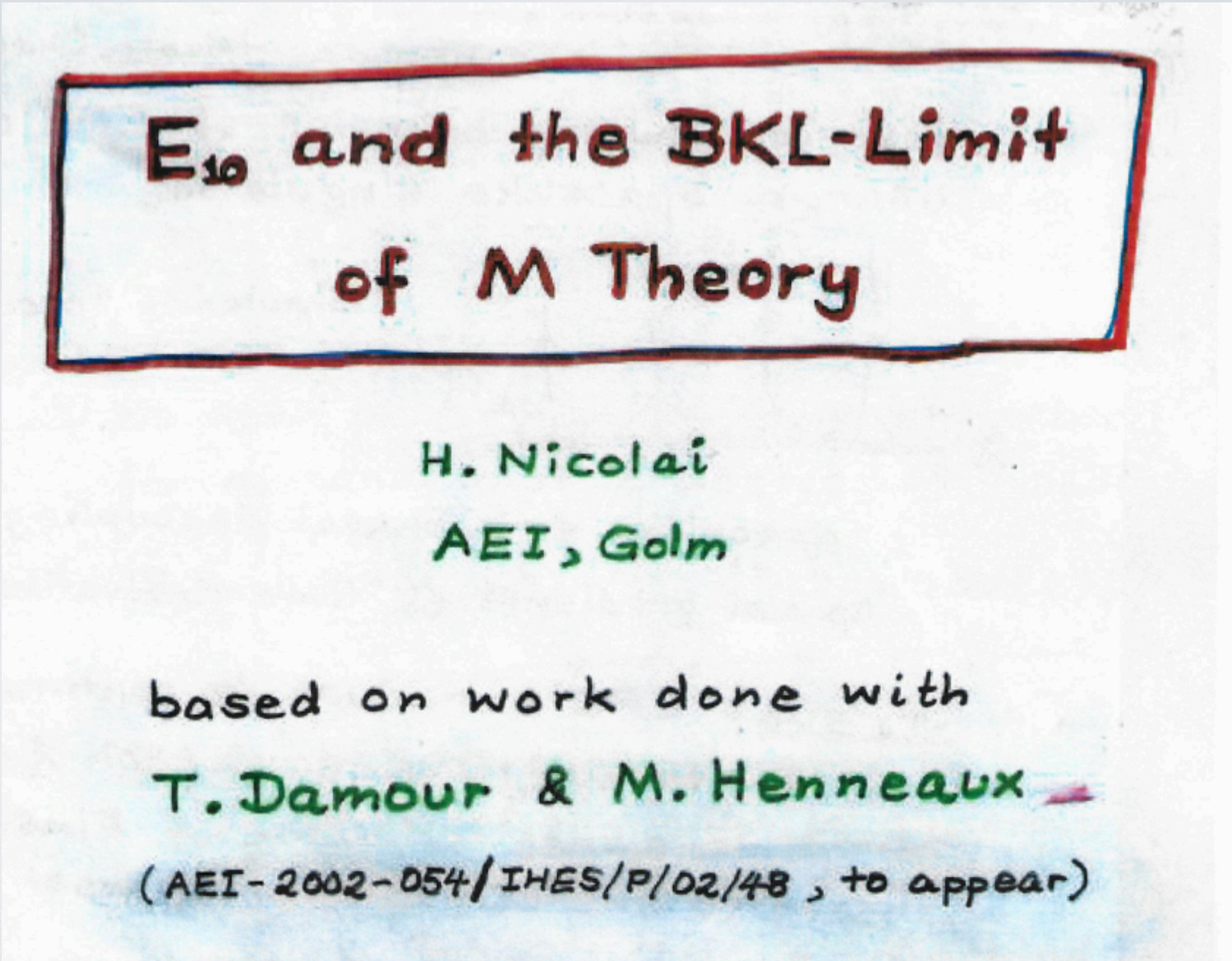
On the Wheeler–DeWitt equation and chaos

Axel Kleinschmidt @ HermannFest 15 Sep 2022

Based on work with Hermann Nicolai et al.

[AK, HN 2202.12676, JHEP]
[AK, HN, Palmkvist 1010.2212, ATMP]
[AK, Köhn, HN 0907.3048, PRD]
[Feingold, AK, HN 0805.3018, J. Algebra]

The E_{10} conjecture


E_{10} and the BKL-Limit of M Theory

H. Nicolai

AEI, Golm

based on work done with

T. Damour & M. Henneaux

(AEI-2002-054/IHES/P/02/48, to appear)

Strings 2002, Cambridge

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The E_{10} conjecture

Main Idea: search for map that relates the time evolution of the geometrical M Theory data at each spatial point [the fields and all their spatial gradients] to a null geodesic motion on the ∞-dimens coset space $E_{10}/K(E_{10})$.

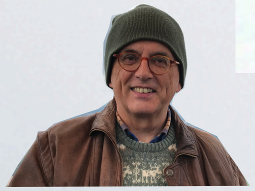
$G_{MN}(t, \mathbf{x})$

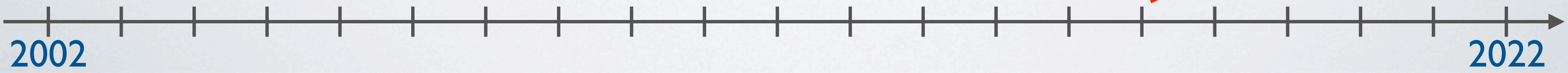
$A_{MNP}(t, \mathbf{x})$

cf. conjectured appearance of E_{10} in $D=1$ reduction of $D=11$ SUGRA

(Julia, 1983)

still a conjecture

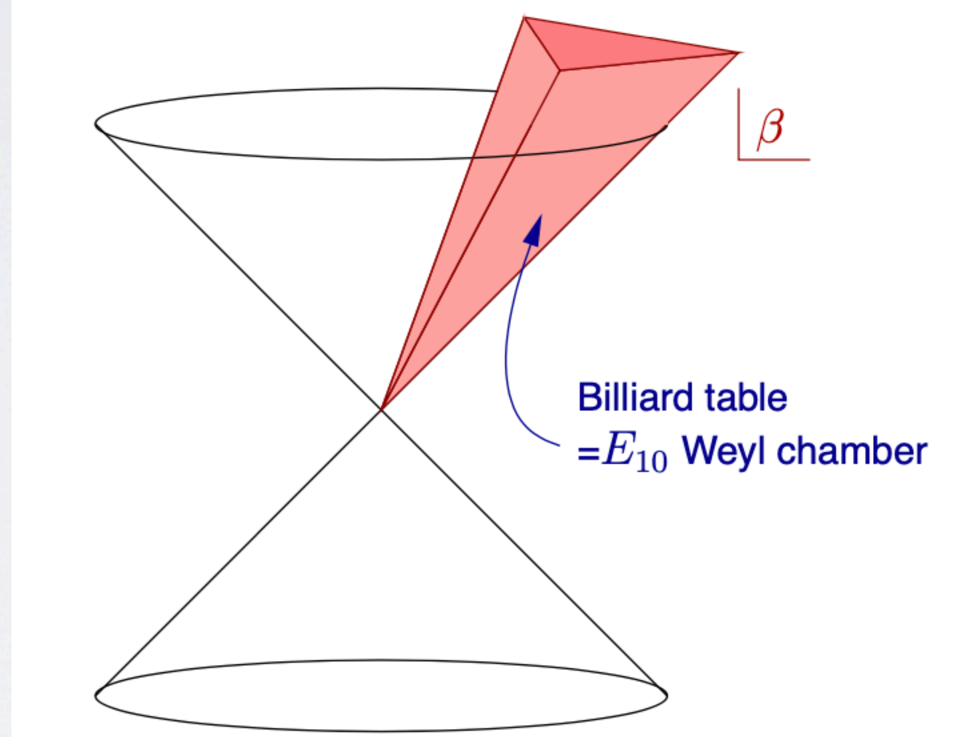




Simpler version: Cosmological billiards

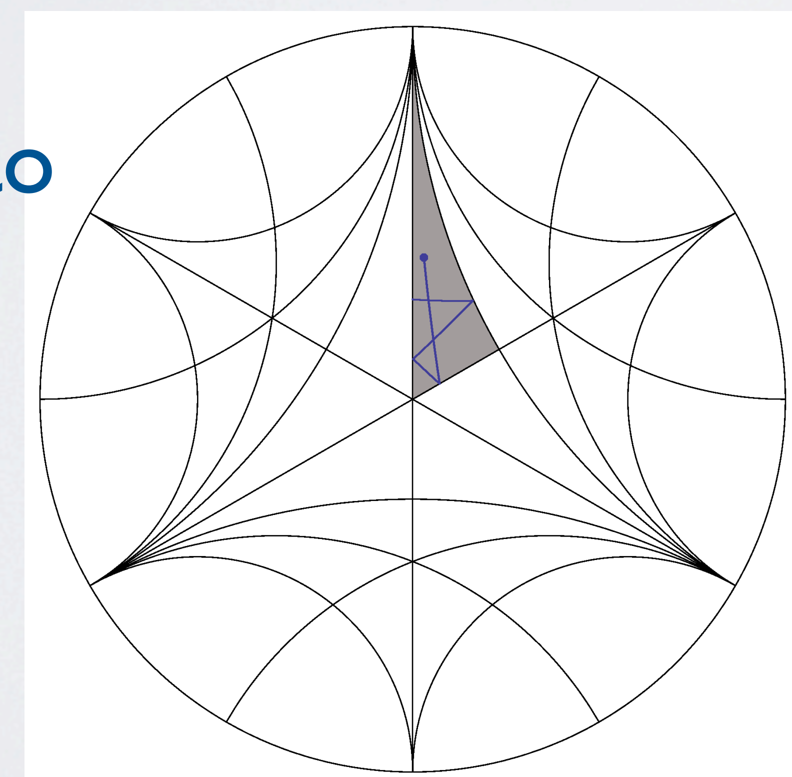
Replace infinite-dimensional symmetric space $E_{10}/K(E_{10})$ by its ten-dimensional Cartan subalgebra: Lorentzian signature $\mathbb{R}^{1,9}$, parametrised by "scale factors" β^a, undergoing free null motion...

Remnant of other supergravity fields near a space-like singularity:
Hard reflecting walls ⇒ billiard picture
Geometry that of E_{10} Weyl group



[Damour, Henneaux 2000]

Projection to hyperbolic space



Chaotic classical motion!
Cf. [BKL, Misner]

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Quantum cosmological billiards

What happens to this picture upon quantisation?

$\mathrm{II}^{\mathrm{CLX}}$

Rome 2009

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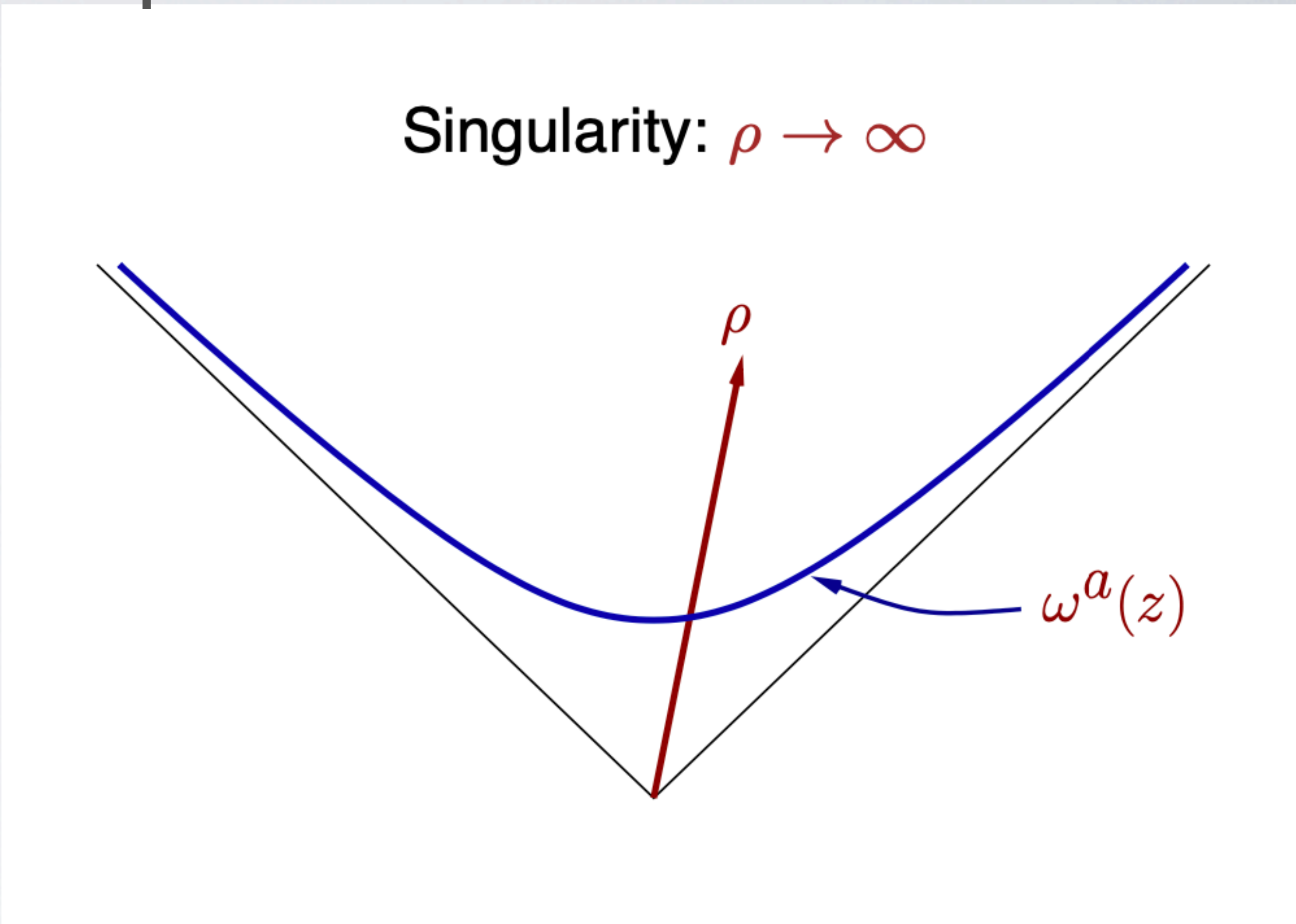
Quantum cosmological billiards

What happens to this picture upon quantisation?



Change coordinates

$$\beta^a = \rho\omega^a, \quad \omega^a G_{ab}\omega^b = -1$$

$$\rho^2 = -\beta^a G_{ab}\beta^b$$

Mini superspace wavefunction satisfies WDW equation (null motion)

$$\left[\rho^{1-d}\frac{\partial}{\partial\rho}\left(\rho^{d-1}\frac{\partial}{\partial\rho}\right) + \rho^{-2}\Delta_{\mathrm{LB}}\right]\Psi(\rho, z) = 0$$

think: $d = 10$ spatial dimensions

Laplace–Beltrami on unit hyperboloid

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Quantum cosmological billiards

$$\left[\rho^{1-d}\frac{\partial}{\partial\rho}\left(\rho^{d-1}\frac{\partial}{\partial\rho}\right)+\rho^{-2}\Delta_{\mathrm{LB}}\right]\Psi(\rho,z)=0$$

Separate variables: $\Psi(\rho,z)=R(\rho)F(z)$

For Laplace eigenfunction

$$-\Delta_{\mathrm{LB}}F(z)=EF(z)$$

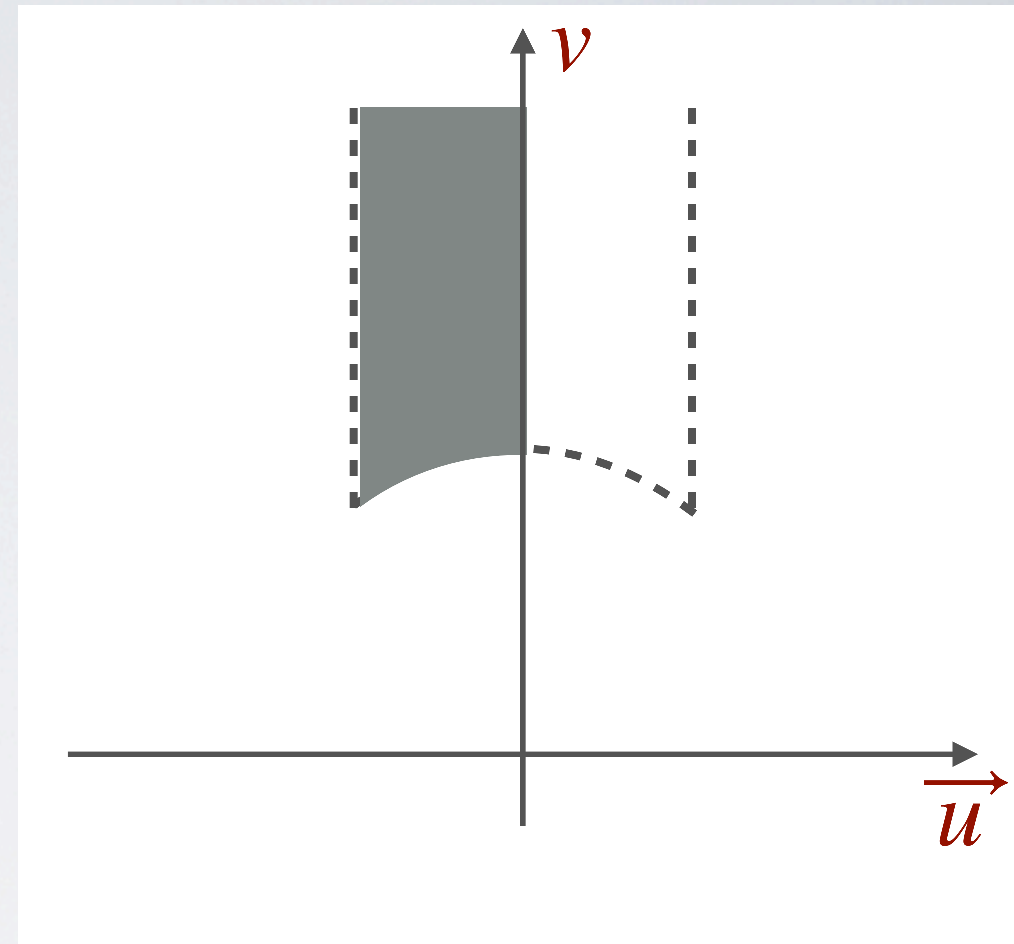
get

$$R_{\pm}(\rho)=\rho^{-\frac{d-2}{2}\pm i\sqrt{E-\left(\frac{d-2}{2}\right)^2}}$$

⇒ Left with spectral problem on unit hyperboloid...

Quantum cosmological billiards

Classical billiard ball is constrained to Weyl chamber
⇒ Dirichlet boundary condition



Use upper half-plane model of
$(d-1)$-dim'l hyperbolic space

$$z = (\vec{u}, v)\,, \quad \vec{u} \in \mathbb{R}^{d-2}, v \in \mathbb{R}_{>0}$$

$$\Rightarrow \quad -\Delta_{\mathrm{LB}} = v^{d-1}\partial_v(v^{3-d}\partial_v) + v^2\partial^2_{\vec{u}}$$

With Dirichlet boundary condition generalise [Iwaniec] to get

$$-\Delta_{\mathrm{LB}}F(z) = EF(z) \quad \Rightarrow \quad E \geq \left(\frac{d-2}{2}\right)^2$$

Quantum cosmological billiards

[w/ M. Köhn]

Recall $R_{\pm}(\rho) = \rho^{-\frac{d-2}{2} \pm i\sqrt{E - \left(\frac{d-2}{2}\right)^2}}$ in separation ansatz

$\Rightarrow$ Inequality $E \geq \left(\dfrac{d-2}{2}\right)^2$ implies that full wavefunction $\Psi(\rho, z)$ vanishes at singularity ($\rho \to \infty$), but remains oscillating and complex. Dilutes infinitely.

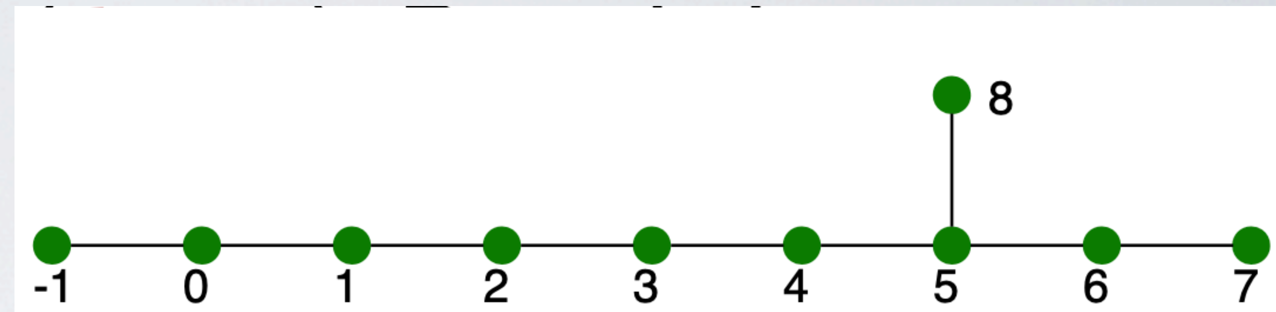
$\Rightarrow$ [DeWitt 1967] quantum-mechanical resolution of singularity?

Other details of the spectrum of allowed E depend on exact shape of billiard table

A mathematical curiosity

Shape of billiard table determined by E_{10} Weyl group acting on $z = (\vec{u}, v)$ with $\vec{u} \in \mathbb{R}^8$ [Damour, Henneaux]

Now $\mathbb{R}^8 \cong \mathbb{O}$ the octonions.
Write $z = u + iv$ with $u \in \mathbb{O}$.
The simple Weyl reflections are



$$w_{-1}(z) = \frac{1}{\bar{z}}\,, \quad w_0(z) = -\,\bar{z} + 1\,, \quad w_j(z) = -\,\varepsilon_j\,\bar{z}\,\varepsilon_j$$

oct. units

⇔ simple E_8 roots

ε_j span a lattice O $\subset \mathbb{O}$, the octavians.

Even part of Weyl group is $W_+(E_{10}) \cong PSL_2(\mathrm{O})$. [w/ A. Feingold] [w/ J. Palmkvist]

Mini superspace wavefunction $\Psi(\rho, z)$ is an odd Maaß form of $PSL_2(\mathrm{O})$ (oct. modular group). For similar ideas in 3+1, e.g. [Graham, Szépfalusy 1990][Perry 2021]

Beyond mini superspace

Go back to full E_{10} conjecture. The β^a are the diagonal components of the spatial metric
$\Rightarrow$ need to include other components and matter fields.

From level decomposition of E_{10} under GL_{10} get infinity of time-dependent fields

$$g_{mn}\,, \quad A_{n_1 n_2 n_3}\,, \quad A_{n_1 \dots n_6}\,, \quad h_{n_1 \dots n_8, m}\,, \quad \dots$$

spatial metric — 3-form — duality — 6-form — "dual graviton"

only fields in $D = 11$ (spatial metric, 3-form)

$\Rightarrow$ Try to find a form that uses all fields "democratically"

Beyond mini superspace

Use Bunster–Henneaux (a.k.a. Henneaux–Teitelboim) formalism for dual fields in field theory

- breaks manifest Lorentz covariance (good)
- Hamiltonian form (maybe good)
- explicitly solves some constraints but does not remove spatial dependence (not so good)

Starting point: Canonical form of $D = 11$ (super-)gravity

$$\mathscr{L}_{\text{can}} = \frac{1}{2}\dot{g}_{mn}\Pi^{mn} + \frac{1}{3!}\dot{A}_{mnp}\Pi^{mnp} - N\mathscr{H} - N^m\mathscr{H}_m - \frac{1}{2}A_{tmn}\mathscr{G}^{mn}$$

Conjugate momenta

Canonical constraints

$m, n, \ldots = 1, \ldots, 10$

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Beyond mini superspace

$$\mathscr{L}_{\text{can}} = \frac{1}{2}\dot{g}_{mn}\Pi^{mn} + \frac{1}{3!}\dot{A}_{mnp}\Pi^{mnp} - N\mathscr{H} - N^m\mathscr{H}_m - \frac{1}{2}A_{tmn}\mathscr{G}^{mn}$$

Focus on matter sector (to start with): Gauß constraint

$$\mathscr{G}^{mn} = -\,\partial_p\left[\Pi^{mnp} + \frac{1}{3\cdot 144}\varepsilon^{mnpk_1\ldots k_7}A_{k_1k_2k_3}F_{k_4\ldots k_7}\right] \stackrel{!}{=} 0$$

Solve locally by

$$\Pi^{mnp} + \frac{1}{3\cdot 144}\varepsilon^{mnpk_1\ldots k_7}A_{k_1k_2k_3}F_{k_4\ldots k_7} = \frac{1}{6!}\varepsilon^{mnpk_1\ldots k_7}\partial_{k_1}A_{k_2\ldots k_7}$$

brings in the dual six-form instead of Π^{mnp}!

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Beyond mini superspace

Resulting flat space action

$$\mathscr{L}_{\text{can}} = \frac{1}{2\cdot 3!\cdot 7!}\dot{A}_{mnp}\varepsilon^{mnpk_1\ldots k_7}F_{k_1\ldots k_7} - \frac{1}{2\cdot 4!\cdot 6!}F_{mnpq}\varepsilon^{mnpqk_1\ldots k_6}\dot{A}_{k_1\ldots k_6}$$

$$+\frac{1}{3!\cdot 864}\dot{A}_{mnp}\varepsilon^{mnpk_1\ldots k_7}A_{k_1k_2k_3}F_{k_4\ldots k_7}$$

$$-\frac{1}{2\cdot 4!}e^2F_{m_1\ldots m_4}g^{m_1n_1}\ldots g^{m_4n_4}F_{n_1\ldots n_4} - \frac{1}{2\cdot 7!}e^2F_{m_1\ldots m_7}g^{m_1n_1}\ldots g^{m_7n_7}F_{n_1\ldots n_7}$$

Treating this canonically gives conjugate momenta Π^{mnp} and $\Pi^{n_1\cdots n_6}$ and a mixed system of first- and second-class constraints.

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The E_{10} conjecture

After some work find Dirac brackets

$$\{F_{m_1\cdots m_4}(\mathbf{x}), F_{n_1\cdots n_4}(\mathbf{y})\}_{\mathrm{DB}} = 0$$

$$\{F_{m_1\cdots m_4}(\mathbf{x}), F_{n_1\cdots n_7}(\mathbf{y})\}_{\mathrm{DB}} = -7\,\varepsilon_{m_1\cdots m_4[n_1\cdots n_6}\partial_{n_7]}\delta(\mathbf{x},\mathbf{y})$$

$$\{F_{m_1\cdots m_7}(\mathbf{x}), F_{n_1\cdots n_7}(\mathbf{y})\}_{\mathrm{DB}} = -\frac{1}{432}\varepsilon_{m_1\cdots m_7 p_1 p_2 p_3}\varepsilon_{n_1\cdots n_7 p_4 p_5 p_6}\varepsilon^{p_1\cdots p_6 q_1\cdots q_4}F_{q_1\cdots q_4}(\mathbf{x})\delta(\mathbf{x},\mathbf{y})$$

[non-commutative variables]

Can be realised as operators

$$\hat{F}_{m_1\ldots m_4}(\mathbf{x}) = -\frac{2}{6!}i\hbar\,\varepsilon_{m_1\ldots m_4 n_1\ldots n_6}\frac{\delta}{\delta A_{n_1\ldots n_6}(\mathbf{x})} + \frac{10}{3}\partial_{[m_1}A_{m_2 m_3 m_4]}(\mathbf{x})$$

$$\hat{F}_{m_1\ldots m_7}(\mathbf{x}) = -\frac{2}{3!}i\hbar\,\varepsilon_{m_1\ldots m_7 n_1 n_2 n_3}\left(\frac{\delta}{\delta A_{n_1 n_2 n_3}(\mathbf{x})} + \frac{1}{12}A_{s_1 s_2 s_3}(\mathbf{x})\frac{\delta}{\delta A_{s_1 s_2 s_3 n_1 n_2 n_3}(\mathbf{x})}\right)$$

$$-\frac{7}{3}\partial_{[m_1}A_{m_2\ldots m_7]}(\mathbf{x}) + \frac{140}{3}A_{[m_1 m_2 m_3}\partial_{m_4}A_{m_5 m_6 m_7]}(\mathbf{x})$$

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Beyond mini superspace

$$\hat{F}_{m_1\ldots m_4}(\mathbf{x}) = -\frac{2}{6!} i\hbar\, \varepsilon_{m_1\ldots m_4 n_1\ldots n_6} \frac{\delta}{\delta A_{n_1\ldots n_6}(\mathbf{x})} + \frac{10}{3}\partial_{[m_1}A_{m_2m_3m_4]}(\mathbf{x})$$

Ignore...
(BKL)

$$\hat{F}_{m_1\ldots m_7}(\mathbf{x}) = -\frac{2}{3!} i\hbar\, \varepsilon_{m_1\ldots m_7 n_1 n_2 n_3} \left(\frac{\delta}{\delta A_{n_1n_2n_3}(\mathbf{x})} + \frac{1}{12} A_{s_1s_2s_3}(\mathbf{x}) \frac{\delta}{\delta A_{s_1s_2s_3n_1n_2n_3}(\mathbf{x})} \right)$$

$$-\frac{7}{3}\partial_{[m_1}A_{m_2\ldots m_7]}(\mathbf{x}) + \frac{140}{3} A_{[m_1m_2m_3}\partial_{m_4}A_{m_5m_6m_7]}(\mathbf{x})$$

Don't know how to relate spatial gradient terms to E_{10}

Implement Hamiltonian constraint $\hat{\mathscr{H}}\Psi = 0$ WDW equation

$$\left[\frac{1}{2\cdot 4!} e^2 F_{m_1\ldots m_4} g^{m_1n_1}\cdots g^{m_4n_4} F_{n_1\ldots n_4} + \frac{1}{2\cdot 7!} e^2 F_{m_1\ldots m_7} g^{m_1n_1}\cdots g^{m_7n_7} F_{n_1\ldots n_7} \right] \Psi = 0$$

Wavefunction $\Psi = \Psi(A_{n_1n_2n_3}, A_{n_1\ldots n_6}, \ldots)$ at fixed $\mathbf{x} = \mathbf{x}_0$

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Relation to E_{10} ?

Consider functional realisation of E_{10}

Functions Φ defined on symmetric space $E_{10}/K(E_{10})$.

Gives partial differential operators in $(g_{mn}, A_{n_1n_2n_3}, A_{n_1\ldots n_6}, \ldots)$ for generators of E_{10}.

Unique second order differential equation invariant under E_{10}

$$\Omega\Phi = 0$$

Ω is the quadratic Casimir of E_{10}. No ordering ambiguities

For the terms considered

Point of the talk!

$$\Omega\Phi = 0 \quad \Leftrightarrow \quad \hat{\mathscr{H}}\Psi = 0$$

E_{10} Casimir WDW equation

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Concluding remarks

- Consider also solutions of the equation. With modularity (U-duality type) get again a vanishing wavefunction at the singularity ⇒ robustness of quantum cosmological billiard
Subtleties for imaginary roots...

- Different view on quantisation:
functional derivatives go to partial derivatives
Not discretisation but relegation of spatial gradients to dual fields ⇒ time- and space-less WDW equation

- Some terms ignored ⇒ need to understand better how to incorporate them by using yet "higher" potentials

- Need to incorporate gravity fully, starting with the dual graviton

- For related work: [Damour, Spindel]

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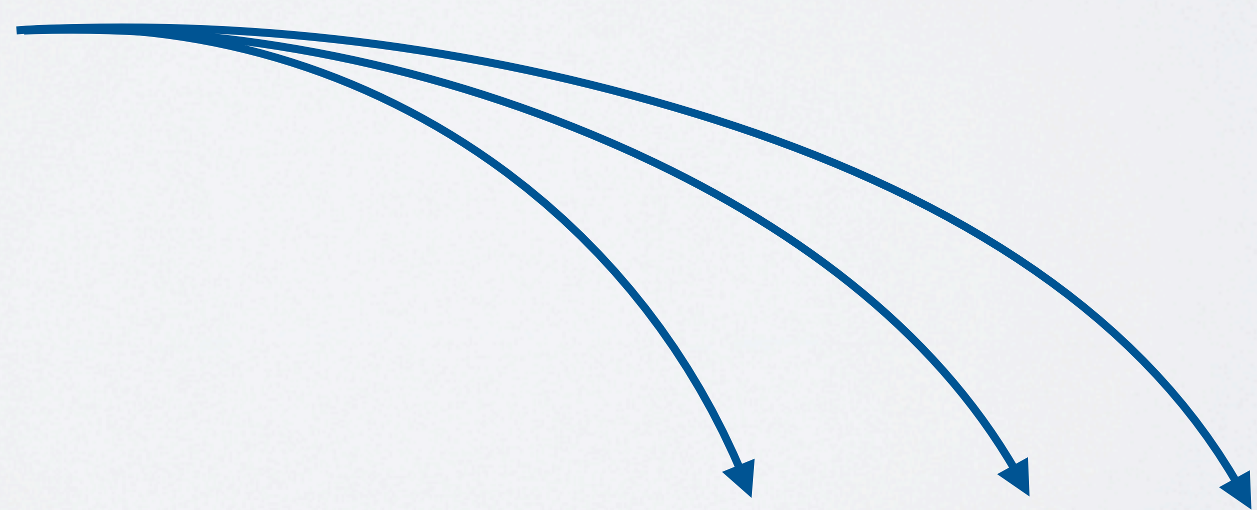
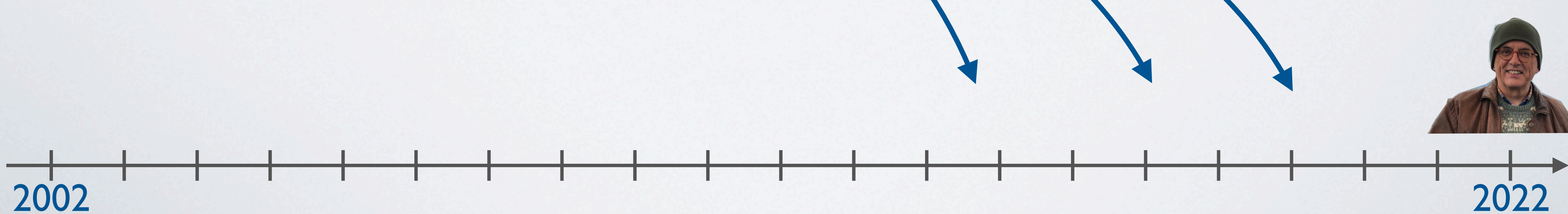
Open questions

- Is E_{10} supergravity or more? Is there a relation to the membrane?
- What about UV-finiteness? Quantisation and $E_{10}(\mathbb{Z})$?
- Do we learn anything about Kac–Moody algebras? Does the exponential growth of root multiplicities relate to an inaccessibility of the singularity?
- Does it all work?





Many happy returns, Hermann!