case of gauge theories

  ④ Summary

① A question raised in 1980 (by Hermann) and answered until 1984 (but not fully...)

- example: Wess-Zumino model in $\mathbb{R}^{1,3}$ (ϕ, ψ, F)

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi + F^* F + \frac{i}{2} \bar\psi \bar\sigma \cdot \partial \psi - \frac{i}{2} \psi \sigma \cdot \partial \bar\psi$$
$$+ W'(\phi) F + W'(\phi)^* F^* - \frac{1}{2} \psi W''(\phi) \psi - \frac{1}{2} \bar\psi W''(\phi)^* \bar\psi$$

integrate out auxiliary (F, F^*) $\rightsquigarrow$ $F^* = -W'$

$$\mathcal{L} = |\partial\phi|^2 - |W'|^2 + \left(\frac{i}{2} \bar\psi \bar\sigma \cdot \partial \psi - \frac{1}{2} \psi W'' \psi + \text{h.c.} \right)$$

integrate out fermions $(\psi, \bar\psi)$ $\rightsquigarrow$ $\det M = e^{\frac{i}{\hbar} \cdot (-i\hbar \operatorname{tr} \ln M)}$

$$S_g[\phi] = \int d^4x \, \{ |\partial\phi|^2 - |W'|^2 \} - i\hbar \operatorname{tr} \ln \begin{pmatrix} W'' & i\sigma\cdot\partial \\ i\bar\sigma\cdot\partial & W''^* \end{pmatrix}$$
$$=: S_g^b[\phi] + \hbar S_g^f[\phi]$$

g = coupling constant(s) inside superpotential $W[\phi]$

$$\langle Y[\phi] \rangle_g := \int \mathcal{D}\phi \; e^{\frac{i}{\hbar} S_g[\phi]} \, Y[\phi] \quad , \quad \langle \mathbb{1} \rangle_g = 1$$

- Q: What trace remains of SUSY? $S_g^b \leftrightarrow S_g^f$?

A [Nicolai 1980]:

$\exists$ (in general nonlocal & nonlinear) map $T_g : \phi \longmapsto \phi'[\phi; g]$

such that

$$\langle Y[\phi] \rangle_g = \langle Y[T_g^{-1}\phi] \rangle_0 \qquad \forall Y \qquad (1)$$

$\rightarrow$ relates interacting to free theory ($g=0$)

- equivalently (RHS: substitute $\phi \to T_g\phi$):

$$\mathcal{D}\phi \; e^{\frac{i}{\hbar} S_g[\phi]} = \mathcal{D}(T_g\phi) \; e^{\frac{i}{\hbar} S_0[T_g\phi]}$$

$$\mathcal{D}\phi \; e^{\frac{i}{\hbar} S_g^b[\phi] + i S_g^f[\phi]} = \mathcal{D}\phi \; e^{\frac{i}{\hbar} S_0[T_g\phi] + \operatorname{tr}\ln\left(\frac{\delta T_g\phi}{\delta\phi}\right)}$$

separate powers in $\hbar$:

$$S_0^b[T_g\phi] = S_g^b[\phi] \qquad \text{"free action condition"} \qquad (2a)$$

$$\underbrace{S_0^f[T_g\phi]}_{\text{constant}} - i \underbrace{\operatorname{tr}\ln\left(\frac{\delta T_g\phi}{\delta\phi}\right)}_{\text{Jacobian}} = \underbrace{S_g^f[\phi]}_{\text{MSS determinant}} \qquad \text{"determinant matching"} \qquad (2b)$$

- an alternative characterization of SUSY
- "supersymmetry without fermions"
- 1979: the map for scalar theories, existence proof (incomplete)

  1980: refinement of the proof, but a gap remains (Golterman 1982)

  1980: extension to gauge theories, construction to $\mathcal{O}(g^2)$ [$\mathcal{O}(g^3)$ in 2020 ! $\mathcal{O}(g^4)$ in 2021]

  1982: examples of <u>lin</u>ear maps in $D \leq 2$, zero-mode obstruction for SYM $\leftrightarrow$ Witten index

  1984: linear maps for $D = 4$ & 6 SYM, doubtful due to "Euclidean light-cone gauge"

  1983/84: constructive (perturbative) proof via coupling flow, requires off-shell SUSY

- infinitesimal version [Lechtenfeld 1984]

$\partial_g (1) \rightsquigarrow$

$$\partial_g \langle Y[\phi] \rangle_g \overset{(1)}{=} \partial_g \langle Y[T_g^{-1}\phi] \rangle_0$$

$$\overset{CR}{=} \langle \partial_g Y[\phi] \rangle_g + \langle \int (\partial_g T_g^{-1}\phi) \cdot \frac{\delta Y}{\delta \phi}[T_g^{-1}\phi] \rangle_0$$

$$\overset{(1)^{-1}}{=} \langle \partial_g Y[\phi] \rangle_g + \langle \int (\partial_g T_g^{-1} \circ T_g)\phi \cdot \frac{\delta Y}{\delta \phi}[\phi] \rangle_g$$

$$=: \langle (\partial_g + R_g[\phi])\, Y[\phi] \rangle_g \qquad (3)$$

with a "coupling flow operator"

$$R_g[\phi] = \int dx \; (\partial_g T_g^{-1} \circ T_g)\phi(x) \frac{\delta}{\delta \phi(x)} \qquad (4)$$

a functional differential operator derived from T_g

- reverse logic: somehow find $R_g \longrightarrow$ construct T_g

$$(T_g^{-1}\phi)(x) = e^{g(\partial_{g'} + R_{g'}[\phi])}\, \phi(x) \Big|_{g'=0} \xrightarrow{\text{invert}} T_g \phi$$

... but see below

- When does R_g exist and how to find it?

restrict to scalar theories (gauge theories in a moment)

if SUSY is realized off-shell then $\exists$ functional $\Delta_\alpha[\phi,\psi]$ such that

$$\partial_g S_{SUSY}[\phi,\psi] = \delta_\alpha \Delta_\alpha[\phi,\psi] \qquad (5)$$

for SUSY transformation δ_α, α = Majorana spinor index

ex. WZ model: $\Delta_\alpha = \frac{1}{2}\int d^4x\, \psi_\alpha\, \partial_g W'(\phi)$

- construction of R_g: uses SUSY Ward identity

$$\partial_g \int\mathcal{D}\phi\int\mathcal{D}\psi\, e^{iS_{SUSY}}\, Y[\phi] = \int\mathcal{D}\phi\int\mathcal{D}\psi\, e^{iS_{SUSY}}\left(\partial_g + i\Delta_\alpha\delta_\alpha\right)Y[\phi]$$

integrate out femions contracts bilinears $\psi\psi$ (femion propagator)

$$R_g[\phi] = i\int dx\, \Delta_\alpha[\phi]\; \delta_\alpha\phi(x)\, \frac{\delta}{\delta\phi(x)} \qquad (6)$$

ex. WZ model for $W = \frac{1}{2}m\phi^2 + \frac{1}{3}g\phi^3 \rightsquigarrow \partial_g W' = \phi^2$, $W'' = m + 2g\phi$

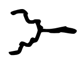
$$R_g[\phi] \simeq \frac{i}{2}\iint d^4x\, d^4y\, \Big(\phi^2(x)\,\psi(x)\,\psi(y) + \phi^{*2}(x)\,\bar\psi(x)\,\psi(y)\Big)_{\alpha\alpha} \frac{\delta}{\delta\phi(y)} + \text{h.c.}$$

spin trace

- "Nicolai rules" [Flume, Lechtenfeld 1983]

$R_g[\phi] \triangleq$  simplified: $(\phi, \phi^*) \sim \phi$

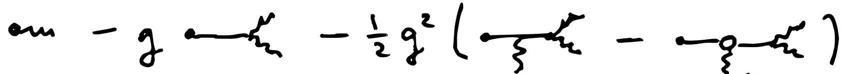
$=$  $+ g$  $+ g^2$  $+ \dots$

where  $\triangleq \phi^2$  $\triangleq (i\partial\!\!\!/ + m + 2g\phi)^{-1}$

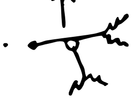
 $\triangleq 2\phi$  $\triangleq (i\partial\!\!\!/ + m)^{-1}$ $\bullet \triangleq \int d^4x$

 $\triangleq \delta/\delta\phi$  $\triangleq \sum_\alpha (\dots)_{\alpha\alpha}$ ← spin trace

linear tree for R_g exponentiates to branched tree for T_g

$T_g \phi \triangleq$  $+ \mathcal{O}(g^3)$

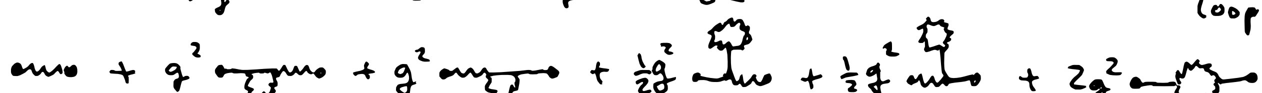
one, two spin traces

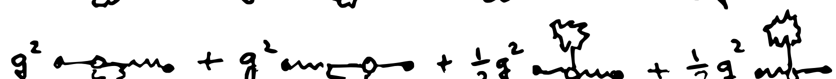
e.g. 

- correlator example:

$\langle \phi(x)\, \phi^*(y) \rangle_g = \langle T_g^{-1}\phi(x)\, T_g^{-1}\phi(y)^* \rangle_0$ ← contract free ϕ's

no fermion loops

$=$ 

better UV  $+ \mathcal{O}(g^4)$

② A more universal answer in 2021

(a) R_g is a derivation $\Longleftrightarrow$ T_g^{-1} acts distributively :

$$R_g Y[\phi] = \int \frac{\delta Y}{\delta \phi} \cdot R_g \phi \quad \Longleftrightarrow \quad T_g^{-1} Y[\phi] = Y[T_g^{-1}\phi]$$

(b) move the map to the other side :

$$(1) \Leftrightarrow \quad \langle Y[\phi] \rangle_0 = \langle Y[T_g \phi] \rangle_g \qquad \text{choose } \partial_g Y = 0$$

$$\partial_g \curvearrowright \quad 0 = \partial_g \langle Y[T_g \phi] \rangle_g =: \partial_g \langle Y_g'[\phi] \rangle_g$$

$$\overset{(3)}{=} \langle (\partial_g + R_g[\phi]) \, Y[T_g \phi] \rangle_g$$

$$\overset{CR}{=} \left\langle \int (\partial_g + R_g[\phi]) \, T_g \phi(x) \cdot \frac{\delta Y}{\delta \phi(x)} [T_g \phi] \right\rangle_g \quad \forall Y$$

$$\Rightarrow \quad (\partial_g + R_g[\phi]) \, T_g \phi(x) = 0 \qquad (7) \qquad \text{"fix point"}$$

this allows one to directly construct $T_g \phi$ from R_g !

was done in 1986 but only perturbatively, until 2021...

- general solution to $(\partial_g + R_g[\phi])\, T_g\phi(x) = 0$:

[Lechtenfeld, Rupprecht 2021]

$$T_g\phi = \mathcal{P}\, e^{-\int_0^g dg'\, R_{g'}[\phi]} \cdot \phi \qquad (8)$$

path-ordered exponential

$$= \sum_{s=0}^{\infty} (-1)^s \int_0^g dg_s \cdots \int_0^{g_3} dg_2 \int_0^{g_2} dg_1\; R_{g_s}[\phi] \cdots R_{g_2}[\phi]\, R_{g_1}[\phi] \cdot \phi$$

- expansion in powers of g

$$R_g[\phi] = \sum_{k=1}^{\infty} g^{k-1}\, r_k[\phi] = r_1[\phi] + g\, r_2[\phi] + g^2 r_3[\phi] + \ldots$$

$$T_g[\phi] = \sum_{\vec{n}} g^n\, c_{\vec{n}}\; r_{n_s}[\phi] \cdots r_{n_2}[\phi]\, r_{n_1}[\phi] \cdot \phi \qquad (9)$$

with $\vec{n} = (n_1, n_2, \ldots, n_s)$, $n_i \in \mathbb{N}$, $\sum_{i=1}^{s} n_i = n$ multi-index

and $c_{\vec{n}} = (-1)^s \left[n_1 \cdot (n_1 + n_2) \cdot (n_1 + n_2 + n_3) \cdots (n_1 + n_2 + \ldots + n_s) \right]^{-1}$

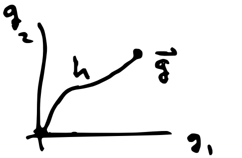
- explicit start of perturbation series:

$$T_g\phi = \phi - g\, r_1\phi - \tfrac{1}{2} g^2 (r_2 - r_1^2)\phi - \tfrac{1}{6} g^3 (2r_3 - r_1 r_2 - 2 r_2 r_1 + r_1^3)\phi + \ldots$$

- generalization to multiple couplings

$\vec{g} = (g^{(i)}) = (g^{(1)}, g^{(2)}, \dots, g^{(n)})$ $\qquad \to \vec{R}_g$

$$\partial_{g^{(i)}} \langle Y[\phi] \rangle_{\vec{g}} = \langle (\partial_{g^{(i)}} + R^{(i)}[\phi]) \, Y[\phi] \rangle_{\vec{g}}$$



$$T_{\vec{g}} \phi = \mathcal{P} \, e^{-\int_0^{\vec{g}} d\vec{g}' \cdot \vec{R}_{g'}[\phi]} \cdot \phi$$

choose path $h : [0,1] \to \{\vec{g}\}$, $s \mapsto \vec{h}(s)$

with $\vec{h}(0) = \vec{0}$, $\vec{h}(1) = \vec{g}$

$$= \mathcal{P} \, e^{-\int_0^1 ds \, \vec{h}'(s) \cdot \vec{R}_{h(s)}} \cdot \phi$$

$$= \sum_{n=0}^{\infty} (-1)^n \int_0^1 ds_n \dots \int_0^{s_3} ds_2 \int_0^{s_2} ds_1 \, [\vec{h}'(s_n) \cdot \vec{R}_{h(s_n)}] \dots [\vec{h}'(s_1) \cdot \vec{R}_{h(s_1)}] \cdot \phi$$

depends on the path h in coupling space, despite

$$[\partial_{g^{(i)}}, \partial_{g^{(j)}}] \langle Y \rangle_{\vec{g}} = 0 \implies \langle \partial_{g^{(i)}} R^{(j)} - \partial_{g^{(j)}} R^{(i)} + [R^{(i)}, R^{(j)}] \rangle_{\vec{g}} = 0$$

- special case: the map is path-independent $\Rightarrow$ unique!

then the power series truncates to a <u>linear</u> map:

$$T_{\vec{g}} \phi = \phi - \vec{g} \cdot \vec{R}_0 \, \phi \qquad \text{sufficient, but necessary?}$$

- partial maps: flow in some couplings, others fixed
- single-coupling flow truncates to a linear map if

$$R_g^2\,\phi\Big|_{\substack{\text{no}\\\text{branch}}} = (\partial_g R_g)\,\phi \quad \rightsquigarrow \quad T_g\phi = \phi - g\,r_1\,\phi \qquad (10)$$

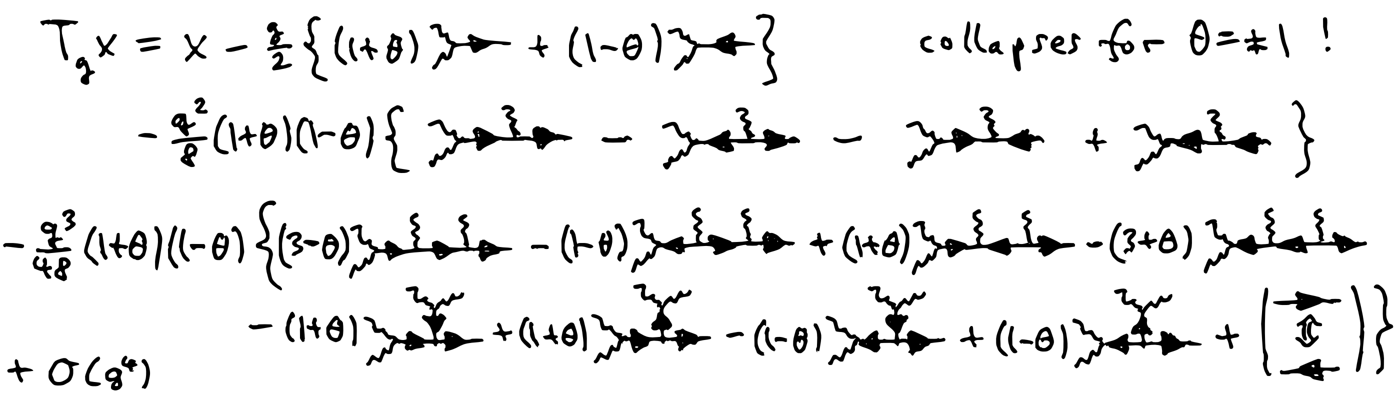
this may happen for special fixed-coupling values!

- example of SUSY quantum mechanics:

$$L = \tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}m^2x^2 - mgx^3 - \tfrac{g^2}{2}x^4 + \bar{\psi}\,[i\partial_t - m - 2gx]\,\psi + i\theta\,(mx+gx^2)\,\dot{x}$$

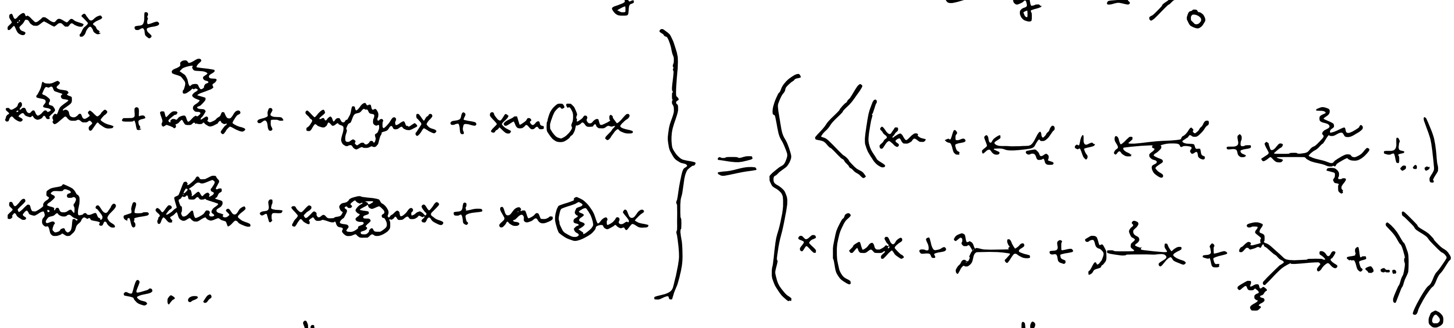
keep m & θ fixed and flow in g

fermion propagator in ω-space: $\frac{-1}{\omega+m} = \rightarrow\,, \quad \frac{1}{\omega-m} = \leftarrow$



- perturbation at large order

$$\langle Y[\phi] \rangle_g = \langle Y[T_g^{-1}\phi] \rangle_0$$



$\Downarrow$

$\#$ diagrams of $\mathcal{O}(g^n) \sim n^{3n/2}$

$\Downarrow$

$\#$ trees in $T_g^{-1}\phi$ to $\mathcal{O}(g^n) \sim c^n$
[Wedderburn-Etherington numbers]

$\#$ free-field contractions
$\sim n!! \sim n^{n/2}$

some constants
α, β, γ

$$\| T_g \phi \| \lesssim \left(1 + \gamma \sum_{n=1}^{\infty} n^{-\beta} (\alpha \|\phi\|)^n g^n \right) \cdot \|\phi\| \qquad (11)$$

$\leadsto$ $T_g\phi$ & $T_g^{-1}\phi$ have a <u>finite</u> convergence radius $\sim (\alpha\|x\|)^{-1}$

(3) The case of gauge theories

- complication: gauge redundancy

  $\longrightarrow$ gauge fixing $\mathcal{G}(A)=0$, ghost fields $c, \bar{c}$
  reduces gauge to BRST symmetry,
  breaks manifest supersymmetry

  | Jacobian = fermion × FP determinant

- $S_{\text{susy}}[A, \lambda, D, c, \bar{c}] = -\frac{1}{g^2} \int dx \, \mathrm{tr} \left\{ \frac{1}{4}\left(F^2 - \theta' F\tilde{F}\right) + \frac{1}{2\xi^2} \mathcal{G}(A)^2 \right.$
  $\left. + \text{fermions} + \text{ghosts} + \text{auxiliaries} \right\}$

  $\theta' = \frac{g^2\theta}{8\pi^2}$ only in $D=4$

  consider $\mathbb{R}^{1,D-1}$, fields in $SU(N)_{\text{adj}}$, gaugini $\lambda \in \mathbb{C}^r$ Majorana

- off-shell map, in any gauge, for $D \leq 4$ (superfields!)

[Lechtenfeld 1984, Malcha, Nicolai 2021, Lechtenfeld, Rupprecht 2021]

- on-shell map, in Landau gauge, via ansatz $\rightsquigarrow$
  multiplicative piece in R_g absent only for $D = 3, 4, 6, 10$!

[D=4: Flume, Lechtenfeld 1983, D≠4 ALMNPP 2020]

- most simple in Landau gauge @ $\partial \cdot A = 0$:

$$\overleftarrow{R}_g[A] = -\frac{1}{2r}\iiint\!\!\int dr\, \frac{\overleftarrow{\delta}}{\delta A_\mu} P_\mu{}^\nu \Big\{ \gamma_\nu \underbrace{\bar\lambda\ \lambda}\, \gamma^{\rho\lambda} \overbrace{(\mathbb{1}+i\theta'\gamma^5)}^{\text{only in } D=4} A_\rho \times A_\lambda \Big\}_{\alpha\alpha} \quad \begin{matrix}\text{spin}\\ \text{trace}\end{matrix}$$

(12) with non-Abelian transversal projector $[\,\underbrace{c\ \bar c} = (\partial\cdot D)^{-1}\,]$

$$P_\mu{}^\nu = \delta_\mu{}^\nu \mathbb{1} - D_\mu \underbrace{c\ \bar c}\, \partial^\nu \quad \Longrightarrow \quad \partial^\mu P_\mu{}^\nu = 0 = P_\mu{}^\nu D_\nu$$

- explicit constuction formula holds true:

$$T_g A = \mathcal{P}\, e^{-\int_0^g dg'\, R_{g'}[A]} \cdot A = A - g\, r_1 A - \tfrac{1}{2} g^2 (r_2 - r_1^2) A + \dots$$

from $R_g[A] = r_1[A] + g\, r_2[A] + g^2 r_3[A] + \dots$ & $\int A \frac{\delta}{\delta A} r_k = k\, r_k$

explicitly evaluated by ALMNPP 2020 to $\mathcal{O}(g^3)$ \
& by H. Malcha 2021 to $\mathcal{O}(g^4)$ } for $\theta' = 0$

$$T_g A_\mu = A_\mu - g\left(\partial^\lambda \Box^{-1} A_\mu \times A_\lambda - \tfrac{1}{2}\theta' \varepsilon_{\mu\nu\rho\lambda} \partial^\nu \Box^{-1} A^\rho \times A^\lambda\right)$$
$$- \tfrac{3}{2} g^2 (1+\theta'^2)\, \partial^\rho \Box^{-1} A^\lambda \times \partial_{[\rho} \Box^{-1} A_\mu \times A_{\lambda]} + \mathcal{O}(g^3) \qquad (13)$$

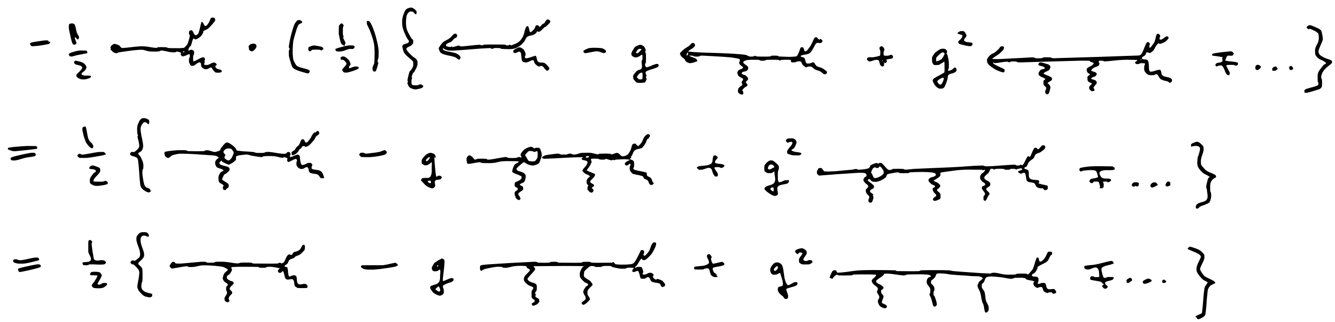
$\rightarrow$ collapse for $\theta' = \pm i$ at $\mathcal{O}(g^2)$! but ghosts kick in at $\mathcal{O}(g^3)$ ...

• surprise at $D=4$ for $\theta' = i$ (or $-i$):

$$\overleftarrow{R}^{\,inv}_{g}[A] = -\frac{1}{8}\iiint\!\!\int \mathrm{tr}\, \frac{\overleftarrow{\delta}}{\delta A_\mu}\, \delta_\mu{}^\nu\, \mathrm{tr}\left\{ \gamma_\nu \underbrace{\bar{\lambda}\ \lambda} \gamma^{\rho\lambda}(\mathbb{1}-\gamma^5)\, A_\rho \times A_\lambda \right\}$$

instead of $P_\mu{}^\nu$

acting on $r_1[A]\cdot A$ yields ( $= \frac{1}{4}\mathrm{tr}\ldots$)



 $+ R^{inv}_{g}[A]\cdot A$

by Fierz identity

$$\tfrac{1}{4}\mathrm{tr}\left(Y\gamma^\mu(1-\gamma^5)\right)\cdot \tfrac{1}{4}\mathrm{tr}\left(\gamma_\mu(1+\gamma^5)Z\right) = \tfrac{1}{8}\mathrm{tr}\left(Y(1+\gamma^5)Z(1-\gamma^5)\right)$$

$$\Rightarrow\quad T^{inv}_{g} A_\mu = A_\mu - g\,\partial^\lambda \Box^{-1} A_\mu \times A_\lambda + i\,\frac{g}{2}\,\varepsilon_{\mu\nu\rho\lambda}\,\partial^\nu \Box^{-1} A^\rho \times A^\lambda \quad !$$

④ Summary

- Supersymmetric theories can be investigated from a novel perspective using Nicolai maps
- Efficient computation of amplitudes via alternative graphical "Nicolai rules"
- Convergent power series for the functional transformation to the free theory
- Works also for super Yang-Mills @ $D = 3, 4, 6, 10$
- Ambiguities: path dependence, θ angle, R-symmetry
- Special θ values project onto single helicity & may collapse the map $\longrightarrow$ hope for $D=4$ $N=4$ SYM?
- Perspectives: SYM amps? integrability? NLσMs? SUGRA?