Affine Generalised Scherk–Schwarz reduction

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Motivations

Consistent truncations are powerful tools to compute solutions in higher dimensions.

- ⋆ Find AdS vacua
- ⋆ Solutions with fixed asymptotics
- ⋆ Kaluza–Klein spectrum [Malek–Samtleben]

First example

↪ $SO(8)$-gauged supergravity in four dimensions [de-Wit–Nicolai]
$\subset$ 11D supergravity on $\mathrm{AdS}_4 \times S^7$ [de-Wit–Nicolai–Pilch–Godazgar's]

Modern technique

↪ Generalised Scherk–Schwarz reduction [Hohm–Samtleben]

Motivations

- Generalisation to two dimensions

  ↪ Involves affine Kac–Moody E_9 [Julia–Nicolai]

  ↪ The D0-brane near horizon $\mathrm{AdS}_2 \times S^8 \times S^1$ [Nicolai–Samtleben]

- AdS_2 - Matrix quantum mechanics duality

  ↪ Membrane matrix model [de Wit–Hoppe–Nicolai]

D0-brane [Banks–Fischler–Shenker–Susskind] ⟷ Supergravity [Sekino–Yoneya]

- A large variety of other AdS_2 vacua

  ↪ Different phases of the Membrane matrix model

  ↪ Other matrix models

Motivations

Understand gauged maximal supergravities in two dimensions

↪ Gauging of the affine Kac–Moody algebra $\mathfrak{e}_9$

⋆ Supersymmetry analysis difficult because of $K(E_9)$

[Nicolai–Samleben,Kleinschmidt–Köhl–Lautenbacher–Nicolai]

⋆ Embedding tensor formalism [Samtleben–Weidner]

⋆ Potential only on $SO(9)$ example [Samtleben–Ortiz]

Exceptional field theory fixes the (pseudo)-Lagrangian uniquely

↪ Determines (geometric) gauged supergravity

Generalised Scherk–Schwarz

The internal metric in the Scherk–Schwarz ansatz decomposes

$$G_{MN}(x,y) = U^I{}_M(y) G_{IJ}(x) U^J{}_N(y)$$

for

$$U^{-1M}{}_I U^{-1N}{}_J (\partial_M U^K{}_N - \partial_N U^K{}_M) = f_{IJ}{}^K$$

solved by group manifold Lie derivative $\mathcal{L}_{E_I} = U^{-1M}{}_I \partial_M$.

Generalises to $E_d/K(E_d)$ symmetric matrix as

$$\mathcal{M}_{MN}(x,y) = \mathcal{U}^I{}_M(y) \mathcal{M}_{IJ}(x) \mathcal{U}^J{}_N(y)$$

with [Berman–Musaev–Thompson, Hohm–Samtleben]

$$r^{-1} \left(\mathcal{U}^{-1M}{}_I \mathcal{U}^{-1N}{}_J \partial_M \mathcal{U}^K{}_N \right) \Big|_{\Lambda_{9-d}} = -\Theta_{I\alpha} T^{\alpha K}{}_J$$

Affine extension

One defines the affine Kac–Moody algebra

$$
\begin{aligned}
[T^A_n, T^B_m] &= f^{AB}{}_C T^C_{n+m} + n\,\delta_{n+m}\eta^{AB} \\
[L_n, T^A_m] &= -m T^A_m \\
[L_n, L_m] &= (n-m)L_{n+m} + \frac{2}{3}n(n^2-1)\delta_{n+m}
\end{aligned}
$$

and

$$
\eta_{k\,\alpha\beta} T^\alpha \otimes T^\beta = \eta_{AB} \sum_n T^A_n \otimes T^B_{k-n} - L_k \otimes 1 - 1 \otimes L_k
$$

invariant under $\widehat{E_8}$ and L_k. We write $\eta_{\alpha\beta} = \eta_{0\,\alpha\beta}$.

Affine extension

One chooses a spectral flowed basis

$$
\begin{aligned}
[T^I_{m\,J}, T^K_{n\;L}] &= \delta^K_J T^I_{m+n\,L} - \delta^I_L T^K_{m+n\,L} + m\delta_{m+n}(\delta^I_J\delta^K_L - \tfrac{1}{9}\delta^i_L\delta^K_j) \\
[T^{I_1I_2I_3}_{m-p/3}, T_{n+p/3\,J_1J_2J_3}] &= 18\,\delta^{[I_1I_2}_{[J_1J_2}\,T^{I_3]}_{m+n\,J_3]} + 6\,(m - \tfrac{p}{3})\delta_{m+n}\delta^{I_1I_2I_3}_{J_1J_2J_3} \\
[T^{I_1I_2I_3}_{m-p/3}, T^{I_4I_5I_6}_{n-p/3}] &= -\frac{1}{6}\varepsilon^{I_1\dots I_9}\,T_{m+n-2p/3\,I_7I_8I_9} \\
[T_{m+p/3\,I_1I_2I_3}, T_{n+p/3\,I_4I_5I_6}] &= \frac{1}{6}\varepsilon_{I_1\dots I_9}\,T^{I_7I_8I_9}_{m+n+2p/3}
\end{aligned}
$$

The $\widehat{\mathfrak{sl}}_9$ subalgebras are conjugate for all

⋆ $p = 0 \bmod 3$ ⤳ $SL(9) \subset E_8$ with $\mathfrak{e}_8 = \mathfrak{sl}_9 \oplus \mathbf{84} \oplus \overline{\mathbf{84}}$

⋆ $p = \pm 1 \bmod 3$.

⋆ $p = 1$ arises in dimensional reduction on T^9.

⋆ $p = -1$ arises in the $S^8 \times S^1$ background.

Basic module

The highest weight module obtained from the E_8 invariant vacuum

$$|\Lambda\rangle = \big(\lambda^0 + \lambda^1_A T^A_{-1} + \lambda^2_{AB} T^A_{-1} T^B_{-1} + \dots\big)|0\rangle$$

with λ^2_{AB} in $\mathbf{1} \oplus \mathbf{248} \oplus \mathbf{3875}$.

E_9 Generalised diffeomorphisms

$$\mathcal{L}_{\Lambda,\Sigma}|V\rangle = \langle\partial_V|\Lambda\rangle|V\rangle - \eta_{\alpha\beta}\langle\partial_\Lambda|T^\alpha|\Lambda\rangle T^\beta|V\rangle - \langle\partial_\Lambda|\Lambda\rangle|V\rangle \\ - \eta_{-1\alpha\beta}\mathrm{Tr}[T^\alpha\Sigma]T^\beta|V\rangle$$

In the $SL(9)$ ($p=1$) decomposition

$$|\Lambda\rangle = \big(\xi^I + \tfrac{1}{2}\lambda_{JK}T_{1/3}^{IJK} + \tfrac{1}{6}\lambda^{*IJKL}T_{2/3\,JKL} + \xi^{*IJ}{}_K T_1^K{}_J + \ldots\big)|0\rangle_I$$

The derivative and constrained ancillary parameter on section

$$\langle\partial| = \langle 0|^I\partial_I \qquad \Sigma = \big(\Sigma^{*J}{}_I + \ldots\big)|0\rangle_J\langle 0|^I$$

and

$$\begin{aligned}\mathcal{L}_{\Lambda,\Sigma}|V\rangle &= \xi^I\partial_I|V\rangle - \partial_I\xi^J T_0^I{}_J|V\rangle + \partial_I\xi^I(L_0 - \tfrac{5}{9})|V\rangle \\ &\quad -\frac{1}{2}\partial_I\lambda_{JK}T_{1/3}^{IJK}|V\rangle - \frac{1}{6}\partial_I\lambda^{*IJKL}T_{2/3\,JKL}|V\rangle \\ &\quad -(\partial_K\xi^{*KJ}{}_I + \Sigma^{*J}{}_I)T_1^I{}_J|V\rangle + \ldots\end{aligned}$$

The coset fields and currents

One defines the E_9 matrix

$$\mathcal{M} = \Big(\rho^{-L_0} e^{-\sigma} V_0 \prod_{n=1}^{\infty} \exp(Y_{nA} T^A_{-n})\Big)^\dagger \Big(\rho^{-L_0} e^{-\sigma} V_0 \prod_{n=1}^{\infty} \exp(Y_{nA} T^A_{-n})\Big)$$

that admits the current

$$\mathcal{J}_M = \mathcal{M}^{-1}\partial_M \mathcal{M} \rightsquigarrow \langle e^M| \otimes \mathcal{J}_M = \langle \mathcal{J}_\alpha| \otimes T^\alpha$$

that expands as

$$\langle \mathcal{J}_\alpha| \otimes T^\alpha = \sum_n \langle J^n_A| \otimes T^A_n + \langle J_0| \otimes L_0 + \langle J_K| \otimes 1$$

Shifted currents

The current expends as

$$\langle \mathcal{J}_\alpha | \otimes T^\alpha = \sum_n \langle J_A^n | \otimes T_n^A + \langle J_0 | \otimes L_0 + \langle J_K | \otimes 1$$

and the shifted current

$$\langle \mathcal{J}_\alpha^- | \otimes T^\alpha = \sum_n \langle J_A^n | \otimes T_{n-1}^A + \langle J_0 | \otimes L_{-1} + \langle \chi | \otimes 1$$

with $\langle \chi |$ satisfies section constraint

$$\langle \chi | = \langle 0 |^I \chi_I$$

and transforms indecomposably in $\langle J_A^n |$ under $\mathfrak{e}_9$.

The potential

One finds the invariant ‘potential’

$$\rho V_{\text{ExFT}} = \tfrac{1}{4}\eta^{\alpha\beta}\langle \mathcal{J}_\alpha | \mathcal{M}^{-1} | \mathcal{J}_\beta\rangle - \rho^{-1}\langle \partial\rho | T^\alpha \mathcal{M}^{-1} | \mathcal{J}_\alpha\rangle - \tfrac{1}{2}\langle \mathcal{J}_\alpha | T^\beta \mathcal{M}^{-1} T^{\alpha\dagger} | \mathcal{J}_\beta\rangle + \tfrac{1}{2}\rho^2 \langle \mathcal{J}^-_\alpha | T^\beta \mathcal{M}^{-1} T^{\alpha\dagger} | \mathcal{J}^-_\beta\rangle$$

that transforms as

$$\mathcal{L}_{\Lambda,\Sigma} V_{\text{ExFT}} = \langle\partial|\Big(|\Lambda\rangle V_{\text{ExFT}}\Big) + \langle\partial|\Big(\rho^{-1}|\Lambda\rangle\langle\partial_\Lambda|\mathcal{M}^{-1}|\partial_\Lambda\rangle - \rho^{-1}\mathcal{M}^{-1}|\partial_\Lambda\rangle\langle\partial_\Lambda|\Lambda\rangle\Big)$$

and matches both eleven dimensional and type IIB supergravity.

Generalised Scherk–Schwarz

We take

$$\mathcal{M}(x,Y) = \mathcal{U}^\dagger(Y) M(x) \mathcal{U}(Y)\,, \quad \rho(x,Y) = r(Y)\varrho(x)$$

and define the $\mathfrak{e}_9$-valued Weitzenböck connection

$$\langle \mathcal{W}_\alpha | \otimes T^\alpha = r^{-1} \langle e^M | \mathcal{U}^{-1} \otimes \partial_M \mathcal{U}\, \mathcal{U}^{-1}$$

with

$$\langle \mathcal{W}_\alpha | \otimes T^\alpha = \sum_n \langle W_A^n | \otimes T_n^A + \langle W_0 | \otimes L_0 + \langle W_K | \otimes 1$$

$$\langle \mathcal{W}^\pm_\alpha | \otimes T^\alpha = \sum_n \langle W_A^n | \otimes T_{n\pm 1}^A + \langle W_0 | \otimes L_{\pm 1} + \langle w^\pm | \otimes 1$$

$\langle w^\pm |$ are constrained and appear in the ansatz of $\langle \chi |$ such that

$$\langle \mathcal{J}^-_\alpha | \mathcal{U}^{-1} \otimes \mathcal{U} T^\alpha \mathcal{U}^{-1} = \langle \mathcal{W}^-_\alpha | \otimes T^\alpha + \varrho^{-2} \langle \mathcal{W}^+_\alpha | \otimes M^{-1} T^{\alpha\dagger} M$$

Gauged algebra

The gauge parameters

$$|\Lambda\rangle = r^{-1}\mathcal{U}^{-1}|\lambda\rangle\,, \quad \Sigma = r\,\mathcal{U}^{-1}\,T^{\alpha}|\lambda\rangle\langle\mathcal{W}^{+}_{\alpha}|\,\mathcal{U}\,,$$

satisfy

$$r\,\mathcal{U}\left(\mathcal{L}_{|\Lambda\rangle,\Sigma}\,r^{-1}\mathcal{U}^{-1}|v\rangle\right) = \eta_{-1\,\alpha\beta}\langle\theta|\,T^{\alpha}|\lambda\rangle\,T^{\beta}|v\rangle + \eta_{\alpha\beta}\langle\vartheta|\,T^{\alpha}|\lambda\rangle\,T^{\beta}|v\rangle$$

with the embedding tensor

$$\langle\theta| = -\langle\mathcal{W}^{+}_{\alpha}|\,T^{\alpha}\,, \quad \langle\vartheta| = \langle\mathcal{W}_{\alpha}|\,T^{\alpha}\,.$$

The potential

Using

$$\mathcal{M}(x,Y)=\mathcal{U}^\dagger(Y)M(x)\mathcal{U}(Y)\,,\quad \rho(x,Y)=r(Y)\varrho(x)$$

in the exceptional field theory potential one gets

$$\begin{aligned}\frac{\varrho}{r}V_{\rm ExFT}=&\frac{1}{2\varrho^2}\langle\theta|M^{-1}|\theta\rangle-2\langle\vartheta|M^{-1}|\vartheta\rangle-\langle\tau|M^{-1}|\theta\rangle\\&+\frac12\eta_{-2\,\alpha\beta}\langle\theta|T^\alpha M^{-1}T^{\beta\dagger}|\theta\rangle+\eta_{-1\,\alpha\beta}\langle\theta|T^\alpha M^{-1}T^{\beta\dagger}|\vartheta\rangle+\frac12\eta_{\alpha\beta}\langle\vartheta|T^\alpha M^{-1}T^{\beta\dagger}|\vartheta\rangle\\&+r^{-1}\langle\partial|\Big(\mathcal{U}^{-1}(T^\beta M^{-1}-M^{-1}T^{\beta\dagger})|\mathcal{W}_\beta\rangle\Big)+\varrho^2\bigg(\frac12\langle\tau|M^{-1}|\tau\rangle\\&+\frac12\eta_{\alpha\beta}\langle\tau|T^\alpha M^{-1}T^{\beta\dagger}|\tau\rangle-2\eta_{-1\,\alpha\beta}\langle\vartheta|T^\alpha M^{-1}T^{\beta\dagger}|\tau\rangle-\eta_{-2\,\alpha\beta}\langle\theta|T^\alpha M^{-1}T^{\beta\dagger}|\tau\rangle\\&+2\eta_{-2\,\alpha\beta}\langle\vartheta|T^\alpha M^{-1}T^{\beta\dagger}|\vartheta\rangle+2\eta_{-3\,\alpha\beta}\langle\theta|T^\alpha M^{-1}T^{\beta\dagger}|\vartheta\rangle+\frac12\eta_{-4\,\alpha\beta}\langle\theta|T^\alpha M^{-1}T^{\beta\dagger}|\theta\rangle\bigg)\end{aligned}$$

with $\langle\tau|=\langle\mathcal{W}^-_\alpha|T^\alpha$ determined from $|\theta\rangle$ and $|\vartheta\rangle$.

The potential

For Lagrangian gaugings one has $\langle\vartheta| = 0$ and one gets

$$\frac{\varrho}{r} V_{\text{ExFT}} = \frac{1}{2\varrho^2}\langle\theta|M^{-1}|\theta\rangle + \frac{1}{2}\eta_{-2\,\alpha\beta}\langle\theta|T^\alpha M^{-1} T^{\beta\dagger}|\theta\rangle + \frac{1}{2}\varrho^2\eta_{-4\,\alpha\beta}\langle\theta|T^\alpha M^{-1} T^{\beta\dagger}|\theta\rangle$$
$$+r^{-1}\langle\partial|\Big(\mathcal{U}^{-1}(T^\beta M^{-1} - M^{-1} T^{\beta\dagger})|\mathcal{W}_\beta\rangle\Big)\,.$$

Giving the supergravity potential

$$V_{\text{gauged}} = \frac{1}{2\varrho^3}\langle\theta|M^{-1}|\theta\rangle + \frac{1}{2\varrho}\eta_{-2\,\alpha\beta}\langle\theta|T^\alpha M^{-1} T^{\beta\dagger}|\theta\rangle + \frac{1}{2}\varrho\,\eta_{-4\,\alpha\beta}\langle\theta|T^\alpha M^{-1} T^{\beta\dagger}|\theta\rangle$$

Compactification on S^4

Consistant truncation with $SO(5)$ invariant $\mathrm{AdS}_7 \times S^4$ vacuum

- ★ Twist matrix $(U^I{}_0, U^I{}_j) \in SL(5)$ [Hohm–Samtleben]
  - ↪ Coordinates $Y^{IJ} \supset Y^{0i}$ and $g_{ij}, \varepsilon^{ij_1j_2j_3}A_{j_1j_2j_3} \in SL(5)$

Sphere Killing vectors

$$K^i_{IJ} = \det g^{\frac{1}{2}-\frac{2}{5}} U^{-1\,0}{}_{[I} U^{-1i}{}_{J]}$$

solving in particular

$$U^{-1\,0}{}_{(I}\partial_i U^{-1i}{}_{J)} - U^{-1i}{}_{(I}\partial_i U^{-1\,0}{}_{J)} = -4r\lambda\delta_{IJ}$$

Compactification type IIB on S^5

Consistant truncation with $SO(6)$ invariant $\mathrm{AdS}_5 \times S^5$ vacuum

- ⋆ Twist matrix $(U^I{}_0, U^I{}_j) \in SL(6) \subset E_6$ [Hohm–Samtleben]
  - ↪ Coordinates $Y^M \supset Y^{IJ} \supset Y^{0i}$ and $g_{ij}, \varepsilon^{ij_1 \dots j_4} C_{j_1 \dots j_4} \in SL(6)$

Sphere Killing vectors

$$K^i_{IJ} = \det g^{\frac{1}{2}-\frac{2}{6}} U^{-1\,0}{}_{[I} U^{-1\,i}{}_{J]}$$

solving in particular

$$U^{-1\,0}{}_{(I} \partial_i U^{-1\,i}{}_{J)} - U^{-1\,i}{}_{(I} \partial_i U^{-1\,0}{}_{J)} = -5 r \lambda \delta_{IJ}$$

Compactification on S^7

Consistant truncation with $SO(8)$ invariant $\mathrm{AdS}_4 \times S^7$ vacuum

[de-Wit–Nicolai–Pilch–Godazgar's]

★ Twist matrix $(U^I{}_0, U^I{}_j) \in SL(8) \subset E_7$ [Hohm–Samtleben]

↪ Coordinates $Y^M \supset Y^{IJ} \supset Y^{0i}$ and $g_{ij}, \varepsilon^{ij_1\dots j_6} A_{j_1\dots j_6} \in SL(8)$

Sphere Killing vectors

$$K^i_{IJ} = \det g^{\frac{1}{2} - \frac{2}{8}} U^{-1\,0}{}_{[I} U^{-1\,i}{}_{J]}$$

solving in particular

$$U^{-1\,0}{}_{(I} \partial_i U^{-1\,i}{}_{J)} - U^{-1\,i}{}_{(I} \partial_i U^{-1\,0}{}_{J)} = -7 r \lambda \delta_{IJ}$$

Compactification on S^8

Type IIA supergravity on S^8 with $\star F_2$ flux

$\hookrightarrow$ Purely gravitational solution in eleven dimensions

$\hookrightarrow$ Twist matrix $\mathcal{U}$ valued in $\widehat{SL(9)}$

- ⋆ $SL(5) \supset SO(5)$ for S^4 compactification
  - $\hookrightarrow$ Coordinates $Y^{IJ} \supset Y^{0i}$ and $g_{ij}, \varepsilon^{ij_1j_2j_3} A_{j_1j_2j_3} \in SL(5)$
- ⋆ $E_6 \supset SL(6) \supset SO(6)$ for S^5 compactification
  - $\hookrightarrow$ Coordinates $Y^M \supset Y^{IJ} \supset Y^{0i}$ and $g_{ij}, \varepsilon^{ij_1\ldots j_4} C_{j_1\ldots j_4} \in SL(6)$
- ⋆ $E_7 \supset SL(8) \supset SO(8)$ for S^7 compactification
  - $\hookrightarrow$ Coordinates $Y^M \supset Y^{IJ} \supset Y^{0i}$ and $g_{ij}, \varepsilon^{ij_1\ldots j_6} A_{j_1\ldots j_6} \in SL(8)$
- ⋆ $E_9 \supset SL(9) \supset SO(9)$ for S^8 compactification
  - $\hookrightarrow$ Coordinates $Y^M \supset Y^{IJ} \supset Y^{0i}$ and $g_{ij}, \varepsilon^{ij_1\ldots j_7} C_{j_1\ldots j_7} \in SL(9)$

Compactification on $S^8 \times S^1$

Conjugate such that $\langle\partial|$ reads in the (p=-1) $SL(9)$

$$\langle\partial| = \langle 0|_J T_{1/3}^{0iJ}\partial_i = \langle\tfrac{1}{3}|^{0i}\partial_i$$

We take

$$\mathcal{U}^{-1} = r^{L_0} e^s\, U^{-1}$$

the embedding tensors

$$\langle\vartheta| = -r^{-1}\partial_i\left(r^{\frac{7}{9}} e^s U^{-1\,0}{}_J U^{-1\,i}{}_K\right)\langle\tfrac{1}{3}|^{JK} = 0$$

and

$$\begin{aligned}\langle\theta| = {} & r^{-\frac{2}{9}} e^s U^{-1\,0}{}_K U^{-1\,i}{}_L \partial_i U^S{}_I U^{-1\,I}{}_R \left(\langle\tfrac{1}{3}|^{[KL} T_{1\,S}^{R]} - \frac{2}{7}\langle\tfrac{1}{3}|^{Q[K} T_{1\,Q}^{L}\delta_S^{R]}\right)\\ & + \frac{1}{8} r^{-\frac{2}{9}} e^s\left(U^{-1\,0}{}_K\partial_i U^{-1\,i}{}_L - U^{-1\,i}{}_K\partial_i U^{-1\,0}{}_L + w_9^+\, U^{-1\,0}{}_K U^{-1\,0}{}_L\right)\langle\tfrac{1}{3}|^{P(K} T_{1\,P}^{L)}\\ & + \frac{9}{14} r^{-\frac{16}{9}} e^s\partial_i\left(r^{\frac{14}{9}} U^{-1\,0}{}_K U^{-1\,i}{}_L\right)\langle\tfrac{1}{3}|^{KL} L_1 = -\frac{1}{56}\Theta_{IJ}\langle\tfrac{1}{3}|^{K(I} T_{1\,K}^{J)}\end{aligned}$$

Scherk–Schwarz solution

Introduce $SO(9)$ embedding coordinates with metric η_{IJ} for $\Theta_{IJ} = \lambda\delta_{IJ}$

$$\delta_{IJ}\mathcal{Y}^I\mathcal{Y}^J = 1$$

The $SL(9)$ twist matrix components are taken to be

$$U^{-1\,i}{}_I = |\mathrm{det} g|^{\frac{1}{9}}\left(g^{ij}\partial_j\mathcal{Y}_I + c^i\mathcal{Y}_I\right)\,,$$
$$U^{-1\,0}{}_I = |\mathrm{det} g|^{-\frac{7}{18}}\mathcal{Y}_I\,,$$

and

$$r = |\mathrm{det} g|^{\frac{1}{2}}\,,\quad e^s = \frac{\lambda}{7}|\mathrm{det} g|^{\frac{7}{18}}\,.$$

where $g_{ij} = \delta^{IJ}\partial_i\mathcal{Y}_I\partial_j\mathcal{Y}_J$ and c^i the 7-form potential

$$g^{ij}\partial_i\mathcal{Y}_I\partial_j\mathcal{Y}_J = \delta_{IJ} - \mathcal{Y}_I\mathcal{Y}_J\,,\quad |\mathrm{det} g|^{-\frac{1}{2}}\partial_i\left(|\mathrm{det} g|^{\frac{1}{2}}g^{ij}\partial_j\mathcal{Y}_I\right) = -8\mathcal{Y}_I$$

and

$$|\mathrm{det} g|^{-\frac{1}{2}}\partial_i\left(|\mathrm{det} g|^{\frac{1}{2}}c^i\right) + |\mathrm{det} g|^{\frac{1}{2}}w_9^+ = 7$$

Scherk–Schwarz solution

With the embedding tensor

$$\langle\theta| = -\frac{1}{56}\Theta_{IJ}\langle 0|^{KI}\, T_{1\,K}^{J}\,,$$

and the coset component

$$\mathcal{V}^{-1} = \cdots\; e^{h^I{}_J T^I_{-1\,J}}\, e^{\frac{1}{6}a^{IJK}\,T_{-1/3\,IJK}}\,\varrho^{L_0}\, e^{\sigma}\, V^{-1}$$

$$\begin{aligned}\langle\theta|\mathcal{V}^{-1} = \frac{1}{56}\Theta_{IJ}e^{\sigma}\Big(& V^{-1\,I}{}_A V^{-1\,J}{}_B\langle 0|_C\, T_{1/3}^{CDA}\, T_{1\,D}^{B} - 8\varrho^{-1}h^I{}_K V^{-1\,J}{}_A V^{-1\,K}{}_B\langle 0|_C\, T_{1/3}^{ABC} \\ & + 28\varrho^{-1/3}a^{IKL}V^{-1\,J}{}_A V^B{}_K V^C{}_L\langle 0|_B\, T_{1\,C}^{A} + 56\varrho^{-4/3}a^{IKL}h^J{}_L V^A{}_K\langle 0|_A \\ & - 7\varrho^{-2/3}a^{IKL}a^{JPQ}V^A{}_K V^B{}_L V^C{}_P V^D{}_Q\langle 0|_A\, T_{2/3\,BCD} \\ & - \frac{1}{36}\varrho^{-1}a^{IK_1K_2}a^{JK_3K_4}a^{K_5K_6K_7}\varepsilon_{K_1\ldots K_7RS}V^{-1\,R}{}_A V^{-1\,S}{}_B\langle 0|_C\, T_{1/3}^{ABC} \\ & - \frac{7}{144}\varrho^{-4/3}a^{IK_1K_2}a^{JK_3K_4}a^{K_5K_6K_7}a^{K_8K_9L}\varepsilon_{K_1\ldots K_9}V^A{}_L\langle 0|_A\Big)\,,\end{aligned}$$

The potential

$$
\begin{aligned}
V_{\text{gauged}} = \frac{e^{2\sigma}}{2\varrho^3}\Theta_{IJ}\Theta_{KL}\Bigg(& \left(2M^{IK}M^{JL} - M^{IJ}M^{KL}\right) + \frac{1}{2}\varrho^{-2/3}\left(a^{IPQ}a^{KRS}M^{JL}M_{PR}M_{QS} - 2a^{IKP}a^{JLQ}M_{PQ}\right) \\
& + 2\varrho^{-2}h^I{}_P h^K{}_Q M^{Q[P}M^{J]L} + \varrho^{-8/3}a^{IPR}h^J{}_P a^{KQS}h^L{}_Q M_{RS} \\
& + \frac{\varrho^{-2}}{72}h^J{}_P a^{KQ_1Q_2}a^{LQ_3Q_4}a^{Q_5Q_6Q_7}\varepsilon_{Q_1\ldots Q_9}M^{IQ_8}M^{PQ_9} \\
& + \frac{3}{8}\varrho^{-4/3}a^{I[M_1M_2}a^{M_3M_4]J}a^{K[N_1N_2}a^{N_3N_4]L}M_{M_1N_1}M_{M_2N_2}M_{M_3N_3}M_{M_4N_4} \\
& + \frac{\varrho^{-2}}{2\cdot 144^2}a^{IN_1N_2}a^{JN_3N_4}a^{N_5N_6N_7}\varepsilon_{N_1\ldots N_9}a^{KP_1P_2}a^{LP_3P_4}a^{P_5P_6P_7}\varepsilon_{P_1\ldots P_9}M^{N_8P_8}M^{N_9P_9} \\
& + \frac{\varrho^{-8/3}}{576}a^{IRP}h^J{}_R a^{KN_1N_2}a^{LN_3N_4}a^{N_5N_6N_7}a^{N_8N_9Q}\varepsilon_{N_1\ldots N_9}M_{PQ} \\
& + \frac{\varrho^{-8/3}}{1152^2}a^{IN_1N_2}a^{JN_3N_4}a^{N_5N_6N_7}a^{N_8N_9Q}\varepsilon_{N_1\ldots N_9}a^{KP_1P_2}a^{LP_3P_4}a^{P_5P_6P_7}a^{P_8P_9S}\varepsilon_{P_1\ldots P_9}M_{QS}\Bigg)
\end{aligned}
$$

Matches Ortiz–Samtleben for $\Theta_{IJ} = \lambda\delta_{IJ}$

$\hookrightarrow$ Nicolai–Samtleben solution $M^{IJ} = \delta^{IJ}$, $a^{IJK} = 0$, $h^I{}_J = 0$

$$
\varrho = \left(\frac{5\lambda\ell}{2}\right)^{\frac{9}{2}} z^{-\frac{9}{5}}\,, \qquad e^{\sigma} = \ell z^{-\frac{7}{5}}
$$

Conclusions

- E_9 exceptional field theory
  - ↪ Pseudo–Lagrangian, extended Virasoro formulation $L_{n\leq 0}$
- Generalised Scherk–Schwarz ansatz
  - ↪ 2D Gauged supergravities that uplift to 11D or type IIB
  - ↪ Kinetic term from the pseudo-Lagrangian [B.-C.-I.-K. to come]
  - ↪ Fate of the Breitenlohner–Maison integrability?
- Gauged supergravity in two dimensions
  - ↪ The potential for non-geometric gaugings?
- AdS_2-Matrix quantum mechanics duality
  - ↪ Membrane matrix model [de Wit–Hoppe–Nicolai]

D0-brane [Banks–Fischler–Shenker–Susskind] ⟷ Supergravity [Sekino–Yoneya]

Dorfman product and Leibniz algebra

The Dorfman product for $\mathbb{X}_i = (|\Lambda_i\rangle, \Sigma_i)$ [Hohm–Samtleben for E_8]

$$\mathbb{X}_1 \circ \mathbb{X}_2 = \Big(\mathcal{L}_{\mathbb{X}_1} \Lambda_2 \ , \ \ \mathcal{L}_{\mathbb{X}_1} \Sigma_2 + \eta_{1\,\alpha\beta} \langle \partial_{\Lambda_1} | T^\alpha | \Lambda_1 \rangle T^\beta | \Lambda_2 \rangle \langle \partial_{\Lambda_1} | + \eta_{\alpha\beta} \mathrm{Tr}(T^\alpha \Sigma_1) T^\beta | \Lambda_2 \rangle \langle \partial_{\Sigma_1} | - | \Lambda_2 \rangle \langle \partial_{\Sigma_1} | \Sigma_1 \Big)$$

that obeys the Leibniz property

$$\mathbb{X}_1 \circ (\mathbb{X}_2 \circ \mathbb{X}_3) = (\mathbb{X}_1 \circ \mathbb{X}_2) \circ \mathbb{X}_3 + \mathbb{X}_2 \circ (\mathbb{X}_1 \circ \mathbb{X}_3)$$

and the algebra

$$[\mathcal{L}_{\mathbb{X}_1}, \mathcal{L}_{\mathbb{X}_2}] = \mathcal{L}_{\mathbb{X}_1 \circ \mathbb{X}_2}$$

Gauged algebra

The generalised frame

$$\langle \mathbb{E}| = \left(r^{-1}\mathcal{U}^{-1}, r\,\mathcal{U}^{-1}T^{\alpha} \otimes \langle \mathcal{W}^{+}_{\alpha}|\mathcal{U} \right)$$

satisfy the Dorfman product

$$\mathbb{E}_A \circ \mathbb{E}_B = -X_{AB}{}^{C}\mathbb{E}_C \;\; = \Theta_{A\alpha} T^{\alpha C}{}_{B}\mathbb{E}_C$$

i.e.

$$\langle \mathbb{E}| \circ \langle \mathbb{E}| = \left(\eta_{-1\,\alpha\beta} \langle \theta| T^{\alpha} + \eta_{\alpha\beta}\langle \vartheta| T^{\alpha} \right) \otimes \langle \mathbb{E}| T^{\beta}$$

provided $\langle \theta|$ and $\langle \vartheta|$ are constant.

$\hookrightarrow$ implies quadratic constraint

$$\begin{aligned} \eta_{-1\,\alpha\beta}\langle \theta| T^{\alpha} \otimes \langle \theta| T^{\beta} + \eta_{\alpha\beta}\langle \vartheta| T^{\alpha} \otimes \langle \theta| T^{\beta} + \langle \vartheta| \otimes \langle \theta| - \langle \theta| \otimes \langle \vartheta| &= 0 \\ \eta_{-1\,\alpha\beta}\langle \theta| T^{\alpha} \otimes \langle \vartheta| T^{\beta} + \eta_{\alpha\beta}\langle \vartheta| T^{\alpha} \otimes \langle \vartheta| T^{\beta} &= 0 \end{aligned}$$

Generalised diffeomorphisms

- ⋆ Diffeomorphisms based on $SL(d)$: $\partial_N Y^M \in GL(d)$

$$\mathcal{L}_\Lambda V^M = \Lambda^N \partial_N V^M - \partial_N \Lambda^M V^N$$

- ⋆ Diffeomorphisms based on $SO(d,d)$: $\partial_N Y^M | \mathbb{R}_+ \times SO(d,d)$

$$\mathcal{L}_\Lambda V^M = \Lambda^N \partial_N V^M - (\partial_N \Lambda^M - \eta_{PN} \eta^{MQ} \partial_Q \Lambda^P) V^N$$

- ⋆ Diffeomorphisms based on G: $\partial_N Y^M | \mathbb{R}_+ \times G$

$$\mathcal{L}_\Lambda V^M = \Lambda^N \partial_N V^M - \eta_{\alpha\beta} (T^\alpha)^P{}_Q \partial_P \Lambda^Q (T^\beta)^M{}_N V^N - (1 - (\lambda, \lambda)) \partial_N \Lambda^N V^M$$

where Y^M are in the representation of highest weight λ.

Generalised diffeomorphisms

- ⋆ Diffeomorphisms based on G: $\partial_N Y^M | \mathbb{R}_+ \times G$

$$\mathcal{L}_\Lambda V^M = \Lambda^N \partial_N V^M - \eta_{\alpha\beta} (T^\alpha)^P{}_Q \partial_P \Lambda^Q (T^\beta)^M{}_N V^N - \big(1 - (\lambda, \lambda)\big) \partial_N \Lambda^N V^M$$

For affine Kac–Moody ($|V\rangle$ is a lattice CFT state)

$$\mathcal{L}_\Lambda |V\rangle = \langle \partial_V | \Lambda \rangle |V\rangle - \eta_{\alpha\beta} \langle \partial_\Lambda | T^\alpha | \Lambda \rangle T^\beta |V\rangle - \langle \partial_\Lambda | \Lambda \rangle |V\rangle$$

Generalised diffeomorphism

- ⋆ $SL(d)$ diffeomorphisms close on all functions
- ⋆ $SO(d,d)$ diffeomorphisms close on section [Hohm–Hull–Zwiebach]

$$\eta^{MN}\partial_M F\,\partial_N G = 0$$

- ⋆ G diffeomorphisms close on section [Berman–Perry,Cederwall–Edlund–Karlsson]

$$\eta_{\alpha\beta}(T^\alpha)^P{}_M(T^\beta)^Q{}_N\partial_P F\partial_Q G = \partial_N F\partial_M G + ((\lambda,\lambda)-1)\partial_M F\partial_N G$$

for $\mathfrak{g}_{+\lambda}$ finite dimensional and otherwise to ancillary transformations. [Cederwall–Palmkvist]

Generalised diffeomorphism

- ⋆ Affine Kac–Moody diffeomorphisms close on section

$$\eta_{\alpha\beta}\langle\partial_F|T^\alpha \otimes \langle\partial_G|T^\beta = \langle\partial_G| \otimes \langle\partial_F| - \langle\partial_F| \otimes \langle\partial_G|$$

including ancillary transformations

$$\mathcal{L}_{\Lambda,\Sigma}|V\rangle = \langle\partial_V|\Lambda\rangle|V\rangle - \eta_{\alpha\beta}\langle\partial_\Lambda|T^\alpha|\Lambda\rangle T^\beta|V\rangle - \langle\partial_\Lambda|\Lambda\rangle|V\rangle \\ - \eta_{-1\alpha\beta}\mathrm{Tr}[T^\alpha\Sigma]T^\beta|V\rangle$$