A Lightcone Embedding of the Building For the Split Real Form of a Hyperbolic Kac-Moody Group

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Information

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Abstract

Let $G_{\mathbb{C}}$ be the complex minimal adjoint Kac-Moody (KM) group associated with a hyperbolic KM Lie algebra, $\mathfrak{g} = \mathfrak{g}(C)$, with $n \times n$ Cartan matrix $C = [c_{ij}]$. In [CFF], Carbone, Feingold and Freyn constructed an embedding of the twin building for $G_{\mathbb{C}}$ into the compact real form $\mathfrak{k}$ of $\mathfrak{g}$ which is generated by the subalgebras $\mathfrak{su}_2^i$, $1 \leq i \leq n$. The corresponding subgroups SU_2^i generate the compact real form K of $G_{\mathbb{C}}$, and their embedding map was only equivariant under the action of K.

Here we present a similar result embedding the twin building for the split real form $G_{\mathbb{R}}$ into $\mathfrak{p}$, where the split real form of the KM Lie algebra, $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k} \oplus \mathfrak{p}$, is decomposed into the ± 1 eigenspaces for the Cartan involution on $\mathfrak{g}_{\mathbb{R}}$. If $\mathfrak{g}$ is E_{10} then $G_{\mathbb{R}}/K_{\mathbb{R}}$ is of interest in physics, and $\mathfrak{p}$ can be viewed as its tangent space, reflecting its geometry.

Introduction

For about 20 years Hermann Nicolai has been my friend, co-author, inspiration, and research supporter through many visits to AEI. I have come to feel like AEI is my second home, a great place to learn and study important mathematics deeply related to physics. I want to express my deepest thanks and appreciation to Hermann, and all the friendly colleagues I have met here over the years.

This work extends the results of Carbone, Feingold and Freyn [CFF] from complex hyperbolic Kac-Moody (KM) groups to their split real forms. We believe there is value in understanding the action of a group on a set, and the theory of buildings was created by J. Tits in order to gain such understanding for groups with a BN pair. The theory of buildings (see [AB] and [RT]) has proven its value in many ways, and been extended to include twin buildings. This brief talk cannot include much background in the theory of buildings, but we hope it will give some idea of how buildings can be applied to the study of hyperbolic KM groups, a topic of great interest for Hermann Nicolai and me.

Kac-Moody Lie Algebra Background (see [K])

A complex KM Lie algebra $\mathfrak{g} = \mathfrak{g}_{\mathbb{C}}(C)$ with $n \times n$ Cartan matrix $C = [c_{ij}]$ is given by generators $\{e_i, f_i, h_i \mid 1 \leq i \leq n\}$ and relations

$$[h_i, h_j] = 0, \quad [h_j, e_i] = c_{ij} e_i, \quad [h_j, f_i] = -c_{ij} f_i, \quad [e_i, f_j] = \delta_{ij} h_i,$$

$$(ad_{e_i})^{1-c_{ij}}(e_j) = 0, \ i \neq j \qquad \text{and} \qquad (ad_{f_i})^{1-c_{ij}}(f_j) = 0, \ i \neq j,$$

where $ad_x(y) = [x, y]$. The scalars here are complex numbers in $\mathbb{C}$.

The standard *Cartan subalgebra* $\mathfrak{h}$ has basis $\{h_i \mid 1 \leq i \leq n\}$.

Under the adjoint action, $ad_h : \mathfrak{g} \to \mathfrak{g}$,
$\mathfrak{h}$ acts simultaneously diagonalizably on $\mathfrak{g}$.

The simultaneous eigenspaces

$$\mathfrak{g}_\alpha \ = \ \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x, \ h \in \mathfrak{h}\}$$

are labelled by certain linear functionals $\alpha \in \mathfrak{h}^*$.

In particular, defining $\alpha_i \in \mathfrak{h}^*$ by $\alpha_i(h_j) = c_{ij}$, we have

$$\mathfrak{g}_0 = \mathfrak{h}, \quad \mathfrak{g}_{\alpha_i} = \mathbb{C}e_i \quad \text{and} \quad \mathfrak{g}_{-\alpha_i} = \mathbb{C}f_i.$$

Kac-Moody Lie Algebra Background

Call $\alpha \in \mathfrak{h}^*$ a *root* when $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$, in which case $\mathfrak{g}_\alpha$ is called the α *root space*, and $Mult(\alpha) = \dim(\mathfrak{g}_\alpha)$ is called the *multiplicity* of root α.

Simple roots are $\Pi = \{\alpha_i \mid 1 \leq i \leq n\}$.

The set of all roots is $\Phi = \{\alpha \in \mathfrak{h}^* \mid \mathfrak{g}_\alpha \neq 0 \text{ and } \alpha \neq 0\}$.

We have the *Cartan decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \quad \text{and} \quad [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta} \ \text{ for } \alpha, \beta \in \Phi.$$

The *Weyl group* $\mathcal{W}$ is generated by the reflections $w_i : \mathfrak{h}^* \to \mathfrak{h}^*$ (or $w_i : \mathfrak{h} \to \mathfrak{h}$) where

$$w_i(\alpha_j) = \alpha_j - c_{ij}\alpha_i \quad (\text{or } w_i(h_j) = h_j - c_{ij}h_i), \qquad 1 \leq i \leq n.$$

$\mathcal{W}$ is an important group of symmetries of the root system Φ, and $Mult(\alpha) = Mult(w\alpha)$ for all $w \in \mathcal{W}$.

Complex Kac-Moody Group G

The *real roots* are the $\mathcal{W}$-orbit of Π,

$$\Phi^{re} = \mathcal{W}(\Pi) = \{w(\alpha_i) \mid w \in \mathcal{W},\ \alpha_i \in \Pi\}$$

and all other roots are called *imaginary*. For $\alpha \in \Phi^{re}$, $\dim(\mathfrak{g}_\alpha) = 1$.

The relations imply that for any $\alpha \in \Phi^{re}$ and $x \in \mathfrak{g}_\alpha$, the operator ad_x is locally nilpotent on $\mathfrak{g}$, so $exp(ad_x) \in GL(\mathfrak{g})$ is well-defined. This gives the (abelian) *real root group*

$$U_\alpha = \langle exp(ad_x) \in GL(\mathfrak{g}) \mid x \in \mathfrak{g}_\alpha \rangle.$$

The *minimal adjoint Kac–Moody group* is $G = \langle U_\alpha \mid \alpha \in \Phi^{re} \rangle$.

There is a nondegenerate invariant symmetric bilinear form $(\cdot, \cdot)$ on $\mathfrak{g}$, a generalization of the Killing form, which satisfies $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ unless $\alpha + \beta = 0$.

Compact Real Form K of Kac-Moody Group G

For $1 \leq j \leq n$, the subgroup $SL_2^j = \langle U_{\alpha_j}, U_{-\alpha_j} \rangle \leq G$ acting on $\mathfrak{h}$ fixes the hyperplane $L_{\mathfrak{h},j} = \{h \in \mathfrak{h} \mid \alpha_j(h) = 0\}$ pointwise, but takes the vector h_j outside of $\mathfrak{h}$.

The family of Cartan subalgebras $SL_2^j \cdot \mathfrak{h}$ has $L_{\mathfrak{h},j}$ in common.

The compact real form $\mathfrak{k}$ of $\mathfrak{g}$ is generated by the subalgebras $\mathfrak{su}_2^j$, each with basis (fixed by the Cartan-Chevalley involution)

$$x_j = \frac{1}{2}(e_j - f_j), \quad y_j = \frac{\mathbf{i}}{2}(e_j + f_j), \quad z_j = \frac{\mathbf{i}}{2}(h_j).$$

The subalgebra $\mathfrak{t} = \mathfrak{h} \cap \mathfrak{k}$ of $\mathfrak{k}$ has real basis $\{z_j \mid 1 \leq j \leq n\}$, and it has a Lorentzian geometry when C has signature $(n-1, 1)$.

The corresponding subgroups SU_2^j generate the compact real form K of G, and SU_2^j acting on $\mathfrak{t}$ fixes the hyperplane $L_{\mathfrak{t},j} = \{z \in \mathfrak{t} \mid \alpha_j(z) = 0\}$ pointwise, but takes z_j outside of $\mathfrak{t}$.

The family of subalgebras $SU_2^j \cdot \mathfrak{t}$ has $L_{\mathfrak{t},j}$ in common, and is indexed by a 2-sphere with antipodes identified.

Split Real Forms $\mathfrak{g}_{\mathbb{R}}$ and $G_{\mathbb{R}}$

The *split real form* $\mathfrak{g}_{\mathbb{R}}$ of KM Lie algebra $\mathfrak{g}$ is the real Lie algebra with the same generators and relations associated with Cartan matrix C. Changing the scalars from $\mathbb{C}$ to $\mathbb{R}$ we similarly get the *split real form* $G_{\mathbb{R}}$ of the KM group. Previous definitions of roots, root spaces, Weyl group, etc, are adjusted.

The Cartan-Chevalley involution, ω, restricted to $\mathfrak{g}_{\mathbb{R}}$ acts by

$$\omega(e_i) = -f_i, \quad \omega(f_i) = -e_i, \quad \omega(h_i) = -h_i, \quad 1 \leq i \leq n,$$

so we get the ± 1 eigenspace decomposition $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k} \oplus \mathfrak{p}$. Clearly, $e_j - f_j \in \mathfrak{k}$, $e_j + f_j \in \mathfrak{p}$ and $h_j \in \mathfrak{p}$, but more generally, $x + \omega(x) \in \mathfrak{k}$ and $x - \omega(x) \in \mathfrak{p}$ for any $x \in \mathfrak{g}_{\mathbb{R}}$.

The geometry of the standard real Cartan subalgebra $\mathfrak{h}_{\mathbb{R}}$ is Lorentzian by assumption that Cartan matrix C is hyperbolic type.

Orbit of $\mathfrak{h}_{\mathbb{R}}$ under K

As in [CFF], we wish to embed the (twin) building for $G_{\mathbb{R}}$ into the K-orbit of the standard Cartan subalgebra, $\mathfrak{h}_{\mathbb{R}}$. Here, K is the group corresponding to $\mathfrak{k}$, and it certainly contains $exp(ad_x)$ for any $x \in \mathfrak{k}$. Since $[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$, it follows that for any $x \in \mathfrak{k}$, we have $exp(ad_x)(\mathfrak{p}) \subseteq \mathfrak{p}$ so $K(\mathfrak{p}) \subseteq \mathfrak{p}$. In particular, since $\mathfrak{h}_{\mathbb{R}} \subseteq \mathfrak{p}$ this says that $K(\mathfrak{h}_{\mathbb{R}}) \subseteq \mathfrak{p}$.

Inside $\mathfrak{h}_{\mathbb{R}}$ we can find a two-sheeted hyperboloid, say, $A = \{h \in \mathfrak{h}_{\mathbb{R}} \mid (h, h) = -1\}$, on which the Weyl group, $\mathcal{W}$, acts so as to tessellate each sheet, $A^{\pm}$, into copies of a fundamental domain. Those are the *chambers* of the *apartment* formed by each sheet. The *twin* structure comes from the duality between the two sheets, A^+ in the forward lightcone and A^- in the backward lightcone. The K-orbit of A provides many apartments, but chambers have boundaries which are shared by families of chambers in different apartments. We can now see that local structure of how Cartans are connected.

Local Structure of Cartans Under K Action

Let's examine in more detail the action of $exp(ad_{r(e_j - f_j)}) \in K$ on $h \in \mathfrak{h}_{\mathbb{R}}$ for any $r \in \mathbb{R}$. We will use the brackets

$$[e_j - f_j, h] = -\alpha_j(h)(e_j + f_j) \quad \text{and} \quad [e_j - f_j, e_j + f_j] = 2h_j.$$

For better comparison with Theorem 4.5 of [CFF], define

$$x_j = \frac{e_j - f_j}{2}, \quad y_j = \frac{e_j + f_j}{2}, \quad z_j = \frac{h_j}{2}$$

a basis of the real Lie subalgebra $\mathfrak{sl}_2^j(\mathbb{R})$ with brackets

$$[x_j, y_j] = z_j, \quad [y_j, z_j] = -x_j, \quad \text{and} \quad [z_j, x_j] = y_j.$$

With a little patience, or using Theorem 4.5 of [CFF], one can compute that for any $r \in \mathbb{R}$, we have

$$exp(ad_{rx_j})(h) = h - \alpha_j(h)\sin(r)y_j + \alpha_j(h)(\cos(r) - 1)z_j$$

so the hyperplane $\{h \in \mathfrak{h}_{\mathbb{R}} \mid \alpha_j(h) = 0\}$ is fixed by these operators.

Local Structure of Cartans Under K Action

It is, of course, also the hyperplane fixed by the simple reflection, $w_j \in \mathcal{W}$, so the intersection of the fixed hyperplane with the two-sheeted hyperboloid is a boundary of the fundamental domain on each sheet.

Applying this formula to z_j, we get

$$exp(ad_{rx_j})(z_j) = \cos(r)z_j - \sin(r)y_j,$$

since $\alpha_j(z_j) = 1$. So the family of Cartan subalgebras sharing that fixed hyperplane, as well as the family of chambers in other apartments that share that chamber boundary, is parametized by a circle, S^1, which matches $\mathbb{R} \cup \infty$ as expected for a building of a group over $\mathbb{R}$.

Definition of a Building

There are many approaches to the definition of a (twin) building (see [AB]), but for a condensed approach see Section 2.3 of [CFF]. Just to give an idea of the basic definition, we have the following:

Definition (Tits Building)

A *Tits building* of type (W, S) consists of a simplicial complex $\mathcal{B}$ together with a collection $\mathcal{A}$ of subcomplexes, each of which is called an apartment, such that

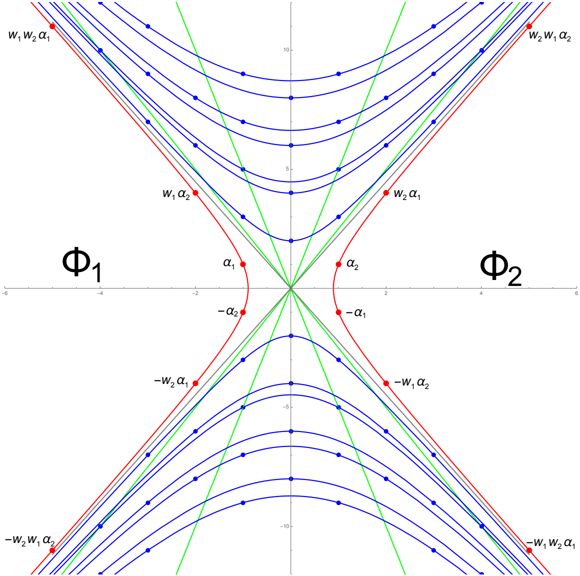
1. Each apartment is a Coxeter complex for the Coxeter system (W, S),
2. Each pair of *chambers*, *i.e.* simplices of maximal dimension in $\mathcal{B}$, is contained in a common apartment,
3. For two apartments A and A' there is an isomorphism $\varphi : A \to A'$, fixing the intersection $A \cap A'$.

Example 1, the rank 2 Fibonacci Hyperbolic

In rank $n = 2$ the Weyl group is the infinite dihedral group, D_∞, each apartment is a line, and each building $\mathcal{B}^\pm$ is a tree. We model each apartment as a copy of the real line tessellated into unit intervals (chambers) $C(n) = [n - \frac{1}{2}, n + \frac{1}{2}]$ for $n \in \mathbb{Z}$, so the vertices are $\mathbb{Z} + \frac{1}{2}$. At each vertex a family of intervals (chambers) is attached, each in a line (apartment) which is tessellated, and in each of those lines chambers are attached, on ad infinitum. The family of chambers attached at any vertex is the projective space $P_1(\mathbb{F})$, where $\mathbb{F}$ is the field over which the group is defined, so for $\mathbb{F} = \mathbb{R}$, we have $P_1(\mathbb{R}) = S^1$ is the circle. Thus, in rank 2 each building $\mathcal{B}^\pm$ is a S^1-tree. See [RT] for more about twin trees, and see [F] for more about the rank 2 Fibonacci hyperbolic KM algebra.

$\mathcal{F}ib$ Root System and non-standard partition of Φ^{re}

$\Phi_1 = W^{even}\{\alpha_1, -\alpha_2\}$ and $\Phi_2 = W^{even}\{-\alpha_1, \alpha_2\}$.

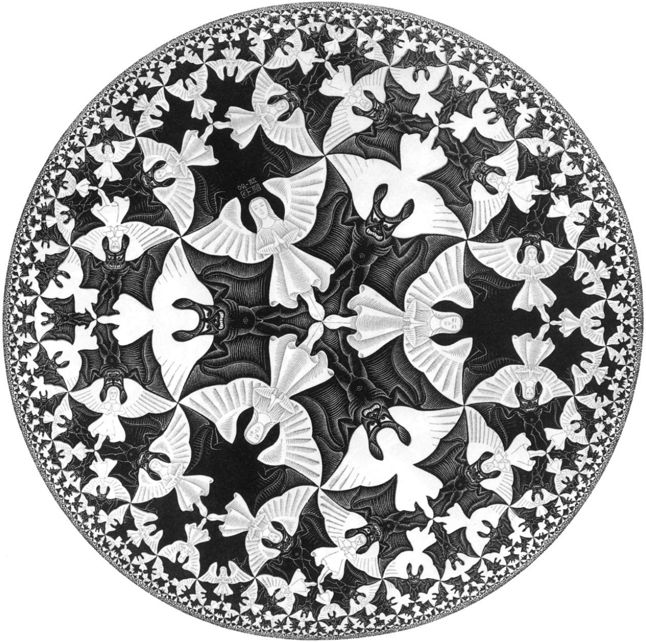


Example 2, the rank 3 Hyperbolic $F = AE_3$

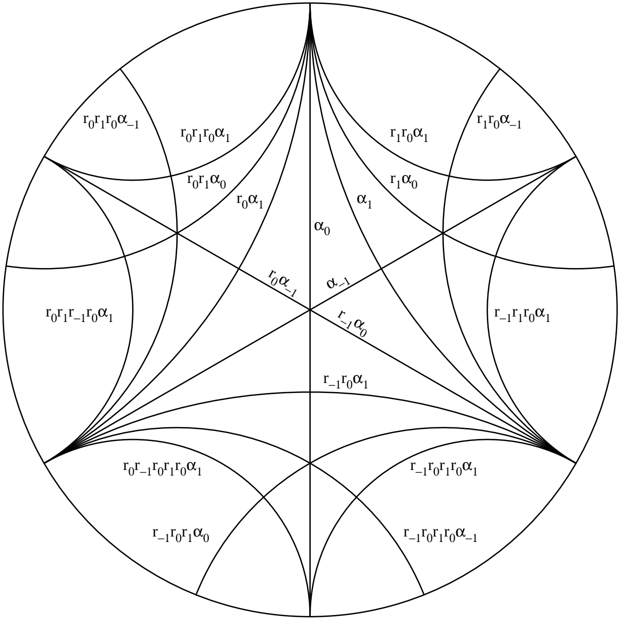
In rank 3 the Weyl group is a hyperbolic triangle group, each apartment is a copy of the Poincaré disk, $\mathcal{P}$, tessellated into hyperbolic triangles by $\mathcal{W}$. The boundary of each triangle is a segment in a hyperbolic geodesic. Along each geodesic segment we have a S^1-family of attached triangles, each in a copy of $\mathcal{P}$, which is tessellated and has attached disks along each geodesic, on ad infinitum.

For the rank 3 Hyperbolic $F = AE_3$, the Weyl group $\mathcal{W} \cong PGL(2, \mathbb{Z})$, and the fundamental domain of its action on each copy of $\mathcal{P}$ is a $(2, 3, \infty)$ hyperbolic triangle, giving the tessellation used by Escher in his famous "Circle Limit IV".

Circle Limit IV



$\mathcal{P}$ Tessellated by $(2, 3, \infty)$ Hyperbolic Triangles



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Thanks for your kind attention!

Happy Birthday, Hermann!