Towards Kostant convexity

And a mathematical confirmation of cosmological billiards

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Overview

1. Topologies on Lie groups
   (Recognizing the Lie group $\mathrm{SL}(n, \mathbb{R})$ with minimal information)

2. Topological Kac–Moody groups
   (Thinking about Kac–Moody groups in exactly the same way)

3. Kac–Moody symmetric spaces
   (Studying the Kac–Moody coset sigma model)

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Amalgams of groups

Let

- $(J, <)$ be a partially ordered set,
- for each $i \in J$ let G_i be a group, and
- for each $i < j$ let
$$\iota_i^j : G_i \hookrightarrow G_j$$
be a group monomorphism such that for all $i < j < k$ one has
$$\iota_i^k = \iota_j^k \circ \iota_i^j.$$

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Then the datum
$$\left((G_i)_{i \in J}, (\iota_i^j)_{i<j}\right)$$
is a called an *amalgam of groups*.

Universal enveloping groups

Let $\mathcal{A} = \left((G_i)_{i \in J}, (\iota_i^j)_{i<j} \right)$ be an *amalgam of groups*.

An *enveloping group* of $\mathcal{A}$ is a tuple $(H, (\phi_i)_{i \in J})$ consisting of

- a group H and
- group homomorphisms $\phi_i : G_i \to H$ such that for all $i < j$ one has

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Examples: direct limits, free products, amalgamated products $A *_C B$

Constructing universal enveloping groups

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Then the tuple $(G, (\psi_i)_{i \in J})$ with

- $G := \left\langle \{x_g : g \in \cup\mathcal{A}\} \mid \begin{array}{c} \forall i < j \in J \; \forall g, h \in G_i : \\ x_g x_h = x_{gh} \text{ and } x_g = x_{\iota_i^j(g)} \end{array} \right\rangle$ and
- $\psi_i : G_i \to G : g \mapsto x_g$

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Theorem 1 (Tits 1974)

Let $G = \mathrm{SL}(n, \mathbb{R})$. Define the amalgam $\mathcal{A}$ via

- $G_{i,i+1} :=$ *block-diagonal* SL_2 *in rows and columns* i *and* $i+1$,
- $G_{i,i+1;j,j+1} :=$ *corresponding block-diagonal* SL_3 *or* $\mathrm{SL}_2 \times \mathrm{SL}_2$
- *and the natural embeddings.*

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- *and the natural embeddings.*

Then G together with the natural embeddings of the above groups form a universal enveloping group of $\mathcal{A}$.

Lie groups as universal enveloping groups

Theorem 2 (Glöckner, Hartnick, K. 2010)

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Endow each $G_{i,i+1} \subset \mathbb{R}^{2\times 2}$ *with the usual (Lie group) topology and let* $\mathcal{O}$ *be the finest group topology on* G *that makes the embeddings*

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Theorems 1 and 2 work for arbitrary split semisimple Lie groups.

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Definition via colimits in the two-spherical situation

Definition 3

Let

- Δ be an arbitrary Dynkin diagram without label ∞,
- $(G_\alpha)_{\alpha \in \Delta}$ be a family of copies of $\mathrm{SL}_2(\mathbb{R})$,
- $(G_{\alpha\beta})_{\{\alpha,\beta\} \in \binom{\Delta}{2}}$ be a family of appropriate algebraically simply connected split real Lie groups, and
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Abramenko–Mühlherr (1997) established the above approach in the category of abstract groups using the simple connectedness of the complex of pairs of opposite simplices of two-spherical twin buildings.

Definition in the general situation

Definition 4

Let

- A be a generalized Cartan matrix and $\mathfrak{g}$ the corresponding (real) Kac–Moody algebra,
- Δ be the corresponding Dynkin diagram,
- $(\overline{G}_\alpha)_{\alpha\in\Delta}$ be a family of copies of $(\mathrm{P})\mathrm{SL}_2(\mathbb{R}) \subset \mathrm{Aut}(\mathfrak{g})$ integrated from the simple root spaces,
- $\overline{G} < \mathrm{Aut}(\mathfrak{g})$ the subgroup generated by the $(\overline{G}_\alpha)_{\alpha\in\Delta}$,
- $G \twoheadrightarrow \overline{G}$ a central extension lifting the rank-1 groups to $\mathrm{SL}_2(\mathbb{R})$'s and the torus to a direct product.

The (algebraically simply connected split real) *topological Kac–Moody group* G of type Δ/A is defined as this abstract group G endowed with the finest group topology making the embeddings of the $\mathrm{SL}_2(\mathbb{R})$'s (considered as Lie groups) continuous.

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- The big cell B_+B_- is open and dense and homeomorphic to $U_+ \times T \times U_-$. (Kac–Peterson 1980s and Hartnick–K.–Mars 2013)

Spin: Computing $\pi_1(G) \cong \pi_1(K)$

Let

- G be a topological Kac–Moody group for a graph Δ,
- K the group of fixed points of the Cartan–Chevalley involution,
- $G_{\alpha\beta} \cong \mathrm{SL}(3, \mathbb{R})$ one of the groups in the defining amalgam,
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\pi_4(K/K_{\alpha\beta}) &\to \pi_3(K_{\alpha\beta}) \to \pi_3(K) \to \pi_3(K/K_{\alpha\beta}) \\
&\to \pi_2(K_{\alpha\beta}) \to \pi_2(K) \to \pi_2(K/K_{\alpha\beta}) \\
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Note that

$$G/P^+_{\alpha\beta} \cong K/(T \cap K)K_{\alpha\beta}$$

by the Iwasawa decomposition.

Generalized flag manifolds

Ghatei–Horn–K.–Weiß 2017 constructed a spin double cover

$$\mathrm{Spin}(\Delta) \twoheadrightarrow K \qquad \text{(conjectured by Damour–Hillmann)}$$

via the gen. spin representations described by Hainke–K.–Levy 2015 based on the E10 example by Damour–Kleinschmidt–Nicolai and de Buyl–Henneaux–Paulot.

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Theorem 5 (Harring–K. 2022)

For simply-laced Δ, the covering map

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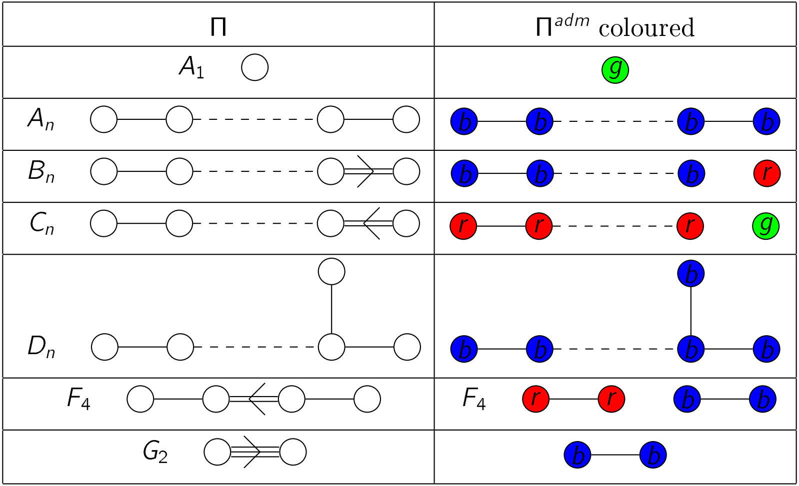
In the symmetrizable case, $\pi_1(G)$ is known.

The fundamental group of a real Kac–Moody group
Harring, K. 2022



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Abstract symmetric spaces following Loos

Definition 6 (Loos)

An abstract symmetric space (of non-compact type) is a set X with a multiplication

$$\mu : X \times X \to X \quad : \quad (x, y) \mapsto x \cdot y$$

satisfying the following axioms:

1. for each $x \in X$ one has $x \cdot x = x$,
2. for each pair of points $x, y \in X$ one has $x \cdot (x \cdot y) = y$,
3. for each triple of points $x, y, z \in X$ one has

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Example: For each group G the map $G \times G \to G : (x, y) \mapsto xy^{-1}x$ satisfies axioms 1, 2, 3.

One-parameter groups without C^1 hypothesis

Theorem 7 (Freyn, Hartnick, Horn, K. 2020)

Let (X, μ) *be a topological space with* continuous μ *satisfying axioms 1, 2, 3, 4.*

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- T_γ *acts sharply transitively on* γ *by Euclidean translations.*

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A *geodesic* $\gamma \subset X$ is defined to be the image of a bijection

$$\phi : \mathbb{R} \to \gamma$$

such that

$$\phi(2x - y) = \mu(\phi(x), \phi(y)) \quad \text{for all } x, y \in \mathbb{R}.$$

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Freyn, Hartnick, Horn, K. 2020 and others

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- G/K is the universal object of the amalgam of embedded sub-symmetric spaces of ranks 1 and 2. (Grüning–K. 2020)

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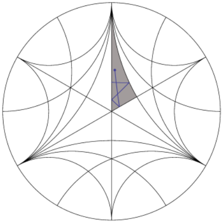
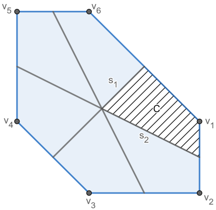
with $k_I \in K \cap \langle U_I, U_{-I} \rangle$ lies in the convex hull of $\{w.a \mid w \in W_I\}$.





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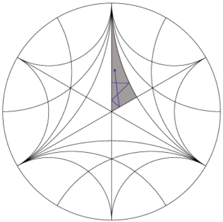

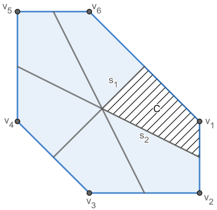

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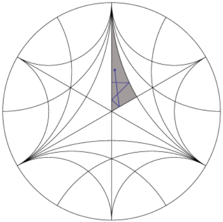

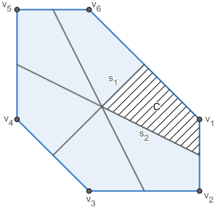

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Once this is done, there is no doubt this will generalize to Kac–Moody groups.