Discrete gravity

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Hermannfest 14 September 2022

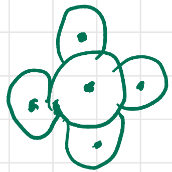
Motivation : • At planck scale points are indistinguishable

• one can think in terms of quanta of geometry in form of spheres or cells.

Aim : Associate geometric invariants with discrete spaces such that these reduce in the continuous limit $l_p \to 0$ to those of differential geometry.

Method : • Imitate lattice gauge theory.

• Replace blocks with cells and assume that every cell is gel like and bounded by 2d cells



Label each cell with set of integers

$$n^\alpha = (n^1, \ldots n^d)$$

• Assume existence of shift operators $\vec{E}_\alpha$ such that

$$\vec{E}_\alpha f(n^\beta) = f(n^\beta + \delta_\alpha^\beta)$$ is shift operator along α-direction ; f is scalar function

- $\vec{E}_\alpha$ form a basis in d-dimensional linear space.

- The inverse shift operator is defined by

$$\vec{E}_\alpha^{-1} f(n^\beta) = f(n^\beta - \delta_\alpha^\beta)$$

- Assume $\vec{E}_\alpha$ acts first with most left operators:

$$\vec{E}_\alpha(n)\, \vec{E}_\beta^{-1}(n)\, f(n) = E_\beta^{-1}(n+l_\alpha) f(n+\alpha) = f(n+l_\alpha - l_\beta)$$

- The d-tangent operator $e_\alpha(n)$ are defined by

$$\vec{e}_\alpha(n) = \frac{1}{2}\left(\vec{E}_\alpha(n) - E_\alpha^{-1}(n)\right)$$

Inspired by Dirac operator, we associate with every cell a Euclidean d-dimensional (real) tangent space $\vec{e}_a$ with inner product $(\vec{e}_a, \vec{e}_b) = \delta_{ab}$ $a, b = 1, \ldots, d$

Inner product invariant under rotation

$$\vec{e}\,'_a(n) = R_a^{\ b}(n)\,\vec{e}_b(n) \qquad R^T R = 1$$

$\vec{e}_a$ are linear combination of $\vec{e}_\alpha$

$$\vec{e}_a(n) = \underbrace{\bar{e}_a^{\ \alpha}(n)}_{\text{inverse of } e_\alpha^{\ a}(n)}\,\vec{e}_\alpha(n) \Rightarrow \vec{e}_\alpha(n) = e_\alpha^{\ a}(n)\,\vec{e}_a(n)$$

We can guess the correct Dirac operator from lattice gauge theory :

$$\sum_n \sum_\alpha \frac{i}{2} \psi^\dagger(n) \gamma^\alpha \left(\psi(n+1_\alpha) - \psi(n-1_\alpha) \right)$$

$$\downarrow$$

$$E_\alpha(n)\psi(n) - E_\alpha^{-1}(n)\psi(n)$$

Spinors $\psi(n)$ transform under local rotations

$$\psi(n) \to R(n)\,\psi(n) \qquad \text{where} \quad R(n) = \exp\left(\frac{1}{4}\lambda^{ab}(n)\gamma_{ab}\right)$$

makes $SO(d)$ local

Analogue of spin connection :

Let $\Gamma_\alpha(n) = \Omega_\alpha(n) E_\alpha(n)$ where $\Omega_\alpha(n) = \exp(\ell^\beta \omega_\beta^{ab}(n) J_{ab})$

$$\Gamma_\alpha(n) \to R(n)\,\Gamma_\alpha(n)\,R^{-1}(n) \Rightarrow \Omega_\alpha(n) \to R(n)\,\Omega_\alpha(n)\,R^{-1}(n+l_\alpha)$$

Thus Dirac action becomes:

$$\sum_n \sum_\alpha \frac{i}{2} \psi^\dagger(n) \gamma^\alpha(n) \left(\gamma_\alpha(n) - \gamma_\alpha^{-1}(n)\right) \psi(n) \underbrace{\mathcal{V}(n)}_{\text{density}}$$

$$\gamma^\alpha(n) = \bar{e}^\alpha_a \gamma^a$$

In analogy with Cartan structure equations:

$$T^a = de^a + \omega^a{}_b \wedge e^b$$

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$$

We define:

$$J_{\alpha\beta}(n) = \frac{1}{\ell^\alpha}\left[\gamma_\alpha(n)\, e_\beta(n)\, \gamma_\alpha^{-1}(n) - e_\beta(n)\right] - \alpha \leftrightarrow \beta$$

$$R_{\alpha\beta}(n) = \frac{1}{2\ell^\alpha \ell^\beta}\left(\gamma_\alpha(n)\, \gamma_\beta(n)\, \gamma_\alpha^{-1}(n)\, \gamma_\beta^{-1}(n) - \alpha \leftrightarrow \beta\right)$$

$$= \frac{1}{2\ell^\alpha \ell^\beta}\left(\Omega_\alpha(n)\, \Omega_\beta(n+\alpha)\, \Omega_\alpha^{-1}(n+\beta)\, \Omega_\beta^{-1}(n) - \alpha \leftrightarrow \beta\right)$$

$R_{\alpha\beta}(n)$ transforms covariantly

For $d = 2, 3, 4$ $\quad R_{\alpha\beta}(n) = R_{\alpha\beta}{}^{cd}(n)\, J_{cd}$

curvature scalar: $\quad R(n) = \bar{e}_a{}^\alpha(n)\, \bar{e}_b{}^\beta(n)\, R_{\alpha\beta}{}^{ab}(n)$

Hermiticity requirement $\quad (\Psi, D\Psi) = (D\Psi, \Psi)$

$$\Rightarrow (\Psi, D\Psi) = i \sum_n V(n)\, \psi^\dagger(n)\, \bar{e}^\alpha(n) \left(\Gamma_\alpha(n) - \Gamma_\alpha^{-1}(n)\right) \psi(n)$$

$$\Rightarrow \quad V(n)\, \bar{e}^\alpha(n)\, \Omega_\alpha(n) = V(n+l_\alpha)\, \Omega_\alpha(n)\, \bar{e}^\alpha(n+l_\alpha)$$

must be satisfied.

Easily show that as $l_\alpha \to 0$, $\quad T_{\alpha\beta}(n) \to T_{\alpha\beta}{}^a$

$$\mathcal{R}_{\alpha\beta}{}^{ab}(n) \to R_{\alpha\beta}{}^{ab}$$

$$R_{\alpha\beta}(n) = \frac{1}{2 l^\alpha l^\beta} \Big(\exp\left(\tfrac{1}{2} l^\alpha \omega_\alpha(n)\right) \exp\left(\tfrac{1}{2} l^\beta \omega_\beta(n+\hat{\alpha})\right) \exp\left(-\tfrac{1}{2} l^\alpha \omega_\alpha(n+\hat{\beta})\right) \exp\left(-\tfrac{1}{2} l^\beta \omega_\beta(n)\right) - \alpha \leftrightarrow \beta \Big)$$

$$\to \quad D_\alpha \omega_\beta - D_\beta \omega_\alpha + \tfrac{1}{2} [\omega_\alpha, \omega_\beta] + O(l)$$

Example 2-dimensions

group $SO(2)$: Clifford algebra $\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\gamma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

generators : $\omega_\mu = \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} = \frac{1}{2}\omega_\mu \tau$ $\tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\Omega_\alpha(n_1, n_2) = e^{\frac{1}{2}\omega_\alpha \tau} = \cos\frac{1}{2}\omega_\alpha(n_1, n_2) + \tau \sin\frac{1}{2}\omega_\alpha(n_1, n_2)$

For $e_\alpha{}^a(n)$: $e_i{}^1(n^1, n^2) = e_i{}^2(n^1, n^2) \equiv e(n^1, n^2)$

we have 2-torsion conditions for 2 unknowns :

$\omega_i(n^1, n^2)$ $\omega_i(n^1, n^2)$

$e(n^1+1, n^2)\sin\omega_i(n^1, n^2) - e(n^1, n^2+1)\cos\omega_i(n^1, n^2) + e(n^1, n^2) = 0$

$e(n^1+1, n^2)\cos\omega_i(n^1, n^2) + e(n^1, n^2+1)\sin\omega_i(n^1, n^2) - e(n^1, n^2) = 0$

Solution:

$$\omega_i(n^1,n^2) = \frac{\pi}{4} - \arcsin\left(\frac{e^2(n^1+1,n^2) - e^2(n^1,n^2+1) + 2e^2(n^1,n^2)}{2\sqrt{2}\; e(n^1+1,n^2)\; e(n^1,n^2)}\right)$$

$$\omega_i(n^1,n^2) = \frac{\pi}{4} - \arccos\left(\frac{e^2(n^1,n^2+1) - e^2(n^1+1,n^2) + 2e^2(n^1,n^2)}{2\sqrt{2}\; e(n^1,n^2+1)\; e(n^1,n^2)}\right)$$

$$R_{ii}^{\;12}(n) = 2\sin\left(\frac{1}{2}\left(D_1\omega_2(n^1,n^2) - D_2\,\omega_1(n^1,n^2)\right)\right)$$

$$R(n) = 2\,\bar{e}_1^{\,1}(n)\,\bar{e}_2^{\,2}(n)\; R_{ii}^{\;12}(n) = 2e^{-2}(n)\; R_{ii}^{\;12}(n)$$

A numerical study shows that this discretization is an excellent approximation.

Numerics with
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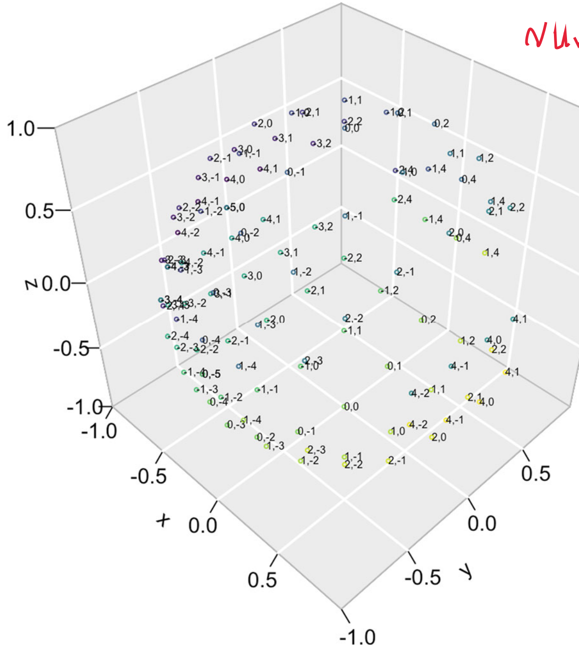

Fig. 1 Two-sphere of radius one formed from the set of the discrete points

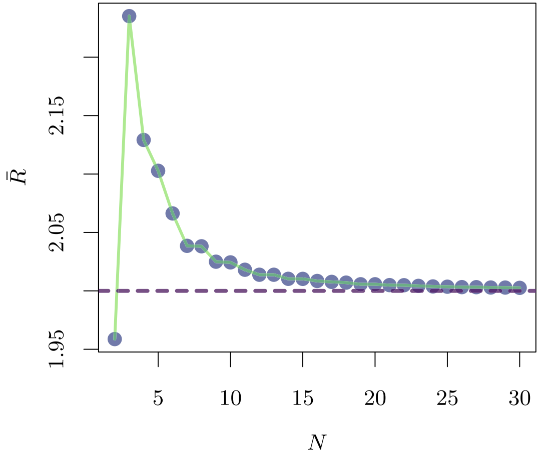

Fig. 2 Mean of the scalar curvature versus N for $a = 1$

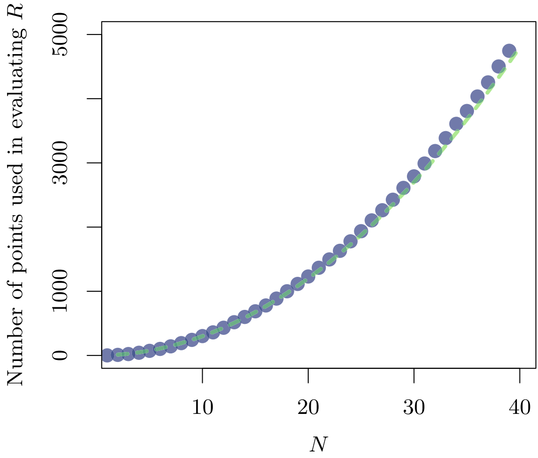

Fig. 3 The estimate of the number of points used in evaluating R, $3N^2$, is represented by the light green curve. The actual number of points satisfying Eq. (6), retrieved numerically, and contributing to the computation of R, is represented by the blue dots

To get continuous limit, consider a line with length L divided into N pieces each with length $\varepsilon = \frac{L}{N}$ then

$$x^\alpha = \varepsilon n^\alpha \rightarrow x = L\frac{n}{N}$$

continuous limit achieved when $N \rightarrow \infty$ to get finite x then $n \rightarrow \infty$ such that $x \rightarrow x_0 = L\frac{n}{N}$ i.e. $\frac{n}{N} = \frac{x_0}{L}$

e.g. $e_\alpha(x) = \frac{1}{2\epsilon_\alpha}\left(E_\alpha(x) - E_\alpha^{-1}(x)\right) f(x) = \frac{f(x+\epsilon_\alpha) - f(x-\epsilon_\alpha)}{2\epsilon_\alpha}$

$$e_\alpha(x) \rightarrow \frac{\partial}{\partial x^\alpha}$$

Density $V_\alpha(n)$ satisfy $V(n)\, \bar{e}^\alpha(n)\, \Omega_\alpha(n) = V(n+1_\alpha)\, \Omega_\alpha(n)\, \bar{e}^\alpha(n+1_\alpha)$

Continuous limit: $V(n) \rightarrow \det e_\alpha^a(x)$.

Conclusions:

1- we have achieved a formulation of discrete gravity based on rotational in tangent space to every cell.

2- Failure of Liebnitz rule for difference equation is avoided by considering shifts of soldering forms satisfying torsion-free condition

3- Definition of curvature gives manifest continuous limit those of differential geometry.

4- In future aim to apply formulation to quanta of geometry present in Noncommutative geometry and to cosmology of expanding universe.