Physical Symmetries, group Actions and group Presentations

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Outline

- I Guessing (duality = « Cremmer-J. ») symmetries and more E_8's
- Hermann's support and collaborations

- II E_8 families: Invariants versus Root spaces
- Fermionic half dimensions and twisted self-duality

- III ADE correspondence to finite polyhedral groups
- Orbifolds
- Towards an affine ADE correspondence

Part I E_8 beginnings

In memory of Eugène Cremmer 1942-2019

- E_8 was first discovered in high energy physics by EC+BJ as a result of the identity 128+120 = 248; see [1]: Nucl.Phys. B159 (1979) 141
- The bosonic action of pure maximal supergravity in 3d with its E_8 symmetry was constructed by BJ in [2]: Proceedings of the International School of Cosmology and Gravitation, Erice 1982, Eds V. De Sabbata & E. Schmutzer, World Scientific 1983 p 215-235.
- Its 4d E_7 ancestor action was presented with fermions in [1] and discovered (first) using the identity 70+63 = 133!

Number of scalar fields + Dim KE = Dim E (both numbers were known)

KE : R-symmetry is the maximal compact subgroup of the U duality group E

S=E/KE and the fermionic middleman (middlegroup) in supergravity.

Number of scalar fields + Dim KE = Dim E

- The present understanding of E remains empirical but now encompasses symmetries of Einstein's gravity (Ehlers since 1980 BJ [3], Geroch BJ [4]…)
- The definition of KE_9 and KGeroch using the Cartan involution and the signature of the generalized Killing form was proposed by BJ in [4] 1981.
- Special features of reductions to two dimensions include:

  the infinite dimensionality of E of KE and of the number of scalar fields

  plus gauge descendants needed? and the emergence of a Witt symmetry

  Diff(S^1) BJ+H.Nicolai [6] on top of the naive duality symmetry expected

  from the growth of The gravity leg [3].
- In [5] 1982 BJ prompted by Feingold and Frenkel's hyperbolic construction proposed E_{10} duality in «one dimension». See BJ Nicolai et al [7] for chaos.

More E_8 intrusions:

- Cvitanovic's 1977 trapezoid. Other stimuli: 't Hooft Parisi Zumino Scherk Gürsey Neveu Gell-Mann (Ramond Garland Frenkel) Olive
- E_9 in the Painlevé I Okamoto space of initial conditions 1979, Sakai 2001
- E_8xE_8 Heterotic string 1985 (D_{16} type I is anomaly free 1984)
- (ADE Itzykson et al 1987 conformal classification)
- E_8 in Zamolodchikov's Ising mass spectrum in magnetic field 1989
- It is important to clarify the origins of E_7, E_8-Ehlers, E_9-Geroch, E_{10}-FF …
- First E_8-family (Cvitanovic 1977…, Deligne-Gross 2002)
- Second E_8-family [1,3], third triangle C,J,Lu and Pope hep-th/9909099
- The R-symmetry arm (leg)
- The twisted self-duality gravitational frontier

Some more references

- [3] Nuffield 1980
- [4] Johns Hopkins 1981
- [5] Chicago 1982 (Proceedings appeared unchanged in 1985)
- [6] +HN 1995
- [7] +HN 2001
- [8] +C,Lu and Pope 1999

I am delighted and most grateful to participate in this celebration of my resonant colleague Hermann Nicolai.

Happy birthday Hermann.

Our interactions resulted in three works still in use. So let me distinguish three periods in our overlap.

Bronze age: 1995 BJ argued to a Max Planck society operative that HN's program was interesting/important after having received a von Humboldt research prize thanks to him.

Iron age: 2022 were BJ to request serious financial support he might be well advised to argue that his program is related to some line of work of HN!

Middle age collaborations with Hermann

1994: Galilean gravitation from null reduction (Gibbons…).

1995 [6]: Twisted self-duality and Witt symmetry in 2d dilaton sector (BJ+Bernard…).

2001 [7]: Chaos in 2d reduced gravity, Feingold-Frenkel hyperbolic symetry (BKL after E10 + Damour+Henneaux)

Part II The main E8 families: Invariants/Roots

Cvitanovic Trapezoid: mT77

								Primitive invariants
$F_4(-52)$	$C_3(-21)$	$A_2(-8)$	$A_1(-3)$					S2,S3
$E_6(-78)$	$A_5(-35)$	$A_2^2(-16)$	$A_2(-8)$	2.U1				V^2,S3
$E_7(-133)$	$D_6(-66)$	$A_5(-35)$	$C_3(-21)$	$A_1^3(-9)$	$A_1(-3)$	U1	←	A2,S4 S2,A3
$E_8(-248)$	$E_7(-133)$	$E_6(-78)$	$F_4(-52)$	$D_4(-28)$	$G_2(-14)$	$A_2(-8)$	$B_1(-3)$	$A_1(-3)$

Faulkner-Ferrar
1971-77

Cvitanovic
1977-79-03

The last line on the previous transparency is the first E_8 family.

It follows an approach using invariant tensors. Key is also the decomposition of E8 by products of two commuting subgroups: E_8xe, E_7xA_1,E_6xA_2,F_4xG_2 (see Tits' magic square and A. Feingold) and D_4xD_4.

Remark: D_4 seems to be related to the Painlevé VI equation

Homework

Explain what simply laced means for a simple complex Lie algebra without using the idea of roots (eg B_n/D_n)

Next 3 slides are Backup

The list of Kervaire manifolds with invariant one is almost known.
(It is a relative of the Hopf invariant one fibration list but over Z_2)
The remaining open question is to decide whether there is one such Kervaire manifold of dimension 126 or not (2016).

Homework

Can you recognize the first three Kervaire manifolds in Cvitanovic’s trapezoid 45 years ago?

Gross-Deligne Magic triangle $H'=Z(H,E_8)$

i=8	j=0								e (i=8)
7	$H_i<K_j$	Triangle	$i\leq j$					e	$A_1(-3)$
6	$E_6(-78)$		$H'_i\cap K_j$				e	U_1	$A_2(-8)$
5	$F_4(-52)$					e	μ_3	$A_1(-3)$	$G_2(-14)$
4=j	$D_4(-28)$				e	μ_2^2	U_1^2	$A_1^3(-9)$	$D_4(-28)$
3-Real	$G_2(-14)$			e	μ_2^2	$A_1(-3)/\mu_2$	$A_2(-8)$	$C_3(-21)$	$F_4(-52)$
2-Cplex	$A_2(-8)$		e	μ_3	U_1^2	$A_2(-8)$	$A_2^2(-16)$	$A_5(-35)$	$E_6(-78)$
1-Quat.	$A_1(-3)$	e	U_1	$A_1(-3)$	$A_1^3(-9)$	$C_3(-21)$	$A_5(-35)$	$D_6(-66)$	$E_7(-133)$
0-Oct.	$e=A_0$	$A_1(-3)$	$A_2(-8)$	$G_2(-14)$	$D_4(-28)$	$F_4(-52)$	$E_6(-78)$	$E_7(-133)$	$E_8(-248)$

→ K>H

Split Magic triangle

hep-th/9909099 C.J.L.P.

Pure	Gravity	line:					A_0	A_1 or R	A_1orR?	1 7/0
						?	R	$A_1.R$	$A_1.R^{aff.}$	2 6/1
Trian	gle:	***8-Rrk***	≤11-D		?	A_1	$A_1.R$	$A_2.A_1$	$A_2.A_1^{af}$	3 5/2
Diago	nal:	D-3=***Rrk***		A_0?	R	$A_1.R$	$A_2.R$	A_4	$A_4^{aff?}$	4 4/3
			?	R	R^2	$A_1^2.R$	$A_3.A_1$	D_5	$D_5^{aff?}$	5 3/4
		?	A_1	$A_1.R$	$A_1^2.R$	$A_2.A_2$	A_5	E_6	$E_6^{aff?}$	6 2/5
	A_0	R	$A_1.R$	$A_2.R$	$A_3.A_1$	A_5	D_6	E_7	$E_7^{aff?}$	7 1/6
A_0	A_1 or R	$A_1.R$	$A_2.A_1$	A_4	D_5	E_6	E_7	E_8	$E_8^{aff?}$	8 0/8
0=11-D 8	1 7	2 6	3 5	4 4	5 3	6 2	7 1	8 0	9	***Rr_3*** 8-Rr_3 D-3 /***N_4***

MAGIC TRIANGLE Pure D=4 SUGRAs (Nuffield Cambridge)

Pure	Gr	avity	line:				A_0	A_1/U_1	$A_1^{aff?}$	1 7/0
		Trian	gle:				U_1/U_1	$A_1.U_1/$ $D_1.U_1$	$A_1R^{aff?}$	1 6/1
		8-Rr $_3$	≤10-D				U_2/U_2	$A_2.A_1/$ $D_2.U_1$	$A_2A_1^{aff}$	1 5/2
		D-2≤***Rr*** $_3$					U_3/U_3	A_4/D_3U_1	$A_4^{aff?}$	1 4/3
							$A_3.A_1/$ $A_3.U_1$	D_5/D_4U_1	$D_5^{aff?}$	2 3/4
							A_5/U_5	E_6/D_5U_1	$E_6^{aff?}$	2 2/5
					D_3A_1/B_2A_1	A_5/C_3	D_6/U_6	E_7/D_6A_1	$E_7^{aff?}$	4 1/6
A_0	R	A_1R/U_1	A_2A_1/A_1U_1	A_4/B_2	D_5/B_2B_2	E_6/C_4	E_7/A_7	E_8/D_8	$E_8^{aff?}$	8 0/8
0=11-D (torus)	1	2	3	4	5	6	7	8	9	Rr_3 7-N'_4 / N_4

The U duality arm

Instead of descending in spacetime dimension (dimensional reduction) one may increase supersymmetry. Supergravity leads to two not only to one E_8 family of real Lie algebras:

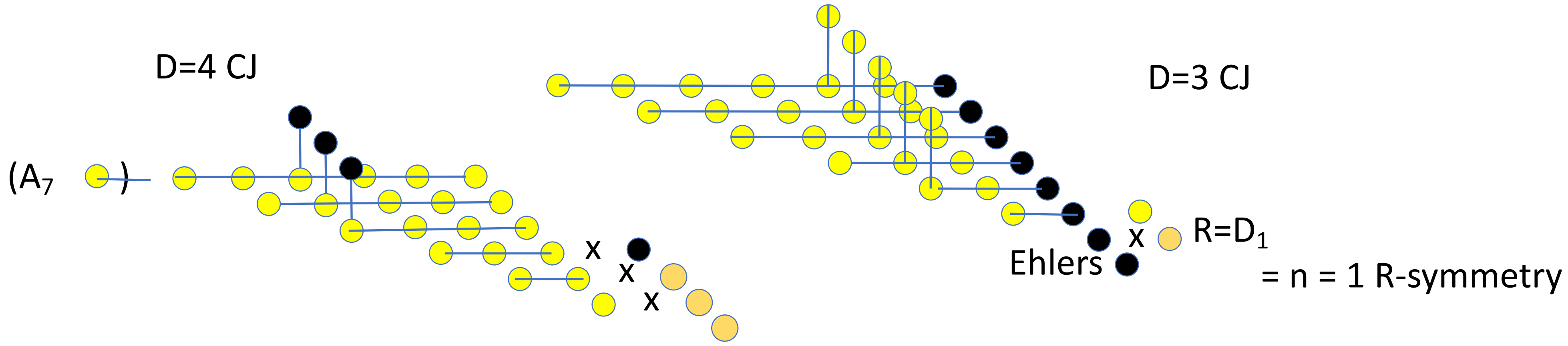
The split one (N=8 column ie gravity leg direction)

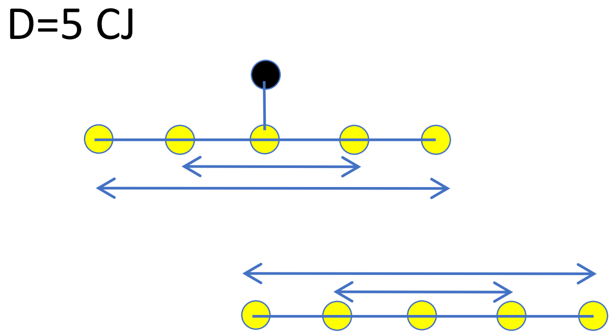
The non-split one (D=3 line ie the R-symmetry « arm » direction).

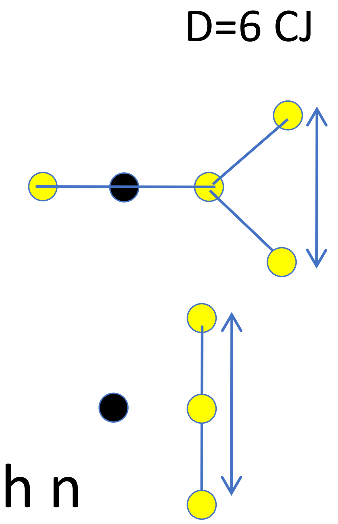
The next slide shows the linear growth of R-symmetry

Apologies: horizontal and vertical directions have been exchanged in the last triangle

Vogan diagrams and Susy (n) reduction: the R-symmetry leg







(compact) R symmetry (leg) increases regularly with n

Yellow is compact (x)
Blue non compact (o)
Beige is $R=D_1$

Twisted Self-duality (after doubling)

- EC+BJ 1979
- H.Nicolai+BJ 1995
- DB+BJ 1997
- EC+H.Lu+C.Pope+BJ 1998-1999
- M.Henneaux+J.Levie+BJ 2012

- VF=*T VF extends to **F=*T F**
- * is Hodge dual and T an internal duality = «twist»
- **F is the p-form curvature $V^{-1}dV$ (Metric is still missing)**

Doubling E_{10}, super-Borcherds and beyond

- Apparently doubling must now be modified; work is still going on.

  See papers of Jakob Palmkvist and collaborators (M.Cederwall,

  A. Kleinschmidt…) . This is another promising program.
- As far as gravity doubling is required one may guess that the number of space dimensions must be doubled.
- Note that this doubling would allow to recover an integer (double) dimension in the non simply laced cases like F_4 mentioned in next backup slide, is this wishful thinking?
- Both are postponed to question session!

Bosonic half dimensions (The imperial sweep M. Duff et al. 2014)

- In the pure supergravity (magic) triangle the (real) U-duality Lie algebras are simply laced (hence my conversion to ADE aficionado).
- However F_4 does appear in three dimensions by considering 3, 5 or 9 2-generator supersymmetries instead of 4-generator ones (deWit).
- It is part of a magic pyramid (Duff et al.)
- If the symmetry between space dimensions and -1/4 fermionic dimensions is to survive one is led to half odd integer space dimensions (BJ 2014 Stony Brook talk)

III ADE correspondences and orbifolds

- Thurston (after Macbeath) analysed surface (and « 3-manifold ») singularities and their topology in particular those of orbifolds.
- Group theory jumps to your face: an orbifold is locally surface/ finite group but in fact group theory is not so successful even with J Conway.

- Message: please stay orbifolded

ADE correspond to finite polyhedral groups

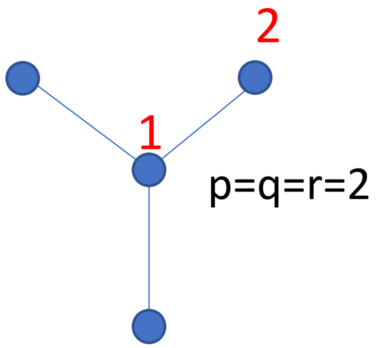
- List of orbifold symbols (14 spherical pattern types)
- 532 432 332 22N MN : S^2 quotients
- *532 *432 *332 *22N *MN and

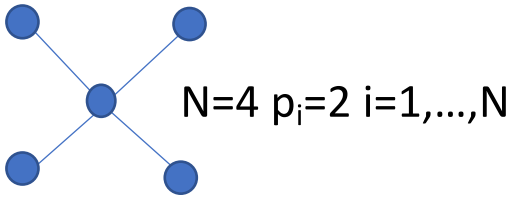
  3*2 2*N N* : D^2 quotients (*stands for boundary)
- Nx : P^2 quotients (x stands for a crosscap)

- The first line gives the finite subgroups of SO(3)
- The correspondence between the E8 Lie group and the icosahedral group I:= 532 can be seen by realizing the singularity C^2/I in the Lie group itself (Brieskorn and Grothendieck)

$T_{pqr} = S(p,q,r) \to S(p_i)$ (S=star, T=Triangle) Triangle/Star groups and Dynkin diagrams

- 1,2 is a p=2 branch
- D_4 , $D_4^{(1)}$: T_{222} , T_{333} S(2,2,2,2)
- E_6 , $E_6^{(1)}$: T_{233} T_{333}
- E_7 , $E_7^{(1)}$: T_{234} T_{244}
- E_8 , $E_8^{(1)}$: T_{235} T_{236}





But D_n ($D_n^{(1)}$ n>4 not a star)

Notations pqr or $p_1 p_2 p_3$ …

- $\sum_{i=1}^{K} (p_i-1)/p_i = 2$ for any number K of i's (or p_i's) more general than

- $\sum_{i=1}^{3} 1/p_i = 1$

Two more ideas to present on the DEEE quartet:

- $D_4^{(1)}$ is as exceptional as affine E_6, E_7 or E_8
- The Lie algebra /finite group ADE (related to Mackay's) correspondence) uses the previous quartet

Backup x 2

Presentations of polyhedral symmetry groups

3 types, given here for E_8

- « Icosahedral » group of order 60: presentation $R^2=S^3=T^5=RST=e$
- Including reflections the « Double icosahedral » group has order 120 and a presentation $(RR'')^2=(RR')^5=(R'R'')^3= e$ using 3 fundamental reflections, it is the Coxeter group H_3.
- « Binary icosahedral » subgroup of Spin 3, not of O(3), order 120. It may be presented by $R^2=S^3=T^5=RST$

  It is used to produce the corresponding singularity by orbifolding C^2

--The ubiquity of $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$ and $D_4^{(1)}$

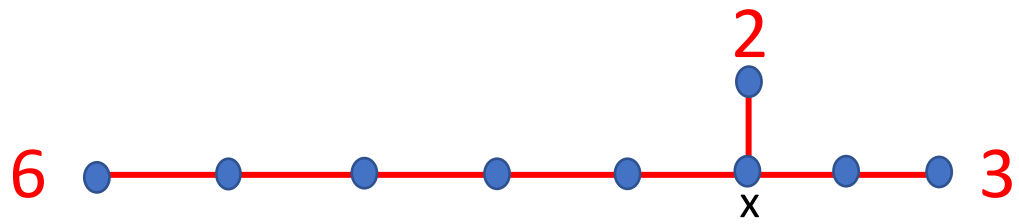
See Abstract: Integrable systems (Etingof), noncommutative geometry, elliptic Painlevé (Briot Bouquet binomial equation), quivers (Painlevé: Boalch), Hall algebras (Schiffmann)

--Abstract orbifolds

In Platonic E_8 spherical crystal
Icosahedral symmetry is noted
*532 (orbifold list)

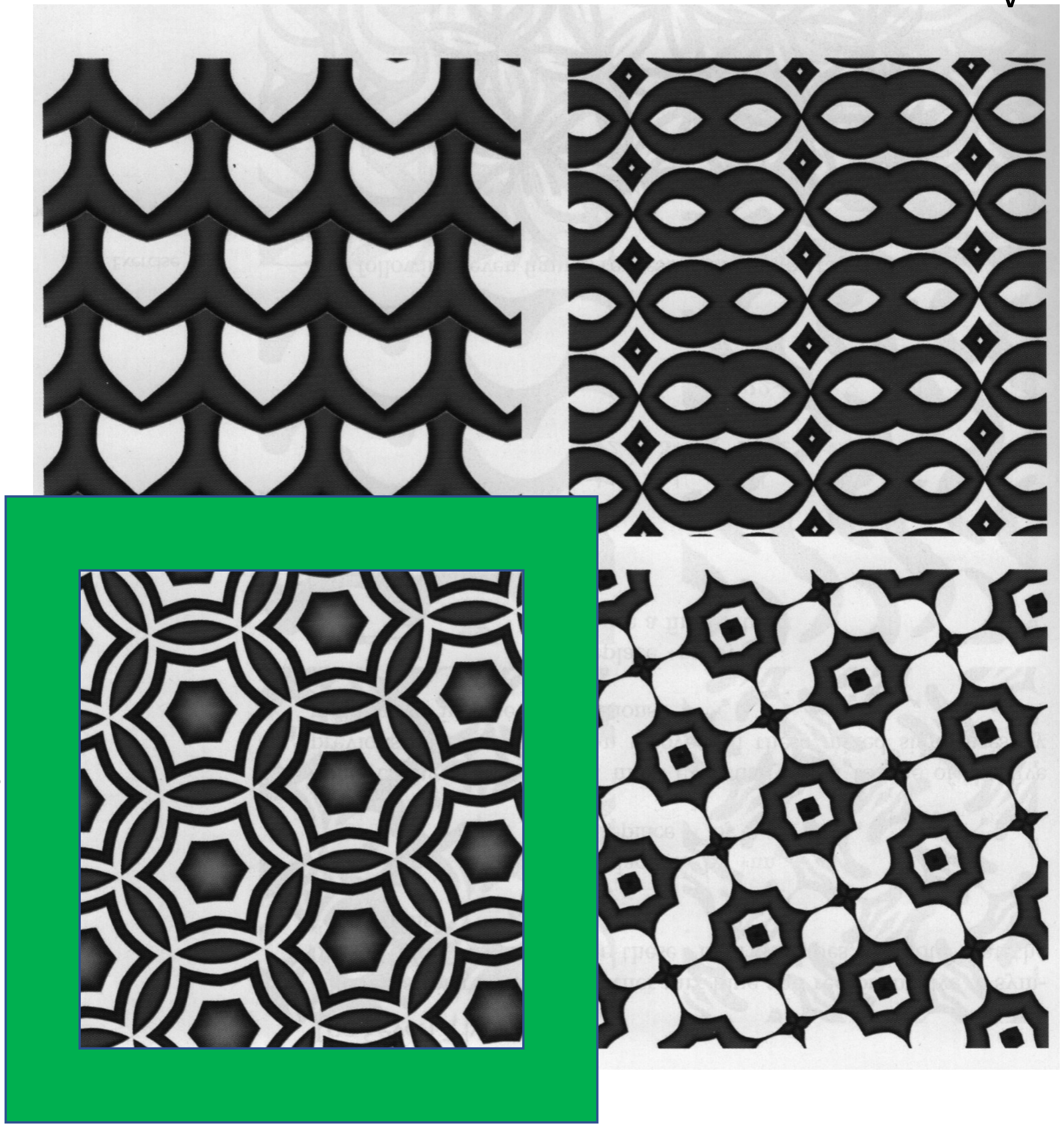
$D_4^{(1)}$ is exceptional too: *2222 Affine D_4

4 of the 17 plane crystals



Platonic E_9
*632

->



Towards an affine ADE correspondence

- List of 17 plane orbifold symbols (2d Crystals Polya…)
- 632 442 333 2222 : are S^2 quotients
- *632 *442 *333 *2222 and

  4*2 3*3 2*22 22* : are D^2 quotients
- 22x : is a P^2 quotient (unorientable)
- o : T^2 (handle)
- ** Annulus
- *x Mobius band
- xx Klein bottle

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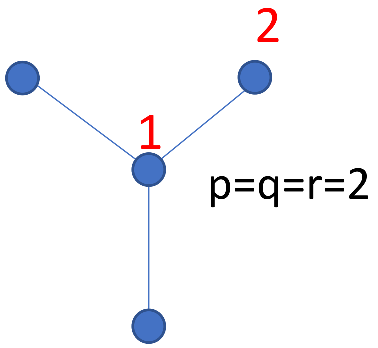

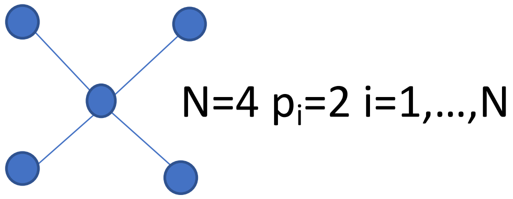

Presentation and correspondence

- 2222 noted C_2 in Polya's notation has a presentation by 4 generating reflections whose total product is trivial
- *2222 is noted D_{2kkkk} in Polya's notation and has a Coxeter diagram that is a square namely that of $A_3^{(1)}$ ie $D_3^{(1)}$ so it seems to be an affine Weyl group!
- By the affine ADE correspondence it should be related to $D_4^{(1)}$
- How? Some generalized singularity analysis in infinite dimension?

Conclusion

- Clearly a lot more work is needed and I look forward to even more fun.
- Happy century (= happy 20**ies) Hermann

Grand Unified Theories

- E6 d_{Adj}=78 rank=6
- SO(10) d_{Adj}=45 rank=5
- SU(5) d_{Adj}=24 rank=4

Homework:

- SU(3)xSU(2) d_{Adj}=11 rank=3
- SU(2)xU(1) d_{Adj}=4 rank=2
- SU(2) d_{Adj}=3 rank=1