Nambu and the Ising Model

Happy Birthday Hermann!

Les Houches (by M. Hindmarsh)

Met Hermann Nicolai at the 1985 Les Houches Summer School. Lectured on "Compactifications of Eleven-Dimensional Supergravity".

Thirty seven years later, Hermann continues to live in more dimensions than most, interrupted with fascinating forays in three dimensions, e.g. , *"Standard Model Fermions and $N = 8$ Supergravity"*, with Krzysztof Meissner (Phys. Rev. D31, (2015) 065029)

After a great dinner at his home, Hermann displayed yet another dimension, his love of music and of playing the piano.

I hope that the beautiful physics from one of the grand masters of the physics keyboard will arouse Hermann's curiosity and sense of beauty

2021: Birth Centenary of Yôichirô Nambu, the Humble Genius

Lars Brink and P. Ramond

Writing a book of Historical/scientific descriptions of his most incisive papers

Expected to discover unknown insights: on September 1, 1949

"A Note on the Eigenvalue Problem in Crystal Statistics".
Prog. Theo. Phys. **5**, 1, (1950)

Strange acknowledgment

"The main part of the present work had been completed nearly two years ago. It is through the kindness of Professor Husimi and Mr. Syôzi of Osaka University that the author enjoys the opportunity of publishing this note."

Publication interrupted by news of the Lamb shift in *Time Magazine of Sept 29, 1947.*

After the war, Nambu was at Tokyo University

Center for statistical and condensed matter physics

Nambu lived in his office, sleeping on his desk, surviving mainly on potatoes.

Ideal Environment for the study the two-dimensional Ising model

Solved analytically, but uncomprehendingly by Lars Onsager in 1944

Asked after the war if anything new had happened in physics Pauli replied

> *"not much, except for Onsager's solution of the Ising Model"*

Nambu expresses Onsager's Ising model as a quantum system of N qubits.

A four-page computation of the $(2N \times 2N)$ transfer matrix eigenvalues

Nambu's ending:

Though as yet no substantial applications has been attempted, nor anything physically new has been derived, it may be hoped that it will do some profit for those who are interested in such problems.

Ising Model Origins

Wilhelm Lenz's seminal paper

"Beitrag zum Verständnis der magnetischen Erscheinungen in festen Körpern"
Physik. Z. XXI, 1920, 613-615

Ferromagnetism may be explained in terms of interacting nearest-neighbor magnets which could flip in opposite directions ("umklapping").

1924: Asked his student Ernst Ising to solve his model

"Beitrag zur Theorie des Ferromagnetismus"
Zeits. f. Phys. 31, 253 (1925)

Found an analytical solution for linear lattice and no ferromagnetic transition

Nambu's Crystal Statistics

Linear Single Spin Array

Conventional Approach

N identical particles with two-valued spin at each site

Periodic boundary conditions and nearest-neighbor interactions

$$P = \sum_{n=1}^{N} P_{n,n+1} = \sum_{n=1}^{N} \frac{1 + \sigma_n \, \sigma_{n+1}}{2}.$$

Independent Fermi oscillators at each site

$$\{a_n, a_n^\dagger\} = 1, \qquad [a_n, a_m^\dagger] = 0, \quad n \neq m,$$

$$P = \sum_{n=1}^{N} \left[a_n^\dagger a_{n+1} + a_{n+1}^\dagger a_n + 2a_n^\dagger a_n a_{n+1}^\dagger a_{n+1} - 2a_n^\dagger a_n + 1\right]$$

P lives in a 2^N-dimensional Hilbert space

Nambu's Approach

P is same if operators at different sites anticommute

Alternate description with new ladder operators

$$\{a_n, a_m\} = \{a_n^\dagger, a_m^\dagger\} = 0; \qquad \{a_n, a_m^\dagger\} = \delta_{m,n}, \quad \forall\ n, m$$

Same Physics

P lives in smaller $2N$-dimensional Hilbert space

Spanned by $2N$ real Grassmann coordinate and momentum

$$\text{Nambu Basis}: \ \{x_n,\, x_m\} = 2\delta_{rs}, \quad n, m = 1, 2, \ldots 2N$$

Applies to solve

linear one spin array

linear two spin array (isotropic $X - Y$) model

and then ...

The Square Ising Model

Nambu's Starting point:

Onsager Operator for Square Lattice

$$\mathcal{H} = \exp\big[H' \sum_{n=1}^{N} s_n s_{n+1}\big] \; \exp\Big[H^* \sum_{n}^{N} c_n\Big],$$

$H' = J'/kT$, $\quad H^*$: Kramers − Wannier dual of $H = J/kT$

H.A. Kramers and G. H. Wannier, Phys. Rev. **60** (1941) 252

Two spins at each site

$$s_n^2 = c_n^2 = 1, \qquad \{\, s_n,\, c_n\} = 0,$$

Commute with one another at different sites

Onsager's *tour de force*

Calculated the $\mathcal{H}$ eigenvalues

Discovered the ferromagnetic transition in the thermodynamic limit

Expressing Onsager's formula in Nambu's basis

New variables $S_n \equiv s_n s_{n+1}, \qquad C_n \equiv c_n$

Commute with one another except at adjacent sites: $\{ S_n, C_{n\pm 1} \} = 0$

Boundary Conditions,

$S \equiv S_1 S_2 \cdots S_N = 1, \; C \equiv C_1 C_2 \cdots C_N = \pm 1$

In the "Nambu basis" (even N): $S_n = i x_{2n} x_{2n+1}, \quad C_n = i x_{2n-1} x_{2n}$

Boundary Conditions

$C = i^N x_1 x_2 x_3 x_4 \cdots x_{2N-1} x_{2N} \equiv X, \quad S = i^N x_2 x_4 x_2 x_5 \cdots x_{2N} x_1 = -X$

$$\mathcal{H} = \exp\left[iH' \sum x_{2n} x_{2n+1} \right] \exp\left[iH^* \sum x_{2n-1} x_{2n} \right] \equiv \mathcal{H}_2 \mathcal{H}_1$$

Ready for Computation

$$\mathcal{H} = \exp\left[iH' \sum x_{2n}x_{2n+1}\right] \exp\left[iH^* \sum x_{2n-1}x_{2n}\right] \equiv \mathcal{H}_2\mathcal{H}_1$$

Product of Rotation Operators in $(n-m)$ plane $\quad \mathcal{U} = e^{\theta x_n x_m}$

$\mathcal{H}$ rotation by an angle iH^* followed by a rotation by iH'

$$\begin{aligned} \mathcal{H}_1 &= \exp\left[iH^*(x_1x_2 + x_3x_4 + \cdots x_{2N-1}x_{2N})\right], \\ \mathcal{H}_2 &= \exp\left[iH'(x_2x_3 + x_4x_5 + \cdots x_{2N}x_1)\right], \end{aligned}$$

$\mathcal{H}_1$ rotates the "odd-even" pairs (x_{2n-1}, x_{2n}),

$$\mathcal{H}_1 : \quad \begin{pmatrix} x_{2n-1} \\ x_{2n} \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} y_{2n-1} \\ y_{2n} \end{pmatrix} = \mathcal{R}(2iH^*) \begin{pmatrix} x_{2n-1} \\ x_{2n} \end{pmatrix},$$

$\mathcal{H}_2$ rotates "even-odd" pairs (y_{2n}, y_{2n+1}),

$$\mathcal{H}_2 : \quad \begin{pmatrix} y_{2n} \\ y_{2n+1} \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} z_{2n} \\ z_{2n+1} \end{pmatrix} = \mathcal{R}(2iH') \begin{pmatrix} y_{2n} \\ y_{2n+1} \end{pmatrix}$$

Rotation matrix

$$\mathcal{R}(2it) = \begin{pmatrix} \cos(2it) & \sin(2it) \\ -\sin(2it) & \cos(2it) \end{pmatrix} = \begin{pmatrix} \cosh(2t) & i\sinh(2t) \\ -i\sinh(2t) & \cosh(2t) \end{pmatrix}$$

Combined Action $\mathcal{H} = \mathcal{H}_2\mathcal{H}_1$ on each pair

$$\begin{pmatrix} x_{2n-1} \\ x_{2n} \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} z_{2n-1} \\ z_{2n} \end{pmatrix} \equiv \lambda \begin{pmatrix} x_{2n-1} \\ x_{2n} \end{pmatrix},$$

Eigenvalue equation for λ

$$\begin{aligned} -bx_{2n-1} + (d-\lambda)x_{2n} + ax_{2n+1} + cx_{2n+2} &= 0, \\ -ax_{2n} + (d-\lambda)x_{2n+1} + bx_{2n+2} + cx_{2n-1} &= 0. \end{aligned}$$

Coefficients

$$a = i\cosh(2H^*)\sinh(2H'), \quad b = i\sinh(2H^*)\cosh(2H')$$

$$c = -\sinh(2H^*)\sinh(2H'), \quad d = \cosh(2H^*)\cosh(2H')$$

Constraints

$$ab - cd = 0, \quad a^2 + b^2 + c^2 + d^2 = 1.$$

Nambu's Elegant solution

$$A \equiv \begin{pmatrix} a & c \\ \lambda_- d & -b \end{pmatrix}, \quad B \equiv \begin{pmatrix} b & \lambda_- d \\ c & -a \end{pmatrix}, \quad \psi_n \equiv \begin{pmatrix} x_{2n-1} \\ x_{2n} \end{pmatrix}$$

Recursion Relation:

$$\psi_{n+1} = A^{-1} B \psi_n \equiv C \psi_n,$$

Solution

$$\psi_{n+1} = C^n \psi_1.$$

Periodicity $\psi_{N+1} = \psi_1$

$$(1 - C^N)\psi_1 = 0 \quad \longrightarrow \quad \det(1 - C^N) = 0.$$

"Well-known" Identity

$$1 - C^N = \prod_{k=1}^{N} (\eta^k - C), \qquad \eta = e^{\frac{2\pi i}{N}}$$

Reduces to N equations

$$\det(\eta^k - C) = 0 \quad \longrightarrow \quad |A\eta^k - B\,| = 0. \quad k = 1, 2 \ldots, N$$

$$\begin{vmatrix} \eta^k a - b & \eta^k c - (\lambda - d) \\ \eta^k (\lambda - d) - c & -\eta^k b + a \end{vmatrix} = 0.$$

$$\lambda^2 - 2\lambda\big[d + c\cos\varphi_k\big] + 1 = 0, \qquad \varphi_k = \frac{2\pi k}{N}$$

Two solutions for each k,

$$\lambda_{k\pm} = \cosh(2\gamma_k) \pm \sinh(2\gamma_k)$$

$$\cosh 2\gamma_k = d + c\cos\varphi_k = \cosh(2H^*)\cosh(2H') - \sinh(2H^*)\sinh(2H')\cos\varphi_k,$$

Same formula as in Onsager's paper:

$$\frac{1}{2}\sum_{r=1}^{n}\gamma_{2r-1} = \frac{1}{2}\sum_{r=1}^{n}\cosh^{-1}\big[\cosh 2H'\cosh 2H^* - \sinh 2H'\sinh 2H^*\cos((2r-1)\pi/2n))\big]$$

Such an enormous simplification suggests that us that it is not just a clever trick; it must point to deeper insights. It would interesting to redo all previous Ising model calculations, especially the emergence of conformal invariance and the Virasoro algebra at the critical point, and generalizations.

Nambu considered the three-dimensional Ising model, where some of the relevant operators are exponentials of quadratics as in the Ising model, but others are exponentials of quartics, such as $e^{ax_1x_2x_3x_4}$. In view of Nambu's many prescient comments, it might be interesting to follow this path. To this day no analytic solution has been found.

There is no better to end this talk than by repeating Nambu's statement at the end of his paper:

Though as yet no substantial applications has been attempted, nor anything physically new has been derived, it may be hoped that it will do some profit for those who are interested in such problems.