Quantum Gravity: The Sound of Symmetry
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D=11 Supermembrane: spectral properties, monodromy and gauge symmetry

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Work in collaboration with

L.Boulton, M.P.García del Moral, A.Restuccia JHEP (2021)

M.P.Garcia del Moral, C. Las Heras, P.León, J.Peña, A.Restuccia, JHEP (2020), arxiv (2022)

M.P.García del Moral, P.León, A.Restuccia , JHEP (2021).

Outline

- Spectral properties of the Hamiltonian of the Regularized SU(N) D=11 Supermembrane in the Light Cone Gauge.
- Dimensional dependence.
- Zero energy state.

- Supermembranes with nontrivial monodromy.

The Hamiltonian of D=11 Supermembrane in the Light Cone Gauge

de Wit, Hoppe, Nicolai

The SU(N) regularized D=11 Supermembrane : J.Goldstone/ J.Hoppe / B.de Wit, J.Hoppe, H.Nicolai/ B.de Wit, Marquard,H.Nicolai/ B.de Wit, Luscher, H.Nicolai.

Reduction to (0+1) dimensions of D= 10 Super Yang Mills theory : M.Claudson, M.Halpern/ R.Flume/ M.Baake, P.Reinicke, V.Rittengerg/ A. Khvedelidze, H.Pavel/ J.Wosiek.

A description of the dynamics of D-0 branes in Superstring theory: E.Witten/ Banks, Fischler, Schrenker, Susskind (BFSS).

Supermembrane

Bergshoeff, Sezgin, Townsend

- Global supersymmetry
- Local diffeomophisms
- Local Kappa symmetry
- Dimension of the target space : D = 4,5,7,11
- Consistency of the D=11 Supermembrane on a general background implies that it must be a solution of the unique D=11 Supergravity.

The spectrum of the regularized SU(N)Hamiltonian in the Light Cone Gauge is continuous from 0 to ∞ .

de Wit, Lüscher, Nicolai

Asymptotic solution for D=11, SU(2) model:

Theorem: The solution in $L^2(\Omega)$ exists and it is unique.

J. Frohlich, G.M.Graf, D.Hasler, J.Hoppe and S.T.Yau

M.B.Halpern, C.Schwartz/ Danielsson, G.Ferretti, B.Sundborg/A.Konechny/M. Bordeman, J.Hoppe,R.Suter/ D.Lundholm Sutter/D.Lungholm/D.Hasler, J.Hoppe/D.Hasler/J.Hoppe,S.-T.Yau/M.Porrati, A.Rosenberg/M.Trzetrzelewski, J.Wosiek/ J.Hoppe, D.Lundholm, M.Trzetrzelewski

Witten Index:

S.Sethi, M.Stern/ M.Staudacher/ P.Yi/Green, Gutperle/Moore, Nekrasov, Shatashvili.

Quantization of the Supermembrane Theory :

O.Lechtenfeld, H.Nicolai

The Hamiltonian associated to the the regularized mass operator of the supermembrane

$$
\begin{aligned}
H &= \frac{1}{2}M = -\Delta + V_{\mathrm{B}} + V_{\mathrm{F}} \\
\Delta &= \frac{1}{2}\frac{\partial^2}{\partial X_A^i \partial X_i^A} + \frac{1}{2}\frac{\partial^2}{\partial Z_A \partial \overline{Z}^A} \\
V_{\mathrm{B}} &= \frac{1}{4} f_{AB}^E f_{CDE}\{X_i^A X_j^B X^{iC} X^{jD} + 4X_i^A Z^B X^{iC} \overline{Z}^D + 2Z^A \overline{Z}^B \overline{Z}^C Z^D\} \\
V_{\mathrm{F}} &= i f_{ABC} X_i^A \lambda_\alpha^B \Gamma_{\alpha\beta}^i \frac{\partial}{\partial \lambda_{\beta C}} + \frac{1}{\sqrt{2}} f_{ABC}(Z^A \lambda_\alpha^B \lambda_\alpha^C - \overline{Z}^A \frac{\partial}{\partial \lambda_{\alpha B}} \frac{\partial}{\partial \lambda_{\alpha C}}).
\end{aligned}
$$

The generators of the local $SU(N)$ symmetry are

$$
\varphi^A = f^{ABC}\left(X_i^B \partial_{X_i^C} + Z_B \partial_{Z^C} + \overline{Z}_B \partial_{\overline{Z}^C} + \lambda_\alpha^B \partial_{\lambda_\alpha^C}\right).
$$

The supercharges

$$Q_\alpha = \left\{ -i\Gamma^i_{\alpha\beta}\partial_{X_i^A} + \frac{1}{2} f_{ABC} X_i^B X_j^C \Gamma^{ij}_{\alpha\beta} - f_{ABC} Z^B \overline{Z}^C \delta_{\alpha\beta} \right\} \lambda^A_\beta$$
$$+ \sqrt{2} \left\{ \delta_{\alpha\beta}\partial_{Z^A} + i f_{ABC} X_i^B \overline{Z}^C \Gamma^i_{\alpha\beta} \right\} \partial_{\lambda^A_\beta}$$

and

$$Q^\dagger_\alpha = \left\{ i\Gamma^i_{\alpha\beta}\partial_{X_i^A} + \frac{1}{2} f_{ABC} X_i^B X_j^C \Gamma^{ij}_{\alpha\beta} + f_{ABC} Z^B \overline{Z}^C \delta_{\alpha\beta} \right\} \partial_{\lambda^A_\beta}$$
$$+ \sqrt{2} \left\{ -\delta_{\alpha\beta}\partial_{\overline{Z}^A} + i f_{ABC} X_i^B Z^C \Gamma^i_{\alpha\beta} \right\} \lambda^A_\beta .$$

The superalgebra

$$
\begin{aligned}
\{Q_\alpha, Q_\beta\} &= 2\sqrt{2}\delta_{\alpha\beta}\overline{Z}^A\varphi_A, \\
\{Q_\alpha^\dagger, Q_\beta^\dagger\} &= 2\sqrt{2}\delta_{\alpha\beta}Z^A\varphi^A, \\
\{Q_\alpha, Q_\beta^\dagger\} &= 2\delta_{\alpha\beta}H - 2i\Gamma^i_{\alpha\beta}X_i^A\varphi_A.
\end{aligned}
$$

Massless Ground State

The massless wave function must obey the Schrödinger equation

$$H\psi = 0 \quad .$$

From the susy algebra

$$\{ Q^\dagger , Q \} \, \psi = 0 \, ,$$

which imply that ψ is a singlet under susy

$$Q \, \psi = 0 \quad , \qquad Q^\dagger \, \psi = 0 \, .$$

Also a singlet under SO(9), in D=11, to have the desired multiplet.
H is a positive, unbounded, (essentially) self adjoint operator with domain in $L^2(R^{9(NN-1)})$.

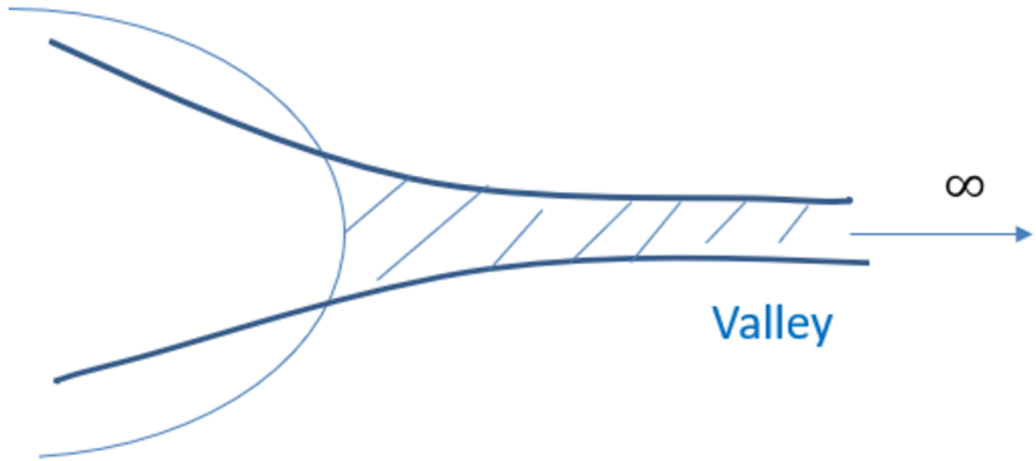
Homogeneous Dirichlet problem

Given g on $\partial\Omega$, let

$$\mathcal{E} = \{ \Lambda \in H^1(\Omega) \, , \; \varphi^A \Lambda = 0 \; , \; \Lambda = g \text{ on } \partial\Omega \}$$

Problem: find $\varphi \in \mathcal{E}$, $H\varphi = 0$ in Ω

$$\mathrm{X} = \{ \Lambda \in H^1(\Omega) \, , \; \varphi^A \Lambda = 0 \; \}$$



Valley $\Omega = \{x \in R^D \ : \ V_B(x) < V_0 \}$

Star shaped region in $\mathbf{R}^{9(NN-1)}$ with valleys extending to infinite/ bounded region

Fermionic potential is unbounded from below in subvarieties of the valleys .

Prescribe a height V_0 Valley $\Omega \equiv \{ X: \; V_B(X) < V_0 \}$

If

$D \geq 7$ for $N = 2$

$D \geq 6$ for $N = 3$

$D \geq 5$ for $N \geq 4$

Then $Vol(\Omega) < \infty$

Convergence of partition function of reduced SU(N) Yang Mills:
P.Austin, J.F.Wheater / D.Lundholm / W.Krauth, M.Staudacher
Direct calculation: L.Boulton, M.García del Moral, A.Restuccia

Theorem (Berger and Schechter): If the volumen of $\Omega \ \cap \ S_x$ tends to zero as $|x| \to \infty$, the embedding of $H_o^1(\Omega)$ into $L^2(\Omega)$ is compact.

(Rellich-Kondrachov theorem: bounded regions)

S_x the ball of radius 1 y center x.

Consequently,

The embedding of $H_o^1(\Omega)$ into $L^2(\Omega)$ is compact (Ω = valley).

Inhomogeneous Dirichlet problem

Given $f \in L^2(\Omega)$, find $\psi \in H_o^1(\Omega) \cap H^2(\Omega)$

satisfying

$$H\,\psi = f \quad \text{in } \Omega$$

$$\psi|_{\partial\Omega} = 0 \quad \text{on } \Omega$$

$$\varphi^{A}\psi = 0$$

Equivalence: Take $f = (\Delta - V)\tilde{g}$, $\tilde{g}$ defined on a neighbourhood of $\partial\Omega$:

$$\tilde{g}|_{\partial\Omega} = g \quad ,$$

then

$\varphi = \psi + \tilde{g}$ is a solution of the homogeneous Dirichlet problem

Supersymmetric model

Lemma. If $\psi \in H^1{}_0(\Omega)$ satisfies Q $\psi = 0$ and $Q^\dagger \psi = 0$ in Ω , then $\psi = 0$ in the closure of Ω.

Due to the regularity properties of elliptic operators the conditions of the Lemma can be extended smoothly to the boundary where $\psi = 0$. These equations are only restrictions to the normal derivative since tangential derivatives are zero .

L.Boulton, M.p.García del Moral, A.Restuccia

$$
\begin{aligned}
Q_8\psi &= \sqrt{2}\partial_{Z^A}\partial_{\lambda_8^A}\psi - i\Gamma^i_{8j}\partial_{X_i^A}\lambda_j^A\psi = 0 \\
Q_8^\dagger\psi &= -\sqrt{2}\partial_{\overline{Z}^A}\lambda_8^A\psi + i\Gamma^i_{8j}\partial_{X_i^A}\partial_{\lambda_j^A}\psi = 0 \\
Q_j\psi &= -i\Gamma^i_{j8}\lambda_8^A\partial_{X_i^A}\psi - i\Gamma^i_{jk}\lambda_j^A\partial_{X_i^A}\psi + \sqrt{2}\partial_{Z^A}\partial_{\lambda_j^A}\psi = 0 \\
Q_j^\dagger\psi &= +i\Gamma^i_{j8}\partial_{X_i^A}\partial_{\lambda_8^A}\psi + i\Gamma^i_{jk}\partial_{X_i^A}\partial_{\lambda_k^A}\psi - \sqrt{2}\partial_{\overline{Z}^A}\lambda_j^A\psi = 0.
\end{aligned}
$$

Here $\Gamma^i_{j8} = -i\delta^i_j$ and $\Gamma^i_{jk} = iC_{ijk}$, where the C_{ijk} are the structure constants of the octonion algebra.

Consider the derivative with respect to ρ^2. Observe that

$$\partial_{Z^A}\psi|_{\partial\Omega} = 2\overline{Z}^A\frac{\partial\psi}{\partial\rho^2}|_{\partial\Omega} \quad \text{and} \quad \partial_{X_i^A}\psi|_{\partial\Omega} = 2X_i^A\frac{\partial\psi}{\partial\rho^2}|_{\partial\Omega}.$$

Write $\partial_{\rho^2}\psi \equiv \psi_{\rho^2}$. Then

$$Q_8\psi = \sqrt{2}\overline{Z}^A\partial_{\lambda_8^A}\psi_{\rho^2} + (X_i^A\lambda_i^A)\psi_{\rho^2} = 0$$

and

$$Q_8^\dagger\psi = -\sqrt{2}Z^A\lambda_8^A\psi_{\rho^2} - X_i^A\partial_{\lambda_i^A}\psi_{\rho^2} = 0.$$

On $\partial\Omega$

$$(X_j^A)^2+2\,Z^A\tilde{Z}^A = R^2$$

Then

$$R^2\psi_{\rho^2}|_{\partial\Omega}=0$$

For $R^2 \neq 0.$ Thus

$$\psi_{\rho^2}|_{\partial\Omega} = 0$$

By virtue of the Cauchy-Kovalevskaya Theorem, it then follows that $\psi = 0$ is the unique solution in a neighbourhood of the boundary. Moreover, since the potential is analytic on Ω this solution can be extended uniquely to the whole ball Ω .

The Dirichlet form

$$H = -\Delta + V \qquad : \qquad H : \; H^2(\Omega) \longrightarrow L^2(\Omega)$$

$$D(u,w) = (\nabla u, \nabla w)_{L^2(\boldsymbol{\Omega})} + (u, Vw)_{L^2(\boldsymbol{\Omega})} \quad ,$$

for $(u,w) \epsilon \, H_o^1(\Omega)$

$$D \; : \quad H_o^1(\Omega) \times H_o^1(\Omega) \longrightarrow \mathrm{C}$$

(i) *if* $u \, \epsilon \, H_o^1(\Omega) \cap H_{loc}^2(\Omega)$

$$D(u,u) = (u, Hu)_{L^2(\boldsymbol{\Omega})} = \|Qu\|_{L^2(\boldsymbol{\Omega})}^2 + \|Q^+ u\|_{L^2(\boldsymbol{\Omega})}^2$$

The Dirichlet problem for the Supermembrane

$$Let \ \ \mathcal{E} = \{ \Lambda \in H^1(\Omega) \, , \ \varphi^A \Lambda = 0 \ , \ \ \Lambda = g \ \text{ on } \ \partial\Omega \}$$

Assume there exists $\psi \in \mathcal{E}$:

$$D(u, \psi) = 0 \ \text{ for all } \ u \in H_o^1(\Omega)$$

which is equivalent to $H\psi = 0$

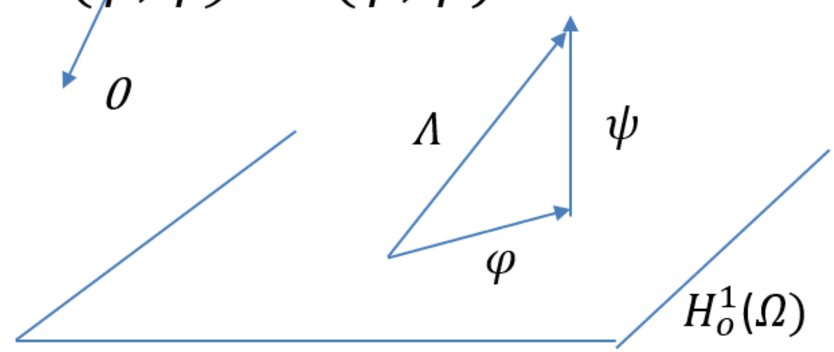
Then for any $\Lambda \in \mathcal{E}$, define $\varphi = \Lambda - \psi$

⇨ $\varphi \in H_o^1(\Omega)$ and

$$D(\Lambda, \Lambda) = D(\psi, \psi) + \underbrace{D(\psi, \varphi)}_{0} + \underbrace{D(\varphi, \psi)}_{0} + D(\varphi, \varphi)$$

⇨ $D(\Lambda, \Lambda) \geq D(\psi, \psi)$

The converse is also valid



Bound on the fermionic potential

There exists positive constants C_1 and $C_2 < 1$:

$$|(\psi, V_F \psi)| \le \int_K \rho \overline{\Psi} \cdot \Psi \le C_1 \int_K \overline{\Psi} \cdot \Psi + C_2 \int_K \nabla \overline{\Psi} \cdot \nabla \Psi$$

for all ψ in $H^1{}_0(\Omega)$.

(Extension of Poincaré Lemma)

Lemma:

The Hamiltonian of the SU(N) D=11 Supermembrane has a coercive Dirichlet form.

D is coercive if there $\exists\, c > 0$ *and* λ :

$$D(u,u) \;\geq\; c\,\|u\|^2_{H^1(\Omega)} + \lambda\|u\|^2_{L^2(\Omega)}$$

for all $u \in H^1_o(\Omega)$.

Theorem: The Hamiltonian H, with domain in $H_o^1(\Omega)$, has a purely discrete spectrum of eigenvalues, each of finite multiplicity, with no accumulation point other than infinity.

This is a direct consequence of the coercivity and the compact embedding, as the resolvent of H becomes a compact operator.

Boulton, García del Moral, Restuccia

Theorem: Given g ϵ $H^2(\Omega)\cap X$ there exists a unique solution $\psi \;\epsilon\; H^1{}_0(\Omega)\cap H^2(\Omega)\cap X$ satisfying the inhomogeneous Dirichlet problem. Consequently, there exists a unique solution $\Phi \;\epsilon\; H^2(\Omega)\cap X$ to the Dirichlet problem.

Boulton, Garcia del Moral, Restuccia

(i) $D(u,w)$ is bounded and coercive for all $u, w \in H_o^1(\Omega)$.

(ii)Supersymmetric structure of *H*.

(iii) The embedding of $H^1(\Omega)$ into $L^2\ (\Omega)$ is compact.

(iv) Lax Milgram: $(u, Tf) = (u, f)$ T: $L^2(\Omega) \longrightarrow H^1\ (\Omega)$ is a bounded operator, then

$$i \circ T \ :\ L^2(\Omega) \longrightarrow L^2(\Omega)$$

is compact. Use the Fredholm theorem.

Matching the internal and the asymptotic solutions

Denote Σ the asymptotic region where the solution ϕ has been determined. It satisfies $Q\,\phi = Q^{\dagger}\,\phi = 0$. Consider the interior región Ω. One can always take the prescribed V_0 large enough so that the boundary $\partial\Omega$ is contained in Σ. Let g be the value of ϕ on $\partial\Omega$. Consider the interior solution ψ with boundary condition g. It can then be proved that in $\Sigma \cap \Omega$, on a neighborhood of $\partial\Omega$: $\phi = \psi$. It then follows that the matching of the solutions is smooth. Consequently, for the SU(2) model, we conclude the existence and uniqueness of the zero eigenstate of the D=11 regularized Hamiltonian.

Spectrum of the Hamiltonian

There exists and it is unique the solution of the Dirichlet problem for the SU(N) model on bounded regions (balls) and on the valleys extending to infinite.

The spectrum of the Hamiltonian with domain in $H^1_0(\Omega)$ is in those cases purely discrete, with finite multiplicity and with no accumulation point other than infinite. It is strictly positive.

The matching of the interior solution with the asymptotic one at the boundary of Ω, for the SU(2) model, shows the existence and uniqueness of the ground state for the SU(2) regularized D=11 Supermembrane.

Explicit calculation of the spectrum of H in the valleys.

The ground state of the SU(N) model and its relation to Yang Mills in the L going to 0 limit.

The Heat kernel of the D = 11 Hamiltonian.

Geometrical framework

- Smooth fiber bundle

  $F \rightarrow E \rightarrow B$

  F the fiber, the flat torus on the target,

  E the total space,

  B the base manifold, a Riemann surface.

- Structure group : G

  area preserving or equivalently symplectomorphisms preserving the area two form .

J.P.Khan

W.Thurston

Monodromy

- There is a natural homomorphism :

  $\Pi_1(B) \rightarrow \Pi_0\,(G)$

  $\Pi_0(G)$, the group of isotopy classes of symplectomorphisms is isomorphic to SL(2,Z)

- Given a representation ρ of $\Pi_1(B)$ on SL(2,Z) it acts on the homology classes of the fiber torus (represented by two integers) as a matrix multiplication on the pair of integers.

  We identify the pair of integers as the KK charges.

M.P.Garcia del Moral, I.Martin, J.Pena, AR.

Classification of symplectic torus bundles

- Given ρ the inequivalent torus bundles are classified by the cohomology classes of $H_2(B, Z_\rho)$.
- Equivalently, by the coinvariant classes of the abelian subgroup of SL(2,Z) determined by ρ.
- Q= (p,q)
- { Q + gQ~ - Q~ }

The Mass operator of the Supermembrane with non trivial winding and a given monodromy

The non trivial winding condition is equivalent to a 2-form flux condition on the target.

The Mass operator can be formulated as a functional of the coinvariants : these are equivalent classes of (p,q) charges and winding terms .

We determine the symmetry relating the (p,q) equivalent charges. Since they determine equivalent torus bundles we identify them as “gauge” equivalent.

When the monodromy is trivial the formulation reduces to the mass formula of the Supermembrane wrapped on a torus with momentum (p,q), which is consistent with the mass formula of BPS states [J.Schwarz].

By dimensional reduction we obtain (p, q) strings with a reduced symmetry according to the monodromy considered.

M.P.García del Moral,C. las Heras, P.León, J.Peña, A.Restuccia