Mode Stability and Black Holes in Supergravity

HermannFest

AEI, Golm, 15th September 2022

With Mirjam Cvetič, Bernard Whiting and Haoyu Zhang

and earlier work also with Gary Gibbons

An important question for black hole solutions in theories of gravity or supergravity concerns their stability under the influence of small perturbations. This has been very extensively studied in four dimensions, and also in higher dimensions. Here, we discuss an approach that was developed first for the four-dimensional Kerr black hole by Bernard Whiting in 1989. This started from the Teukolsky equation that gives a gauge-invariant description of the gravitational perturbations around the Kerr background, separating variables, and then studying the separated modes in order to establish that no exponentially growing perturbations could arise. Whiting's technique involved finding a transformation that mapped the difficult problem of studying modes in a spacetime with an ergo-region and its associated super-radiant phenomena into a simpler problem in a geometry with no ergo-region. This can in fact be studied already for the simpler case of a massless Klein-Gordon field. Here, we examine the analogous problem in a higher dimension, for the five-dimensional Myers-Perry black hole, and the charged rotating black holes of five-dimensional supergravity.

Mode Stability for the Kerr Black Hole

- Especially since it has become clear that rotating black holes are ubiquitous throughout the universe, the question of whether or not they are stable has become one of considerable importance. One approach to investigating this question for the Kerr black hole was developed by Bernard Whiting in the late 1980s. The starting point was the Teukolsky equation, which provides a gauge-invariant way of analysing small fluctuations of the spacetime geometry around the Kerr background:

$$\Big\{\partial_r(\Delta\partial_r) - \frac{1}{\Delta}((r^2+a^2)\partial_t + a\partial_\phi - (r-M)s)^2 - 4s(r+\mathrm{i}\,a\cos\theta)\partial_t + \frac{1}{\sin\theta}\,\partial_\theta(\sin\theta\,\partial_\theta)$$
$$+\frac{1}{\sin^2\theta}\,(a\sin^2\theta\,\partial_t + \partial_\phi + \mathrm{i}\,s\cos\theta)^2\Big\}\,\Psi = 0\,, \qquad (1)$$

 where (t, r, θ, ϕ) are the usual Kerr coordinates, M is the mass, a is the rotation parameter, and $\Delta = r^2 - 2Mr + a^2$ is the function in the Kerr metric that vanishes on the horizon. When $s = \pm 2$, Ψ represents the gauge-invariant Newman-Penrose quantities that characterise the fluctuations of the geometry around the Kerr background. If $s = 0$, we have the massless Klein-Gordon equation in the Kerr background, while for $s = \pm 1$ we have the Maxwell equations. For $s = \pm\frac{1}{2}$ and $s = \pm\frac{3}{2}$ we have the squares of the massless Dirac and Rarita-Schwinger equations, respectively.

- Although the $s = \pm 2$ case is the one of principal interest for studying black hole stability, many of the relevant features are present already in the $s = 0$ case. In higher dimensions, where no analogue of the Teukolsky analysis of gauge-invariant gravitational perturbations is known, we can again gain insights by looking at the massless Klein-Gordon equation.

Mode Stability for the Kerr Black Hole

- If one sets $a = 0$ in Kerr, giving Schwarzschild, one can straightforwardly establish Klein-Gordon mode stability by constructing the conserved energy functional for eqn (1) with $s = 0$. In this case the Killing vector ∂_t is timelike everywhere outside the horizon, and from this one can prove positivity of the energy and hence mode stability.

- In the Kerr case, however, there is a term

$$\Big(\frac{1}{\sin^2\theta} - \frac{a^2}{\Delta}\Big)\Big|\frac{\partial\Psi}{\partial t}\Big|^2$$

  in the energy functional that is negative within the ergosphere $r = M + \sqrt{M^2 - a^2\cos^2\theta}$ outside the horizon at $r_+ = M + \sqrt{M^2 - a^2}$, and the previous simple proof of mode stability will hence fail.

- Whiting overcame this problem by finding (how?) a rather subtle integral transformation of the radial functions in the separated wave equation. By then "unseparating" the variables again, after the transformation, a four-dimensional wave equation in an auxiliary metric is obtained. The remarkable feature of this auxiliary metric is that ∂_t is timelike everywhere outside $r = r_+$, meaning that there is no ergoregion and hence one has a positive energy functional that allows a simple proof of mode stability for the transformed modes.

- The mode stability proof is then completed by establishing a one-to-one relation between putative unstable modes in the original Klein-Gordon equation and in the transformed wave equation.

- Whiting's proof for the mode stability of the $s = \pm 2$ Teukolsky equation proceeds almost identically.

- In this talk, we shall outline an analogous mode-stability proof for the massless Klein-Gordon field in five-dimensional rotating black holes in gravity and supergravity.

Five-Dimensional Myers-Perry Black Hole

- The higher-dimensional analogues of the Kerr black hole were constructed by Meyers and Perry. In five dimensions, it is given by

$$\begin{aligned} ds^2 &= -\frac{\Delta}{\rho^2}(dt - a\sin^2\theta\, d\phi - b\cos^2\theta\, d\psi)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2\, d\theta^2 \\ &\quad + \frac{\sin^2\theta}{\rho^2}[a dt - (r^2+a^2)\, d\phi]^2 + \frac{\cos^2\theta}{\rho^2}[b dt - (r^2+b^2)\, d\psi]^2 \\ &\quad + \frac{1}{r^2\rho^2}[ab dt - b(r^2+a^2)\sin^2\theta d\phi - a(r^2+b^2)\cos^2\theta\, d\psi]^2 \end{aligned}$$

where

$$\Delta = \frac{(r^2+a^2)(r^2+b^2)}{r^2} - 2M\,, \qquad \rho^2 = r^2 + a^2\,\cos^2\theta + b^2\,\sin^2\theta\,.$$

Here a and b are the two independent rotation parameters, with ϕ and ψ being the two associated azimuthal angles (each with period 2π). The latitude coordinate θ ranges over $0 \le \theta \le \frac{1}{2}\pi$. The outer horizon is at the largest root of $\Delta = 0$.

- Defining $dr_* = (r^2+a^2)(r^2+b^2)dr/(r^2\,\Delta)$, and advanced coordinates (v, ϕ_+, ψ_+) by

$$dv = dt + dr_*\,, \qquad d\phi_+ = d\phi + \frac{a\,(r^2+b^2)\,dr}{r^2\,\Delta}\,, \qquad d\psi_+ = d\psi + \frac{b\,(r^2+a^2)\,dr}{r^2\,\Delta}\,, \quad \text{gives}$$

$$\begin{aligned} ds^2 &= -dv^2 + 2dr\,(dv - a\sin^2\theta\, d\phi_+ - b\cos^2\theta\, d\psi_+) + \rho^2\, d\theta^2 \\ &\quad + \frac{2M}{\rho^2}(dv - a\sin^2\theta\, d\phi_+ - b\cos^2\theta\, d\psi_+)^2 + (r^2+a^2)\,\sin^2\theta\, d\phi_+^2 + (r^2+b^2)\,\cos^2\theta\, d\psi_+^2\,. \end{aligned}$$

It is regular on the future horizon. Similarly, retarded coordinates with $du = dt - dr_*$ give metric regular at future null infinity.

Separation of Klein-Gordon Equation

- Defining new radial and latitude coordinates x and y by

$$r^2 = 2M\epsilon_+ \epsilon_- \, x + \tfrac{1}{2}M(\epsilon_+ - \epsilon_-)^2\,, \quad y = \cos^2\theta\,, \quad \epsilon_\pm^2 = 1 - \frac{(a \pm b)^2}{2M}$$

(inner horizon at $x = 0$, outer horizon at $x = 1$), the Klein-Gordon equation separates by writing

$$\Psi = e^{-\mathrm{i}\,\omega t + \mathrm{i}\,m\phi + \mathrm{i}\,n\psi}\, \frac{X(x)}{\sqrt{x(x-1)}}\, \frac{Y(y)}{\sqrt{y(1-y)}}\,.$$

$X(x)$ and $Y(y)$ satisfy

$$\frac{X''}{X} + \frac{\kappa+\Lambda}{x} + \frac{\kappa-\Lambda}{x-1} + \frac{\frac{1}{4}-\beta^2}{x^2} + \frac{\frac{1}{4}-\gamma^2}{(x-1)^2} = 0\,,$$
$$\frac{Y''}{Y} + \frac{\hat\kappa+\hat\Lambda}{y} + \frac{\hat\kappa-\hat\Lambda}{y-1} + \frac{\frac{1}{4}-\hat\beta^2}{y^2} + \frac{\frac{1}{4}-\hat\gamma^2}{(y-1)^2} = 0\,,$$

where $(\kappa, \Lambda, \beta, \gamma, \hat\kappa, \hat\Lambda, \hat\beta, \hat\gamma)$ are certain constants depending on the black hole parameters (M, a, b), on the three separation constants (ω, m, n), and on the fourth separation constant σ (entering in $\Lambda = \sigma + \cdots$ and $\hat\Lambda = \sigma + \cdots$).

- The differential equations arising here are of the form of confluent Heun equations. It is difficult to establish results about mode stability by means of a direct approach. Instead, we follow a similar strategy to that employed by Whiting for the proof of Kerr mode stability.

- First, we discuss what mode stability actually means:

Mode Stability

- The equation, in this case the Klein-Gordon equation in the five-dimensional Myers-Perry black hole background, is said to exhibit *mode stability* if it is the case that every mode of the separated solutions that obeys appropriate spatial fall-off conditions has the property that it decays at large times. In other words, small disturbances die off, rather than grow, as t goes to infinity.

- In the present case, the spatial region of interest is the region between the outer future horizon $\mathcal{H}^+$ (at $x = 1$) and future null infinity $\mathscr{I}^+$ (at $x = \infty$). We are thus concerned with modes that are:

  1) **Ingoing** at $\mathcal{H}^+$, so $\sim e^{-\mathrm{i}\,\omega(t+r_*)}$ near $x = 1$;
  2) **Outgoing** at $\mathscr{I}^+$, so $\sim e^{-\mathrm{i}\,\omega(t-r_*)}$ near $x = \infty$;
  3) The frequency ω should be such that $\mathrm{Im}(\omega) \le 0$.

- Conditions (1) and (2) ensure that there is nothing entering the region $1 \le x \le \infty$ from either of the boundaries $\mathcal{H}^+$ or $\mathscr{I}^+$. Condition (3) is the requirement that the mode should not grow exponentially at large t.

- In fact our analysis is concerned entirely with the case where $\mathrm{Im}(\omega)$ is non-zero. The standard modes are of this kind; they are the so-called *quasi-normal modes*, and they have $\mathrm{Im}(\omega) < 0$. Our task, in fact, is to show that there are no modes obeying conditions (1) and (2) with $\mathrm{Im}(\omega) > 0$.

- A more challenging task would be to establish mode stability also on the real axis, i.e. allowing for the possibility that $\mathrm{Im}(\omega) = 0$. The analysis would be more delicate here; for standard quasi-normal modes the exponential die-off makes it much simpler to deal with the boundary conditions.

Showing Mode Stability

- A strategy that one could try is to construct a conserved energy functional for the Klein-Gordon field by calculating its energy-momentum tensor $T^{\mu\nu}$ and hence the conserved current $J^\mu = T^{\mu\nu} K_\nu$ where $K = \partial_t$, and hence the conserved energy $\mathcal{E} = \int \sqrt{-g}\, J^0\, d^4x$.

- If the energy functional were positive definite it would bound $|\partial_t \Psi|^2$, hence showing there could not exist any mode that increased exponentially in time. Unfortunately, because of the existence of an ergoregion outside the horizon (where ∂_t is not timelike), the energy functional is not positive definite, and one cannot get such a bound, at least directly.

- After some experimentation, it turns out to be useful to make an integral transform of the radial wave function $X(x)$, introducing at the same time a new radial coordinate z, such that the new radial wave function is $h(z)$ given by

$$h(z) = e^{-\tilde{\alpha} z}\, e^{\tilde{\nu}/z}\, z^{1+\tilde{\gamma}} \int_1^\infty e^{2\tilde{\alpha} x z}\, x^{-\frac{1}{2}-\beta}\, (x-1)^{-\frac{1}{2}-\gamma}\, X(x)\, dx\,,$$

where $\tilde{\nu} = -\kappa/(2\tilde{\alpha})$, $\tilde{\gamma} = -\beta - \gamma$, and $\tilde{\alpha} = \mathrm{i}\, \sqrt{M\epsilon_-}\, \omega/(2\sqrt{2})$. Defining also $\tilde{\kappa} = \beta - \gamma$ and $\tilde{\Lambda} = \frac{1}{2} - \beta^2 - \gamma^2 - \Lambda$, the new radial equation is

$$\Big(\partial_z^2 - \tilde{\alpha}^2 + \frac{2\tilde{\alpha}\,\tilde{\kappa}}{z} + \frac{\tilde{\Lambda}}{z^2} + \frac{2\tilde{\gamma}\,\tilde{\nu}}{z^3} - \frac{\tilde{\nu}^2}{z^4}\Big) h(z) = 0\,.$$

- The new radial variable z ranges from $0 \le z \le \infty$ as the original variable x ranges from $x = \infty$ (near $\mathscr{I}^+$) to $x = 1$ (near $\mathcal{H}^+$, the future horizon). After some analysis one finds

Near $\mathcal{H}^+$, $x = 1$, $z = \infty$: $\quad X(x) \sim (x-1)^{\frac{1}{2}-\gamma}$, $\quad h(z) \sim e^{\tilde{\alpha} z}\, z^{-\tilde{\kappa}}$

Near $\mathscr{I}^+$, $x = \infty$, $z = 0$: $\quad X(x) \sim x^{\frac{1}{4}}\, e^{2\mathrm{i}\sqrt{2\kappa\, x}}$, $\quad h(z) \sim e^{\tilde{\nu}/z}\, z^{1+\tilde{\gamma}}$

Unseparation of Variables

- As a check, we see that a putative unstable mode with $\mathrm{Im}(\omega) > 0$ implies $\mathrm{Re}(\tilde{\alpha}) < 0$ and so indeed $h(z)$ will go to zero on $\mathcal{H}^+$. We also have $\tilde{\nu} = \mathrm{i}\,\sqrt{M}\,\epsilon_+\,\omega/(2\sqrt{2})$, so $\mathrm{Re}(\tilde{\nu}) < 0$ for an unstable mode, and therefore $h(z)$ goes to zero on $\mathscr{I}^+$.

- We have thus established that if an unstable mode exists, the associated transformed radial wavefunction $h(z)$ will satisfy appropriate fall-off conditions on $\mathcal{H}^+$ and $\mathscr{I}^+$.

- We now perform an "unseparation of variables," by defining wavefunctions

$$\Phi(t,z,y,\phi,\psi) = \frac{1}{z\,\sqrt{y(1-y)}}\,h(z)\,Y(y)\,e^{-\mathrm{i}\,\omega t + \mathrm{i}\,m\phi + \mathrm{i}\,n\psi}\,,$$

  and derive the wave equation that $\Phi(t,z,y,\phi,\psi)$ satisfies, given the known ODEs for $h(z)$ and $Y(y)$. This can be interpreted as a massless Klein-Gordon equation

$$\partial_\mu(\sqrt{-\hat{g}}\,\hat{g}^{\mu\nu}\,\partial_\nu\Phi) = 0$$

  in the background of a certain auxiliary metric $\hat{g}_{\mu\nu}$.

- Calculating the energy-momentum tensor for Φ, and hence the conserved current $J^\mu = T^{\mu\nu}\,K_\nu$, we find the conserved energy $\mathcal{E} = \int \sqrt{-\hat{g}}\,J^0\,d^4x$, where

$$\sqrt{-\hat{g}}J^0 = \frac{1}{2z^2}\Big\{P\,|\partial_t\Phi|^2 + z^2\,|\partial_z\Phi|^2 + y(1-y)\,|\partial_y\Phi|^2 + \frac{1}{4(1-y)}\,|\partial_\phi\Phi|^2 + \frac{1}{4y}\,|\partial_\psi\Phi|^2\Big\}\,,$$

  where P is a function of z and y that is manifestly positive in the whole integration region. Since every term is integrable for any putative unstable mode, and each contribution to $\mathcal{E}$ is non-negative, it follows that the integral of the $P\,|\partial_t\Phi|^2$ term is bounded from above by the conserved energy $\mathcal{E}$. Therefore Φ cannot grow exponentially in time, and so there cannot in fact exist any exponentially unstable modes.

- The absence of an ergoregion in the metric $\hat{g}_{\mu\nu}$ is crucial in this proof.

The Transformed Metric

- The transformed metric $d\hat{s}^2$ takes the form, up to an unimportant constant factor, of

$$d\hat{s}^2 = (1+z)^{-10/3}\, d\tilde{s}^2\,,$$

where

$$\begin{aligned} d\tilde{s}^2 &= -\Big(2z\,dt + (a+b)(y d\phi + (1-y)d\psi) + (a-b)[(1-y)d\phi - y d\psi]z^2\Big)^2 \\ &\quad +2M(1+z)^4\Big[\frac{dz^2}{4z^2} + \frac{dy^2}{4y(1-y)} + (1-y)d\phi^2 + y d\psi^2\Big]\,. \end{aligned}$$

- This is quite analogous to the transformation of the four-dimensional Kerr metric that was found by Whiting. (His was a little bit different because he made a differential transformation of the separated angular equation too, that was needed for the higher-spin Teukolsky equations but that could have been omitted for $s=0$.)

- In both four dimensions and in our present five-dimensional case, the key feature is that the original wave equation in the original metric is transformed (with a 1-1 mapping of modes) to the wave equation in the hatted metric. And furthermore that the hatted metric has no ergoregion, so in the transformed theory the proof of the absence of unstable modes is elementary, as we saw in the previous slide.

Mode Stability for Five-Dimensional Supergravity Black Holes

- The most general charged analogues of the five-dimensional Myers-Perry black hole arise in maximal $\mathcal{N} = 8$ supergravity; with 3 electric charges. The solutions can be constructed by employing the $SL(3, R)$ global symmetry of the theory reduced to three dimensions, "boosting" from the Myers-Perry black hole as a seed solution. The metric is given by

$$
\begin{aligned}
ds^2 &= (H_1 H_2 H_3)^{1/3}\,(\tilde{x}+\tilde{y})\Big(-\Phi(dt+\mathcal{A})^2 + ds_4^2\Big),\\
ds_4^2 &= \frac{d\tilde{x}^2}{4X} + \frac{d\tilde{y}^2}{4Y} + \frac{U}{G}\Big(d\chi - \frac{Z}{U}\,d\sigma\Big)^2 + \frac{XY}{U}\,d\sigma^2,\\
X &= (\tilde{x}+a^2)(\tilde{x}+b^2) - 2M\,\tilde{x}, \qquad Y = -(a^2-\tilde{y})(b^2-\tilde{y}),\\
G &= (\tilde{x}+\tilde{y})(\tilde{x}+\tilde{y}-2M), \qquad U = \tilde{y}\,X - \tilde{x}\,Y, \qquad Z = ab\,(X+Y),\\
\mathcal{A} &= \frac{2M\pi_c\,(\tilde{x}+\tilde{y})}{G}\,[(a^2+b^2-\tilde{y})d\sigma - ab\,d\chi] - \frac{2M\pi_s}{\tilde{x}+\tilde{y}}\,(ab\,d\sigma - \tilde{y}\,d\chi),\\
\Phi &= \frac{G}{(\tilde{x}+\tilde{y})^3\,H_1H_2H_3}, \qquad H_i = 1 + \frac{2Ms_i^2}{\tilde{x}+\tilde{y}}. \qquad i = 1,2,3,
\end{aligned}
$$

where $\pi_c = c_1c_2c_3$, $\pi_s = s_1s_2s_3$, and $s_i = \sinh\delta_i$ $c_i = \cosh\delta_i$, where δ_i are the three boost parameters. $\tilde{x}$, $\tilde{y}$, σ and χ are related to the previous Myers-Perry coordinates by

$$
\tilde{x} = 2M\epsilon_+\epsilon_-\,x + \tfrac{1}{2}M(\epsilon_+ - \epsilon_-)^2, \quad \tilde{y} = (a^2-b^2)y + b^2, \quad \sigma = \frac{a\phi - b\psi}{a^2-b^2}, \quad \chi = \frac{b\phi - a\psi}{a^2-b^2}.
$$

- The proof of mode stability proceeds almost identically to that for Myers-Perry, the only difference in the energy functional $\mathcal{E}$ being that the prefactor P of $|\partial_t\Phi|^2$ is modified. It is

$$
P = \frac{\epsilon_+^2\,M}{8z^2} + \frac{\epsilon_-^2\,M\,z^2}{8} + \frac{(\pi_c+\pi_s)M}{2z} + \frac{(\pi_c-\pi_s)M\,z}{2} + \tfrac{1}{4}M\Big(3 + \sum_i 2s_i^2\Big) - \tfrac{1}{8}(a^2-b^2)(1-2y).
$$

The positivity that holds for $\delta_i = 0$ (Myers-Perry) is true *a fortiori* for $\delta_i > 0$.

Some Further Applications of Supergravity Black Holes

- In recent years it has been understood that there is large rotating black hole lurking at the centre of just about every galaxy, and also there are smaller black holes whose presence can be detected through the gravitational radiation generated when mergers take place.
- Now that these black holes can actually be observed, either directly or indirectly, it raises the possibility of testing the predictions of general relativity in these extreme conditions. For example, one can probe the extent to which the observational results are in line with the predictions for the Kerr metric.
- To do this one can consider some parameterised family of metrics that reduces to the Kerr metric when the parameters are turned off, and to compare the physical predictions from the different members of the family with the actual observational data. For randomly chosen such families, the chances are the metrics will suffer from some pathologies, such as requiring an energy-momentum tensor that violates a positive-energy criterion, or having causality violation.
- The charged black holes of supergravity provide nice parameterised families of black hole metrics that are known to be free of such pathologies. They are solutions of a theory with a well-posed initial-value problem; they have positive energy; the propagation of time-dependent solutions is causal; the Hamilton-Jacobi equation for null geodesics is integrable; and the Klein-Gordon equation is separable.
- In various papers in collaboration with Gary Gibbons, Mirjam Cvetič and Bernard Whiting, we have used the supergavity black holes as "foils" (to use a term coined by Thibault Damour) to test some of the predictions against observation. Some examples include calculating the shape of the "shadow" cast by the black hole and the characteristics of the photon sphere, and calculating the Love numbers that characterise the response of the black hole to tidal perturbations.

And Finally...

HAPPY BIRTHDAY Hermann!!!!!