Hidden Hyperbolic Kac-Moody Structures in Supergravity

(work with M. Henneaux, H. Nicolai, A. Kleinschmidt, P. Spindel, B. Julia,…)

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HermannFest- The Sound of Symmetry

Conference in honour of Hermann Nicolai's 70th birthday

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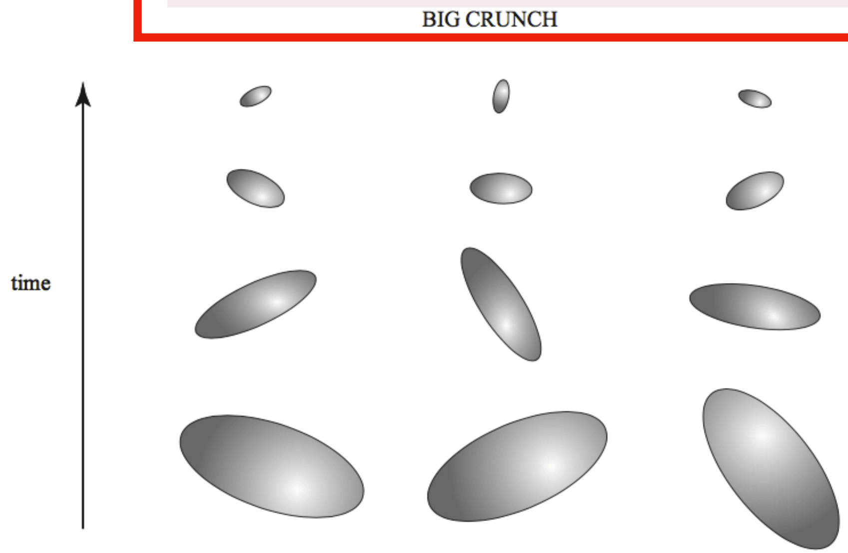
Belinsky-Khalatnikov-Lifshitz (1969) chaotic oscillatory approach near a generic inhomogeneous spacelike singularity in D=3+1 GR

$$ds^2 = -dt^2 + (a^2\,\ell_i\,\ell_j + b^2\,m_i\,m_j + c^2\,n_i\,n_j)\,dx^i\,dx^j$$



Exponential parametrisation:

$$a = e^{-\beta^1}\,;\ b = e^{-\beta^2}\,;\ c = e^{-\beta^3}$$

Billiard description (Misner'69, Chitre'72,…, TD-Henneaux-Nicolai…)

Lagrangian for beta dynamics

$$\mathcal{L} = \frac{1}{2} G_{ab}\,\dot\beta^a\,\dot\beta^b - V(\beta)$$

Potential

kinetic metric (in any d=D-1)

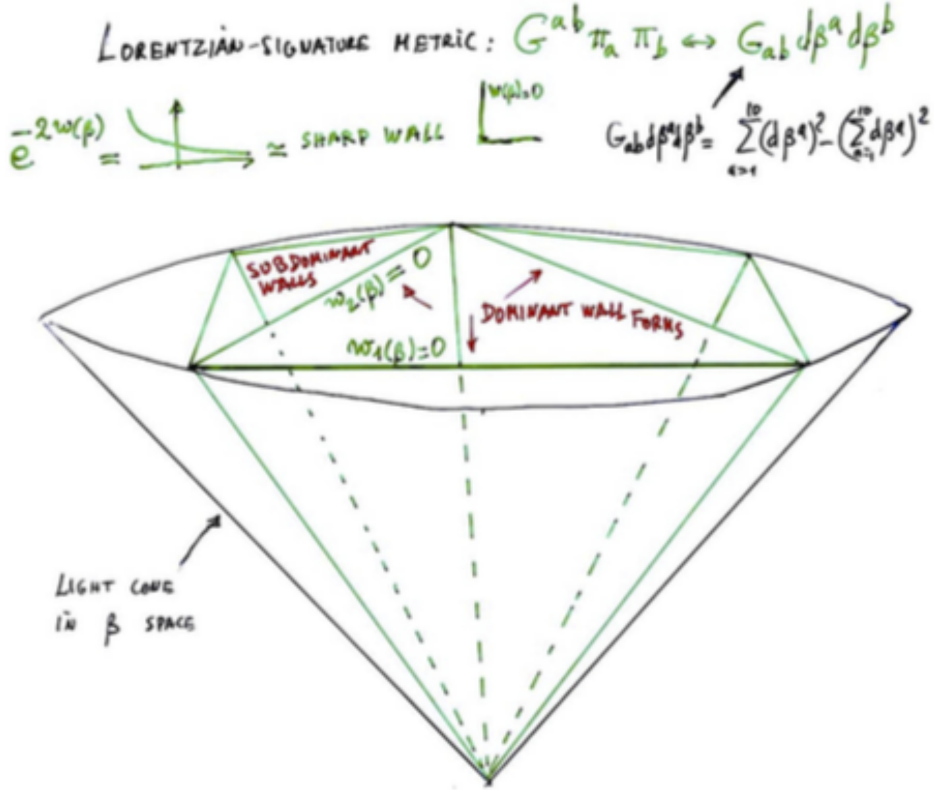
$$G_{ab}\,\dot\beta^a\,\dot\beta^b = \sum_a (\dot\beta^a)^2 - \left(\sum_a \dot\beta^a\right)^2$$

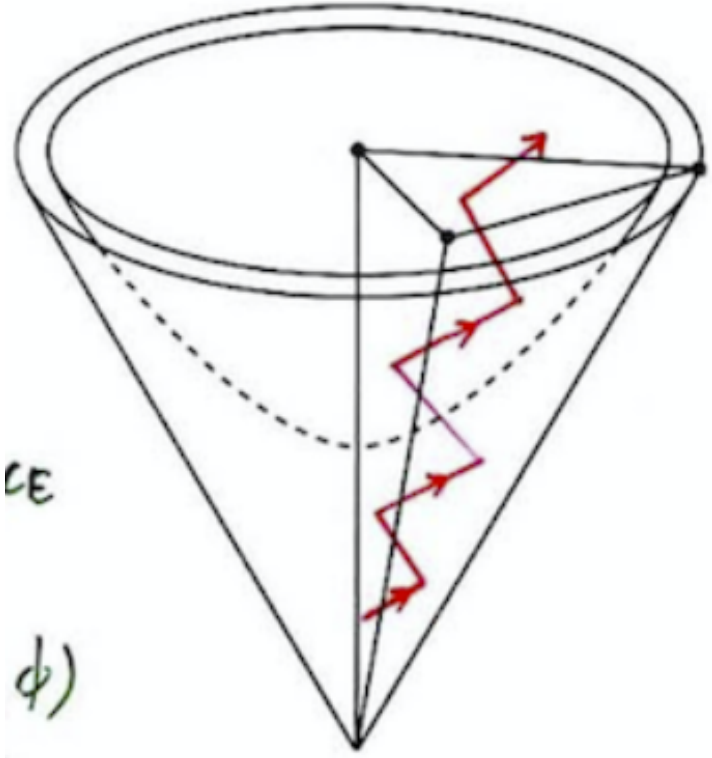
$$V(\beta) = a^4 + b^4 + c^4 - 2(a^2 b^2 + b^2 c^2 + c^2 a^2)$$
$$= \sum c_A(\dots)\, e^{-2 w_A(\beta)}$$

gravitational wall forms in beta space

$$w^{(g)}_{abc}(\beta) = \sum_e \beta^e + \beta^a - \beta^b - \beta^c$$

Billiards in beta space





Appearance of E_10 in maximal SUGRA

E_{10}, BE_{10} and Arithmetical Chaos in Superstring Cosmology

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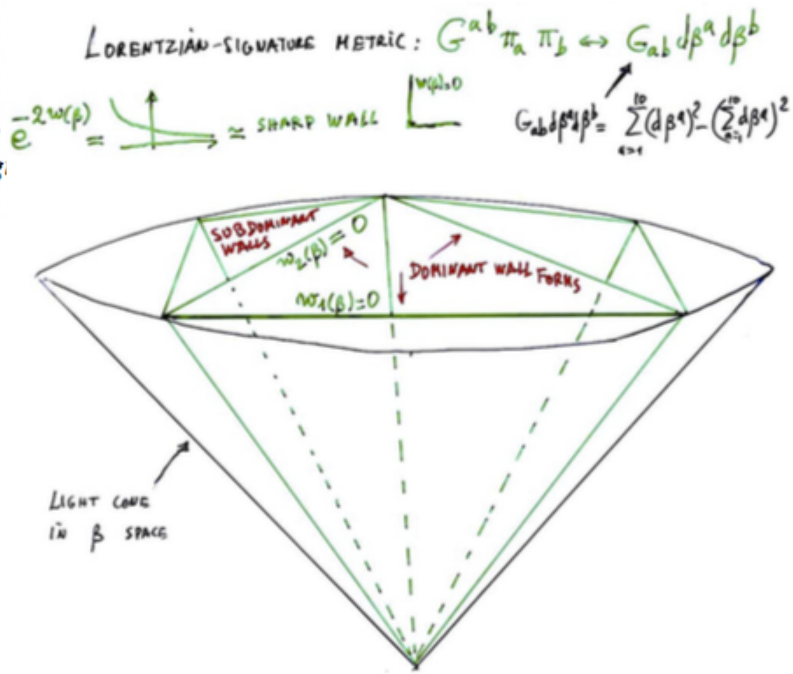


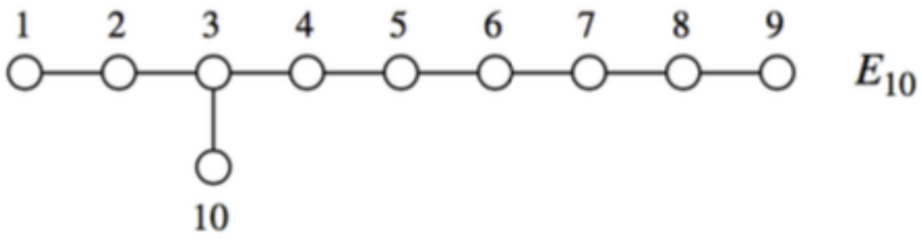
$$S = \int d\tau \left[G_{\mu\nu} \frac{d\beta^\mu}{d\tau} \frac{d\beta^\nu}{d\tau} - V(\beta^\mu) \right]$$

$$V(\beta) \simeq \sum_A C_A e^{-2w_A(\beta)}.$$

liards. The first block (with 2 SUSY's in $D = 10$) is $\mathcal{B}_2 = \{M, \mathrm{IIA}, \mathrm{IIB}\}$ and its ten walls are (in the natural variables of M theory $\beta^\mu = \beta^\mu_M$).

$$\mathcal{B}_2 \colon w_i^{[2]}(\beta) = -\beta^i + \beta^{i+1} \qquad (i = 1, \ldots, 9), \qquad w_{10}^{[2]}(\beta) = \beta^1 + \beta^2 + \beta^3. \tag{4}$$

10 walls
in the
10-dim
Lorentzian
beta space
making up the Weyl chamber of E10



Kac-Moody algebras

Generalization of the well-known "triangular" structure of $A_1 = so(3) = su(2) = sl(2)$: diagonalizable (Cartan) generator: J_z, and raising/lowering generators: $J_\pm = J_x \pm i\, J_y$ with $[J_z, J_+] = +J_+ \,; \quad [J_z, J_-] = -J_- \,; \quad [J_+, J_-] = 2\, J_z$

Rank r: r mutually commuting Cartan generators h_i and r simple raising (e_i) and lowering (f_i) generators:

$$[h_i, h_j] = 0\,; [h_i, e_j] = A_{ij}\, e_j\,; \quad [h_i, f_j] = -A_{ij}\, f_j\,; \quad [e_i, f_j] = \delta_{ij}\, h_j$$

Serre relations: $ad_{e_i}^{1-A_{ij}}\, e_j = 0$; $ad_{f_i}^{1-A_{ij}}\, f_j = 0$

A_{ij} = Cartan matrix: $A_{ii} = +2$, $A_{ij} \in -\mathbb{N}$

Roots: α = linear form on Cartan: $h = \sum_i \beta^i h_i \to \alpha(h) = \alpha_i \beta^i$

$$E_\alpha \sim [e_{i_1}[e_{i_2}[e_{i_3}, \ldots]]] \qquad \alpha = n_1 \alpha^{(1)} + n_2 \alpha^{(2)} + \ldots + n_r \alpha^{(r)}$$

$e_i = E_{\alpha^{(i)}}$ simple roots; $\left[h, E_\alpha^{(s)}\right] = \alpha(h)\, E_\alpha^{(s)}$ $\qquad A_{ij} = \dfrac{2(\alpha^{(i)}, \alpha^{(j)})}{(\alpha^{(i)}, \alpha^{(i)})}$

The gravity/coset conjecture: sound of E10 symmetry?

Bosonic story

E_{10} and a Small Tension Expansion of M Theory

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A formal "small tension" expansion of $D = 11$ supergravity near a spacelike singularity is shown to be equivalent, at least up to 30th order in height, to a null geodesic motion in the infinite-dimensional coset space $E_{10}/K(E_{10})$, where $K(E_{10})$ is the maximal compact subgroup of the hyperbolic Kac-Moody group $E_{10}(\mathbb{R})$. For the proof we make use of a novel decomposition of E_{10} into irreducible representations of its SL(10, $\mathbb{R}$) subgroup. We explicitly show how to identify the first four rungs of the E_{10} coset fields with the values of geometric quantities constructed from $D = 11$ supergravity fields and their spatial gradients taken at some comoving spatial point.

Fermionic story

Hidden symmetries and the fermionic sector of eleven-dimensional supergravity

Thibault Damour [a], Axel Kleinschmidt [b], Hermann Nicolai [b]

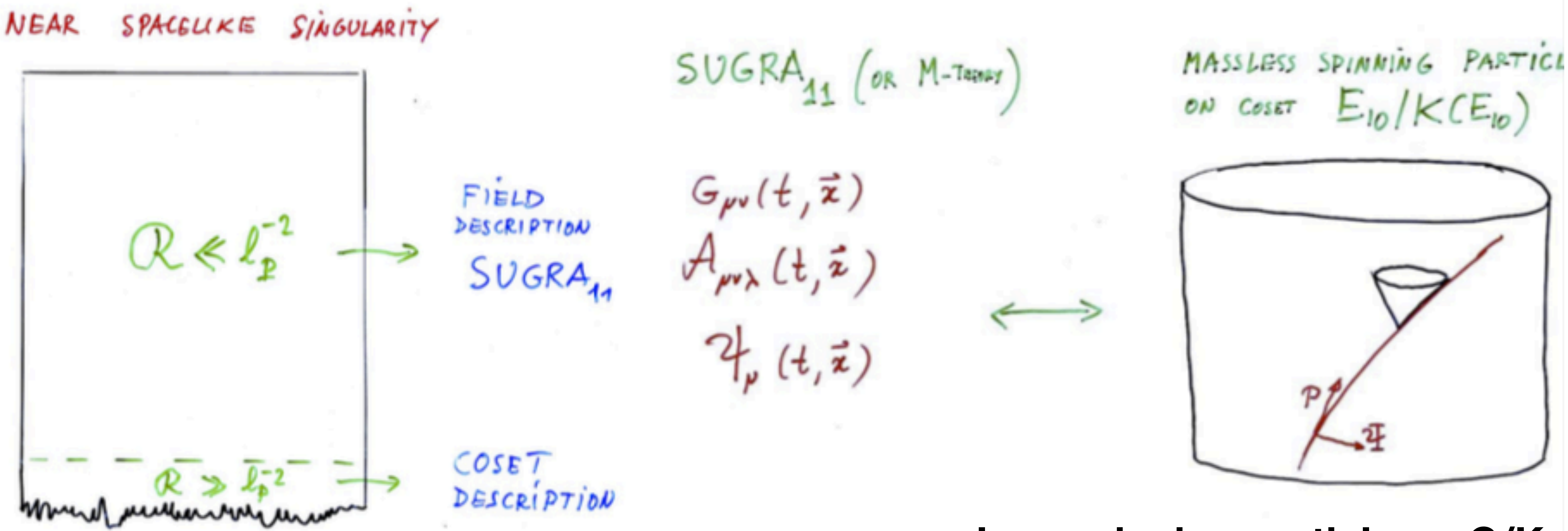
We study the hidden symmetries of the fermionic sector of $D = 11$ supergravity, and the rôle of $K(E_{10})$ as a generalised 'R-symmetry'. We find a consistent model of a massless spinning particle on an $E_{10}/K(E_{10})$ coset manifold whose dynamics can be mapped onto the fermionic and bosonic dynamics of $D = 11$ supergravity in the near space-like singularity limit. This E_{10}-invariant superparticle dynamics might provide the basis of a new definition of M-theory, and might describe the 'de-emergence' of spacetime near a cosmological singularity.

Extended E_8 invariance of 11-dimensional superg

Sophie de Buyl,* Marc Henneaux† and Louis Paulot

ABSTRACT: The hyperbolic Kac-Moody algebra E_{10} has repeatedly been suggested to pl
a crucial role in the symmetry structure of M-theory. Recently, following the analysis
the asymptotic behaviour of the supergravity fields near a cosmological singularity, tl
question has received a new impulse. It has been argued that one way to exhibit t
symmetry was to rewrite the supergravity equations as the equations of motion of t
non-linear sigma model $E_{10}/K(E_{10})$. This attempt, in line with the established result th
the scalar fields which appear in the toroidal compactification down to three spacetir
dimensions form the coset $E_8/SO(16)$, was verified for the first bosonic levels in a lev
expansion of the theory. We show that the same features remain valid when one includ
the gravitino field.

Basic idea: existence of two `dual' descriptions: gravity and coset



massless spinning particle on G/K

Supergravity Description	Coset Description
$G_{\mu\nu}(t,\mathbf{x}), \mathcal{A}_{\mu\nu\lambda}(t,\mathbf{x}), \psi_\mu(t,\mathbf{x})$ in (T^{10} ?) compactified space	$g(t) \in E_{10}(\mathbb{Z})\backslash E_{10}(\mathbb{R})/K_{10}(\mathbb{R})$
$\mathcal{R} \ll \ell_P^{-2}$	$\mathcal{R} \gg \ell_P^{-2}$

Coset model: null geodesic on G/K

(super)gravity $\leftrightarrow$ massless (spinning) particle on G/K

$g(t) \in G/K$; velocity $v \equiv \partial_t g\, g^{-1} \in \mathrm{Lie}(G)$ is decomposed into $v = \mathcal{P} + \mathcal{Q}$ where $\mathcal{Q} \in \mathrm{Lie}(K)$ and $\mathcal{P} = v^{\mathrm{sym}} = \frac{1}{2}(v + v^T) \in \mathrm{Lie}(G) - \mathrm{Lie}(K)$

Coset Action for massless particle:

$$S^{\mathrm{coset}}_{1_{\mathrm{BOS}}} = \int \frac{dt}{n(t)} \frac{1}{4} \langle \mathcal{P}(t), \mathcal{P}(t) \rangle$$

$n(t)$: coset lapse $\rightarrow$ constraint $\langle \mathcal{P}(t), \mathcal{P}(t) \rangle = 0$

For hyperbolic (or more generally Lorentzian) Kac-Moody algebras the coset G/K is an infinite dimensional Lorentzian space of signature $- + + + + + \ldots$

Evidence for Gravity/Coset correspondence

Damour, Henneaux, Nicolai 02; Damour, Kleinschmidt, Nicolai 06; de Buyl, Henneaux, Paulot 06; Kleinschmidt, Nicolai 06

Insert in $S_1^{\rm COSET} = \int dt \{ \frac{1}{4n(t)} \langle P(t), P(t) \rangle - \frac{i}{2} (\Psi(t) \mid \mathcal{D}^{\rm vs} \Psi(t))_{\rm vs} + \dots \}$ the $GL(10)$ level expansion of the coset element

$$g(t) = \exp(h_b^a(t)\, K_a^b) \times$$

$$\times \exp\left[\frac{1}{3!} A_{abc}(t)\, E^{abc} + \frac{1}{6!} A_{a_1 \dots a_6}(t) E^{a_1 \dots a_6} + \frac{1}{9!} A_{a_0 | a_1 \dots a_8}(t)\, E^{a_0 | a_1 \dots a_8} + \dots \right].$$

Agreement (up to height 29) of EOM of $g^{ab}(t) = (e^h)^a_c (e^h)^b_c$, $A_{abc}(t)$, $A_{a_1 \dots a_6}(t)$, $A_{a_0 | a_1 \dots a_8}(t)$, and $\Psi_a^{\rm coset}(t)$ with supergravity EOM (including lowest spatial gradients) for $G_{\mu\nu}(t, \mathbf{x})$, $\mathcal{A}_{\mu\nu\lambda}(t, \mathbf{x})$, $\psi_\mu(t, \mathbf{x})$ with dictionary:

$g^{ab}(t) = G^{ab}(t, \mathbf{x}_0)$, $\qquad \dot{A}_{abc}(t) = \mathcal{F}_{0abc}(t, \mathbf{x}_0)$,

$DA^{a_1 \dots a_6}(t) = -\frac{1}{4!} \varepsilon^{a_1 \dots a_6 b_1 \dots b_4} \mathcal{F}_{b_1 \dots b_4}(t, \mathbf{x}_0)$,

$DA^{b | a_1 \dots a_8}(t) = \frac{3}{2} \varepsilon^{a_1 \dots a_8 b_1 b_2} C^b_{b_1 b_2}(t, \mathbf{x}_0)$

and $\Psi_a^{\rm coset}(t) = G^{1/4} \psi_a(t, \mathbf{x}_0)$

Moreover, $\exists$ roots in E_{10} formally associated with the infinite towers of higher spatial gradients of $G_{\mu\nu}(t, \mathbf{x})$, $\mathcal{A}_{\mu\nu\lambda}(t, \mathbf{x})$, $\psi_\mu(t, \mathbf{x})$

Bosonic (evolution) EOM of SUGRA_11

$D = 11$ spacetime, zero-shift slicing ($N^i = 0$) time-independent spatial coframe $\theta^a(x) \equiv E^a{}_i(x)dx^i$, $i = 1, \ldots, 10$; $a = 1, \ldots, 10$ choose time coordinate x^0 s.t. lapse $N = \sqrt{G}$ with $G := \det G_{ab}$

structure constants of frame: $d\theta^a = \frac{1}{2}C^a{}_{bc}\,\theta^b \wedge \theta^c$; frame derivative $\partial_a \equiv E^i{}_a(x)\partial_i$; 3-form $\mathcal{A}$; 4-form $\mathcal{F} = d\mathcal{A}$; $2G_{ad}\Gamma^d_{\ bc} = C_{abc} + C_{bca} - C_{cab} + \partial_b G_{ca} + \partial_c G_{ab} - \partial_a G_{bc}$

$$ds^2 = -N^2(dx^0)^2 + G_{ab}\theta^a\theta^b$$

$$\mathcal{F} = \frac{1}{3!}\mathcal{F}_{0abc}dx^0 \wedge \theta^a \wedge \theta^b \wedge \theta^c + \frac{1}{4!}\mathcal{F}_{abcd}\theta^a \wedge \theta^b \wedge \theta^c \wedge \theta^d$$

$$\partial_0(G^{ac}\partial_0 G_{cb}) = \frac{1}{6}G\mathcal{F}^{a\beta\gamma\delta}\mathcal{F}_{b\beta\gamma\delta} - \frac{1}{72}G\mathcal{F}^{\alpha\beta\gamma\delta}\mathcal{F}_{\alpha\beta\gamma\delta}\delta^a_b - 2GR^a{}_b(\Gamma, C)$$

$$\partial_0(G\mathcal{F}^{0abc}) = \frac{1}{144}\varepsilon^{abca_1a_2a_3b_1b_2b_3b_4}\mathcal{F}_{0a_1a_2a_3}\mathcal{F}_{b_1b_2b_3b_4} + \frac{3}{2}G\mathcal{F}^{de[ab}C^{c]}{}_{de} - GC^e{}_{de}\mathcal{F}^{dabc} - \partial_d(G\mathcal{F}^{dabc})$$

$$\partial_0\mathcal{F}_{abcd} = 6\mathcal{F}_{0e[ab}C^e{}_{cd]} + 4\,\partial_{[a}\mathcal{F}_{0bcd]}$$

Fermionic (evolution) EOM of SUGRA_11 (linear level in Psi)

In the gauge $\psi_0^{(11)} = \Gamma_0\Gamma^a\psi_a^{(11)}$, the equation of motion of the rescaled gravitino $\psi_a^{(10)} := g^{1/4}\psi_a^{(11)}$ (neglecting cubic terms) reads

$$\begin{aligned}
\mathcal{E}_a \quad = \quad & \partial_t\psi_a^{(10)} + \omega_{t\,ab}^{(11)}\psi^{(10)b} + \frac{1}{4}\omega_{t\,cd}^{(11)}\Gamma^{cd}\psi_a^{(10)} \\
- \quad & \frac{1}{12}F_{tbcd}^{(11)}\Gamma^{bcd}\psi_a^{(10)} - \frac{2}{3}F_{tabc}^{(11)}\Gamma^b\psi^{(10)c} + \frac{1}{6}F_{tbcd}^{(11)}\Gamma_a{}^{bc}\psi^{(10)d} \\
+ \quad & \frac{N}{144}F_{bcde}^{(11)}\Gamma^0\Gamma^{bcde}\psi_a^{(10)} + \frac{N}{9}F_{abcd}^{(11)}\Gamma^0\Gamma^{bcde}\psi_e^{(10)} - \frac{N}{72}F_{bcde}^{(11)}\Gamma^0\Gamma_{abcdef}\psi^{(10)f} \\
+ \quad & N(\omega_{a\,bc}^{(11)} - \omega_{b\,ac}^{(11)})\Gamma^0\Gamma^b\psi^{(10)c} + \frac{N}{2}\omega_{a\,bc}^{(11)}\Gamma^0\Gamma^{bcd}\psi_d^{(10)} - \frac{N}{4}\omega_{b\,cd}^{(11)}\Gamma^0\Gamma^{bcd}\psi_a^{(10)} \\
+ \quad & Ng^{1/4}\Gamma^0\Gamma^b\left(2\partial_a\psi_b^{(11)} - \partial_b\psi_a^{(11)} - \frac{1}{2}\omega_{c\,cb}^{(11)}\psi_a^{(11)} - \omega_{0\,0a}^{(11)}\psi_b^{(11)} + \frac{1}{2}\omega_{0\,0b}^{(11)}\psi_a^{(11)}\right).
\end{aligned}$$

Apart from the last line, this is equivalent to the $K(E_{10})$-covariant equation

$$0 = \overset{\text{vs}}{\mathcal{D}}\Psi(t) := \left(\partial_t - \overset{\text{vs}}{\mathcal{Q}}(t)\right)\Psi(t).$$

expressing the parallel propagation of the $K(E_{10})$ vector-spinor $\Psi(t)$ along the $E_{10}/K(E_{10})$ worldline of the coset particle, with the $K(E_{10})$ connection $\mathcal{Q}(t) := \frac{1}{2}(v(t) - v^T(t)) \in \mathrm{Lie}(\mathrm{K}(\mathrm{E}_{10}))$, with $v(t) = \partial_t g g^{-1} \in \mathfrak{e}_{10} \equiv \mathrm{Lie}(E_{10})$.

Quantum (bosonic) coset model

Iwasawa parametrization of coset element:

$$\mathcal{V}(t) = \exp(\sum \beta^a H_a)\ \exp(\sum \nu^\alpha E_\alpha)$$

classical Hamiltonian

$$H(\beta^a, \pi_a, \nu, p) = n\left[\frac{1}{2}G^{ab}\pi_a\pi_b + \sum_{\alpha>0}\sum_{s=1}^{\mathrm{mult}(\alpha)}\left(\Pi_{\alpha,s}(\nu^\alpha, p_\alpha)\right)^2 \exp\left(-2\alpha(\beta)\right)\right]$$

Quantum Bosonic Coset Model: in configuration space β^a, ν^α

$$\Box_{E_{10}/K_{10}}\Psi(\beta^a, \nu^\alpha) = 0$$

$$\left[-G^{ab}\,\partial_{\beta^a}\,\partial_{\beta^b} - \sum_{\alpha>0} e^{-2\alpha(\beta)}\,\partial^2_{\nu^\alpha} + \dots\right]\Psi(\beta^a, \nu^\alpha) = 0$$

Infinite-dimensional Klein-Gordon type equation: $- + + + + + \dots$

From Hull-Townsend '95 expect $E_{10}(\mathbb{Z})$ symmetry, i.e.

$\Psi(\beta, \nu)$ = automorphic function over $E_{10}(\mathbb{Z})\backslash E_{10}(\mathbb{R})/K_{10}(\mathbb{R})$

(Ganor '99; Brown, Ganor, Helfgott '04) Kac-Moody-Eisenstein series

(Fleig, Kleinschmidt, 12, 13)

Finite-dimensional truncations: quantum cosmo billiards (Kleinschmidt-Nicolai'06)

E10 WDW eq (Kleinschmidt-Nicolai'22)

Shortcomings of coset/gravity correspondence

spatial gradients at levels ell >= 3 beyond dual fields ?

spatial gradients of gravitino?

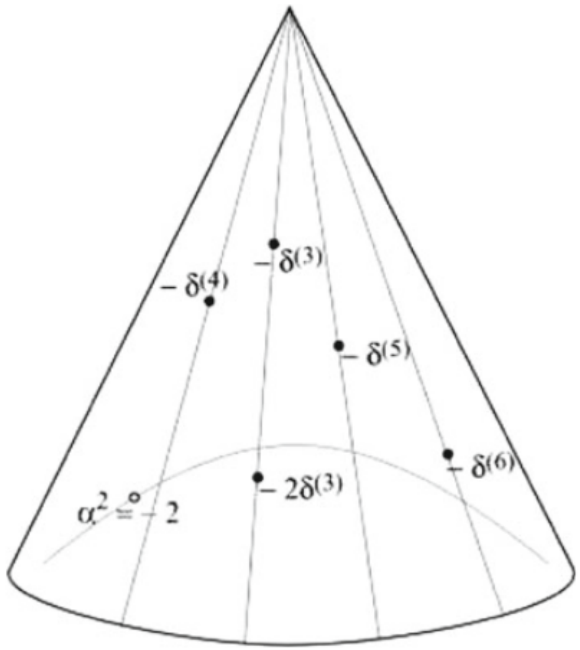
constraints and subgroups of E10?

Sugawara-Type Constraints in Hyperbolic Coset Models

Thibault Damour[1], Axel Kleinschmidt[2], Hermann Nicolai[3]

$$\mathfrak{L}_\alpha = \sum_{\beta \in \Delta^{\text{hyp}}} \sum_{s,s'} M_{s,s'}(\alpha,\beta) J^{(s)}_{\alpha-\beta} J^{(s')}_{\beta}$$

$$\{\mathfrak{L}_\alpha, \mathfrak{L}_\beta\} = \sum_\gamma J_{\alpha+\beta-\gamma} \mathfrak{L}_\gamma \,.$$



Beyond linear gravitino equation: TD-Spindel ’13,’17,'22

Hidden Kac-Moody Structures in the Fermionic Sectors of SUGRA4 and SUGRA5

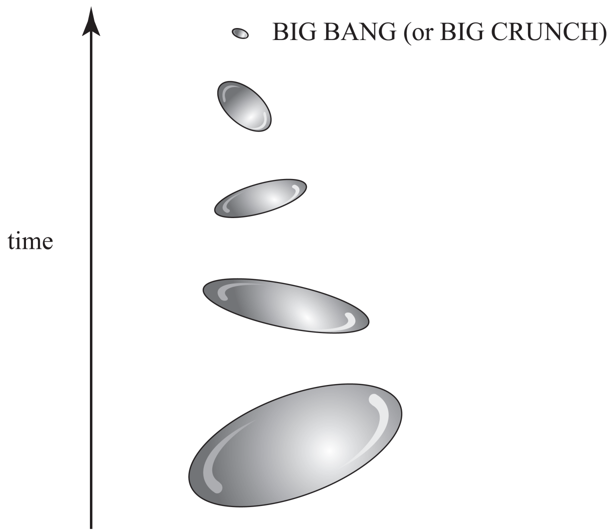
(TD-Spindel ’13,’17,’22)

SUGRA4 Reduction to SU(2)-homogeneous (Bianchi IX) BKL-type model

$$ds^2 = -N(t)^2 dt^2 + g_{ij}(t)(\tau^i + N^i(t)dt)(\tau^j + N^j(t)dt)$$

τ^a: left-invariant one-forms on $SU(2) \approx S_3$: $d\tau^a = \frac{1}{2} C^a_{bc} \tau^b \wedge \tau^c$; here $C^a_{bc} = \varepsilon_{abc}$ plays the role of a nonabelian “gravitational flux”, or constant momentum of (coset) dual graviton.

squashed 3-sphere



- 6 bosonic dof: Gauss-decomposition of the metric: $g_{bc} = \sum_{\hat{a}=1}^{3} e^{-2\beta^a} S^{\hat{a}}{}_b(\varphi_1, \varphi_2, \varphi_3) S^{\hat{a}}{}_c(\varphi_1, \varphi_2, \varphi_3)$
$\beta^a = (\beta^1(t), \beta^2(t), \beta^3(t))$ cologarithms of the squashing parameters a, b, c of 3-sphere $a = e^{-\beta^1}$, $b = e^{-\beta^2}$, $c = e^{-\beta^3}$ and three Euler angles: $\varphi_a = (\varphi_1(t), \varphi_2(t), \varphi_3(t))$ parametrizing the orthogonal matrix $S^{\hat{a}}{}_b$

- and 12 fermionic dof: Gravitino components in specific gauge-fixed orthonormal frame $\theta^{\hat{\alpha}}$ canonically associated with the Gauss-decomposition $\theta^{\hat{0}} = N(t)dt$, $\theta^{\hat{a}} = \sum_b e^{-\beta^a(t)} S^{\hat{a}}{}_b(\varphi_c(t))(\tau^b(x) + N^b(t)dt)$
- redefinitions of the gravitino field:

$$\Psi^A_{\hat{\alpha}}(t) := g^{1/4} \psi^A_{\hat{\alpha}} \quad \text{and} \quad \Phi^a_A := \Sigma_B \gamma^{\hat{a}}_{AB} \Psi^B_{\hat{a}} \quad \text{(no summation on } \hat{a}\text{)}$$

- 3×4 gravitino components Φ^a_A, $a = 1, 2, 3$; $A = 1, 2, 3, 4$.

Quantization: wave function $\Psi(\beta^a, \varphi^a, \sigma)$

3 log-scale factors → β^a

3 angles → φ^a

$$\left[\hat{\beta}^a, \hat{\pi}_b\right] = i\hbar\delta^a_b \qquad \widehat{\pi}_a = -i\,\frac{\partial}{\partial\beta^a}\ ; \quad \widehat{p}_{\varphi^a} = -i\,\frac{\partial}{\partial\varphi^a}$$

anticommutation relation of fermions:

$$\boxed{\widehat{\Phi}^a_A\,\widehat{\Phi}^b_B + \widehat{\Phi}^b_B\,\widehat{\Phi}^a_A = G^{ab}\,\delta_{AB}}$$

← **quantized gravitino zero-mode**

Here G_{ab} is the Lorentzian-signature quadratic form which defines the kinetic terms of of the gravitino, as well as those of the \beta^a's

$$G_{ab}\,d\beta^a\,d\beta^b \equiv \sum_a (d\beta^a)^2 - \left(\sum_a d\beta^a\right)^2$$

Algebra of fermions=Clifford algebra Spin(8^+ , 4^-)

The wave function of the universe $\boxed{\Psi_\sigma(\beta,\varphi)}$
is a 64-dimensional spinor of Spin(8,4) and the gravitino operators \Phi_A^a are 64 x 64 ``gamma matrices'' acting on $\Psi_\sigma(\beta,\varphi)\,,\ \sigma = 1,\cdots,64$

NB: for SUGRA_11 Fermions have 2^160 ~ 10^48 components !!

Dirac Quantization of the Constraints

$$\widehat{\mathcal{S}}_A \, \Psi = 0 \, , \qquad \widehat{H} \, \Psi = 0 \, , \qquad \widehat{H}_a \, \Psi = 0$$

Diffeomorphism constraint:

$$\widehat{H}_a \, \Psi = 0 \Leftrightarrow -i \, \frac{\partial}{\partial \varphi^a} \, \Psi = 0$$

Crucial SUSY constraints:
4x64 1st order PDEs for
the 64 functions Psi_sigma(beta)

$$\widehat{\mathcal{S}}_A \, \Psi(\beta) = 0$$

$$\widehat{\mathcal{S}}_A \widehat{\mathcal{S}}_B + \widehat{\mathcal{S}}_B \widehat{\mathcal{S}}_A \sim \widehat{H} \delta_{AB} + O(\widehat{\mathcal{S}}_C)$$

$$\widehat{\mathcal{S}}_A = -\frac{1}{2} \sum_a \widehat{\pi}_a \, \Phi^a_A + \frac{1}{2} \sum_a e^{-2\beta^a} (\gamma^5 \, \Phi^a)_A$$
$$- \frac{1}{8} \coth \beta_{12} (\widehat{S}_{12} (\gamma^{12} \, \widehat{\Phi}^{12})_A + (\gamma^{12} \, \widehat{\Phi}^{12})_A \, \widehat{S}_{12})$$
$$+ \, \mathrm{cyclic}_{(123)} + \frac{1}{2} (\widehat{\mathcal{S}}^{\mathrm{cubic}}_A + \widehat{\mathcal{S}}^{\mathrm{cubic}\,\dagger}_A)$$

with

$$\widehat{S}_{12}(\widehat{\Phi}) = \frac{1}{2} [(\bar{\widehat{\Phi}}^3 \, \gamma^{\hat{0}\hat{1}\hat{2}} (\widehat{\Phi}^1 + \widehat{\Phi}^2)) + (\bar{\widehat{\Phi}}^1 \, \gamma^{\hat{0}\hat{1}\hat{2}} \, \widehat{\Phi}^1)$$
$$+ (\bar{\widehat{\Phi}}^2 \, \gamma^{\hat{0}\hat{1}\hat{2}} \, \widehat{\Phi}^2) - (\bar{\widehat{\Phi}}^1 \, \gamma^{\hat{0}\hat{1}\hat{2}} \, \widehat{\Phi}^2)]$$

$$\widehat{\mathcal{S}}^{\mathrm{cubic}}_A \sim \sum \Phi \, \Phi \, \Phi$$

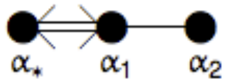
$$\widehat{\mathcal{S}}^{\mathrm{cubic}}_A = \frac{1}{4} \sum_a (\bar{\widehat{\Psi}}_0, \gamma^{\hat{0}} \, \widehat{\Psi}_{\hat{a}}) \, \gamma^{\hat{0}} \, \widehat{\Psi}^A_{\hat{a}} - \frac{1}{8} \sum_{a,b} (\bar{\widehat{\Psi}}_{\hat{a}} \, \gamma^{\hat{0}} \, \widehat{\Psi}_{\hat{b}}) \, \gamma^{\hat{a}} \, \widehat{\Psi}^A_{\hat{b}}$$
$$+ \frac{1}{8} \sum_{a,b} (\bar{\widehat{\Psi}}_0, \gamma^{\hat{a}} \, \Psi_{\hat{b}}) (\gamma^{\hat{a}} \, \Psi^A_{\hat{b}} + \gamma^{\hat{b}} \, \Psi^A_{\hat{a}}) \, ,$$

Hidden Kac-Moody (AE3) Structures in the Hamiltonian

$$2\,\widehat{H} = G^{ab}(\widehat{\pi}_a + i\,A_a)(\widehat{\pi}_b + i\,A_b) + \widehat{\mu}^2 + W_g^{\mathrm{bos}}(\beta) + \widehat{W}_g^{\mathrm{spin}}(\beta) + \widehat{W}_{\mathrm{sym}}^{\mathrm{spin}}(\beta)\,.$$

$G_{ab} \leftrightarrow$ metric in Cartan subalgebra of AE_3

$$(A_{ij}) = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$



$$W_g^{\mathrm{bos}}(\beta) = \frac{1}{2}\,e^{-2\alpha^g_{11}(\beta)} - e^{-2\alpha^g_{23}(\beta)} + \mathrm{cyclic}_{123}$$

$$\widehat{W}_g^{\mathrm{spin}}(\beta, \widehat{\Phi}) = e^{-\alpha^g_{11}(\beta)}\widehat{J}_{11}(\widehat{\Phi}) + e^{-\alpha^g_{22}(\beta)}\widehat{J}_{22}(\widehat{\Phi}) + e^{-\alpha^g_{33}(\beta)}\widehat{J}_{33}(\widehat{\Phi})\,.$$

Linear forms $\alpha^g_{ab}(\beta) = \beta^a + \beta^b \Leftrightarrow$ six level-1 roots of AE_3

$$\widehat{W}_{\mathrm{sym}}^{\mathrm{spin}}(\beta) = \frac{1}{2}\,\frac{(\widehat{S}_{12}(\widehat{\Phi}))^2 - 1}{\sinh^2 \alpha^{\mathrm{sym}}_{12}(\beta)} + \mathrm{cyclic}_{123}\,,$$

Linear forms $\alpha^{\mathrm{sym}}_{12}(\beta) = \beta^1 - \beta^2$, $\alpha^{\mathrm{sym}}_{23}(\beta) = \beta^2 - \beta^3$, $\alpha^{\mathrm{sym}}_{31}(\beta) = \beta^3 - \beta^1$
$\Leftrightarrow$ three level-0 roots of AE_3

Fermionic Operators coupled to AE3 roots Kac-Moody

$$\begin{aligned} \widehat{S}_{12}(\widehat{\Phi}) &= \frac{1}{2}[(\bar{\widehat{\Phi}}{}^3\gamma^{\hat{0}\hat{1}\hat{2}}(\widehat{\Phi}^1 + \widehat{\Phi}^2)) + (\bar{\widehat{\Phi}}{}^1\gamma^{\hat{0}\hat{1}\hat{2}}\,\widehat{\Phi}^1) \\ &+ (\bar{\widehat{\Phi}}{}^2\gamma^{\hat{0}\hat{1}\hat{2}}\,\widehat{\Phi}^2) - (\bar{\widehat{\Phi}}{}^1\gamma^{\hat{0}\hat{1}\hat{2}}\,\widehat{\Phi}^2)]\,, \end{aligned}$$

$$\widehat{J}_{11}(\widehat{\Phi}) = \frac{1}{2}[\;\bar{\widehat{\Phi}}{}^1\gamma^{\hat{1}\hat{2}\hat{3}}(4\widehat{\Phi}^1 + \widehat{\Phi}^2 + \widehat{\Phi}^3) + \;\bar{\widehat{\Phi}}{}^2\gamma^{\hat{1}\hat{2}\hat{3}}\,\widehat{\Phi}^3]$$

$$\widehat{S}_{12}(\widehat{\Phi}) = \frac{1}{2}[(\bar{\widehat{\Phi}}^3\gamma^{\hat{0}\hat{1}\hat{2}}(\widehat{\Phi}^1+\widehat{\Phi}^2))+(\bar{\widehat{\Phi}}^1\gamma^{\hat{0}\hat{1}\hat{2}}\,\widehat{\Phi}^1) + (\bar{\widehat{\Phi}}^2\gamma^{\hat{0}\hat{1}\hat{2}}\,\widehat{\Phi}^2)-(\bar{\widehat{\Phi}}^1\gamma^{\hat{0}\hat{1}\hat{2}}\,\widehat{\Phi}^2)]\,,$$

$$\widehat{J}_{11}(\widehat{\Phi}) = \frac{1}{2}[\ \bar{\widehat{\Phi}}^1\gamma^{\hat{1}\hat{2}\hat{3}}(4\widehat{\Phi}^1+\widehat{\Phi}^2+\widehat{\Phi}^3)+\ \bar{\widehat{\Phi}}^2\gamma^{\hat{1}\hat{2}\hat{3}}\,\widehat{\Phi}^3]$$

- $\widehat{S}_{12}$, $\widehat{S}_{23}$, $\widehat{S}_{31}$, $\widehat{J}_{11}$, $\widehat{J}_{22}$, $\widehat{J}_{33}$ generate (via commutators) a 64-dimensional representation of the (infinite-dimensional) "maximally compact" sub-algebra $K(AE_3) \subset AE_3$. [The fixed set of the (linear) Chevalley involution, $\omega(e_i) = -f_i$, $\omega(f_i) = -e_i$, $\omega(h_i) = -h_i$, which is generated by $x_i = e_i - f_i$.]

In the middle of the Weyl chamber (far from all the hyperplanes $\alpha_i(\beta) = 0$):

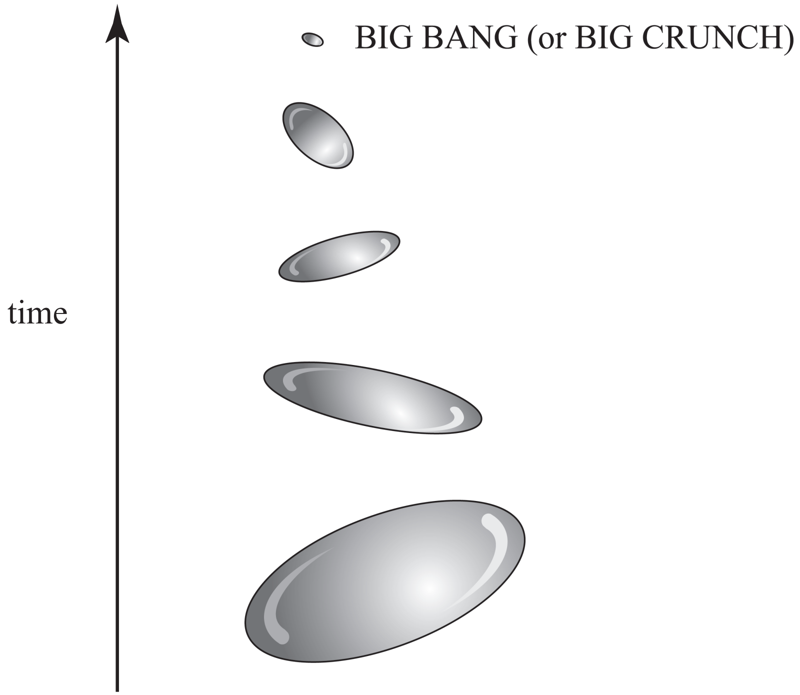
$$2\,\widehat{H} \simeq \widehat{\pi}^2 + \widehat{\mu}^2$$

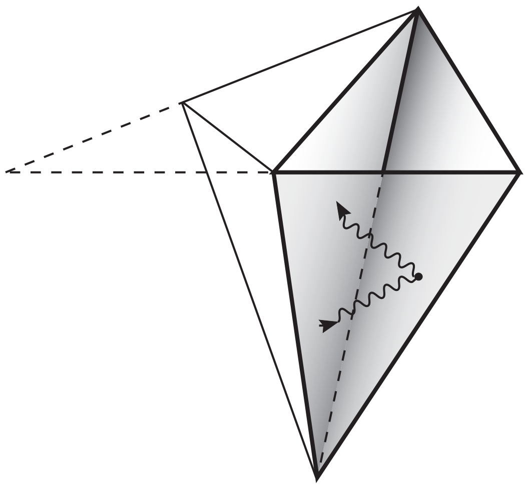
where $\widehat{\mu}^2 \sim \sum \widehat{\Phi}^4$ gathers many complicated quartic-in-fermions terms (including $\sum \widehat{S}_{ab}^2$ and the infamous ψ^4 terms of supergravity).

Remarkable Kac-Moody-related facts:

- $\widehat{\mu}^2 \in$ *Center* of the algebra generated by the $K(AE_3)$ generators $\widehat{S}_{ab}$, $\widehat{J}_{ab}$
- $\widehat{\mu}^2$ is $\sim$ the square of a very simple operator $\in$ Center

Quantum Fermionic generalization of Belinsky-Khalatnikov-Lifshitz chaotic oscillations (quantum fermionic billiard, `a la TD-Hillmann'09)





The Dirac-like SUSY equations for the propagation of the spinorial wave function in \beta-space

$$\widehat{\mathcal{S}}_A \, \Psi = \left(\frac{i}{2} \, \Phi_A^a \, \frac{\partial}{\partial \beta^a} + \ldots \right) \Psi = 0$$

leads to ``reflection operators'' on three hyperplanes in beta-space which have the form

$$\widehat{\mathcal{R}}_{\alpha_i} = \exp \left(-i \, \frac{\pi}{2} \, \widehat{\varepsilon}_{\alpha_i} \, \widehat{J}_{\alpha_i} \right)$$

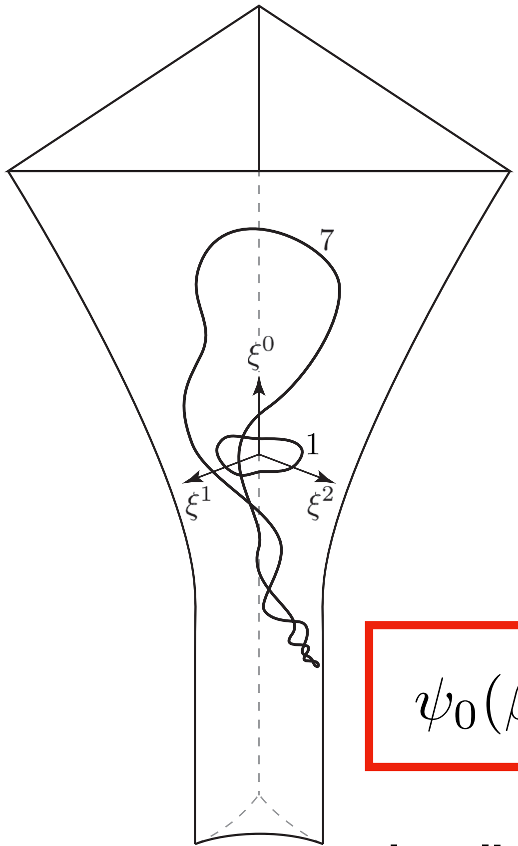
where the operators J

$$\widehat{J}_{\alpha_i} = \{ \widehat{S}_{23} \, , \, \widehat{S}_{31} \, , \, \widehat{J}_{11} \}$$

define a representation of the compact subalgebra of the hyperbolic Kac-Moody AE_3 (spinorial extension of the Weyl group of AE_3)

-> conjecture of « duality » between gravity and Kac-Moody coset

the quantum SUGRA effects quartic in fermions dominate the dynamics near a small-volume singularity and can generically lead to a quantum avoidance of a singularity, i.e. a bounce of the universe



$$\rho_4 = \sum \psi^4 = -C V_3^{-2} = -C(abc)^{-2}$$

$$g_{ij} \sim \mathrm{diag}(a^2, b^2, c^2); a = e^{-\beta^1}; \mathrm{etc}$$

in particular: « ground-state »-like wave function:

$$\psi_0(\beta) = abc \left[(b^2 - a^2)(c^2 - b^2)(a^2 - c^2)\right]^{3/8} e^{-\frac{1}{2}(a^2+b^2+c^2)} |0\rangle_-$$

localized around a,b,c ~ 1=L_Planck; vanishes both as a,b,c->0 and -> infinity

Hidden Kac-Moody Structures in the Fermionic Sectors of SUGRA5

Reducing SUGRA5 (Chamseddine-Nicolai'80, Cremmer'80) to one timelike dimension (TD-Spindel '22)

$$L = \frac{1}{4}R(\mathring{\omega}) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{6\sqrt{3}}\eta^{\mu\nu\lambda\rho\sigma}A_\mu F_{\nu\lambda}F_{\rho\sigma} + \frac{1}{2}(\bar{\psi}_\mu\gamma^{\mu\nu\rho}\mathcal{D}_\nu(\mathring{\omega})\psi_\rho - \overline{\mathcal{D}_\nu(\mathring{\omega})\psi_\mu}\gamma^{\mu\nu\rho}\psi_\rho)$$
$$- i\frac{\sqrt{3}}{4}(\bar{\psi}_\mu\gamma^{\mu\nu\rho\sigma}\psi_\nu + \bar{\psi}^\rho\psi^\sigma - \bar{\psi}^\sigma\psi^\rho)F_{\rho\sigma} + \bar{\psi}_{[\mu}\gamma^\mu\psi_{\alpha]}\bar{\psi}^{[\nu}\gamma_\nu\psi^{\alpha]} - \frac{1}{2}\bar{\psi}_{[\mu}\gamma_{|\nu|}\psi_{\rho]}\bar{\psi}^{[\mu}\gamma^{|\rho|}\psi^{\nu]}$$
$$- \frac{1}{4}\bar{\psi}_{[\mu}\gamma^\nu\psi_{\rho]}\bar{\psi}^{[\mu}\gamma_\nu\psi^{\rho]} + \frac{1}{4}\bar{\psi}_\mu\psi_\nu\bar{\psi}_\rho\gamma^{\mu\nu\rho\sigma}\psi_\sigma + \frac{3}{8}(\bar{\psi}_\mu\psi_\nu - \bar{\psi}_\nu\psi_\mu)\bar{\psi}^\mu\psi^\nu.$$

zero-modes: minisuperspace $g_{\mu\nu}(t)$, $A_\mu(t)$, and $\psi_\mu(t)$

upper triangular

Iwasawa decomp of spatial 4-bein: $\theta^{\hat{a}}_i = e^{-\beta^{\hat{a}}}(\mathcal{N})^{\hat{a}}{}_i$

4 diagonal scale factors

$$B_{\hat{a}} \equiv A_i(\mathcal{N}^{-1})^i{}_{\hat{a}}$$

$$\mathcal{L} = \mathcal{L}_R + \mathcal{L}_{F^2} + \mathcal{L}_{RS} + \mathcal{L}_{F\Psi^2} + \mathcal{L}_{\Psi^4}$$

Hamiltonian:

$$H^{\text{tot}} = H^{(0)} + H^{(2)} + H^{(4)} + \bar{\Psi}'_{\hat{0}}\mathcal{S} + \bar{\mathcal{S}}\Psi'_{\hat{0}};$$

$$H^{(0)} = G^{ab}\pi_a\pi_b + 2\sum_{a<b} e^{-2(\beta^b-\beta^a)}(P_{ab})^2 + \frac{1}{2}\sum_a e^{-2\beta^a}(P^a)^2,$$

$$H^{(2)} = +2\sum_{a<b} e^{-(\beta^b-\beta^a)} P_{ab} J_{ab}(\Psi) - \frac{1}{\sqrt{3}}\sum_a e^{-\beta^a} P^a J_a(\Psi)$$

$$H^{(4)} = \frac{1}{2}\sum_{a<b}(J_{ab}(\Psi))^2 + \frac{1}{6}\sum_a (J_a(\Psi))^2 - L^{\text{cg}}_{\Psi^4};$$

$$J_{ab}(\Psi) = (G_{cd} - 2\alpha_c^{(ab)}\alpha_d^{(ab)})\Phi^{\dagger c}\left(\frac{i\gamma^{ab}}{2}\right)\Phi^d,$$

$$J_a(\Psi) = (G_{cd} - 2\alpha_c^{(a)}\alpha_d^{(a)})\Phi^{\dagger c}\left(\frac{3\gamma^a}{2}\right)\Phi^d,$$

$$\alpha^{(ab)}(\beta) \equiv \alpha_c^{(ab)}\beta^c \equiv \beta^b - \beta^a,$$

$$\alpha^{(a)}(\beta) \equiv \alpha_c^{(a)}\beta^c \equiv \beta^a.$$

$$\mathcal{S} = \mathcal{S}^{(1)} + \mathcal{S}^{(3)}$$

$$\mathcal{S}^{(1)} = \sum_a \pi_a\Phi^a - \sum_{a<b} e^{-(\beta^b-\beta^a)} P_{ab}\gamma^{ab}(\Phi^b - \Phi^a) - i\frac{\sqrt{3}}{2}\sum e^{-\beta^a} P^a\gamma^a\Phi^a,$$

$$\mathcal{S}^{(3)} = -\frac{1}{2}\sum_{\hat{p},\hat{q}}(\bar{\Psi}_{\hat{p}}\gamma_{\hat{q}}\Psi_{\hat{q}} - \bar{\Psi}_{\hat{q}}\gamma_{\hat{q}}\Psi_{\hat{p}})\gamma_{\hat{0}}\Psi_{\hat{p}} - \frac{1}{2}\sum_{\hat{q},\hat{p}>\hat{q}}(\bar{\Psi}_{\hat{p}}\gamma_{\hat{0}}\Psi_{\hat{q}} - \bar{\Psi}_{\hat{q}}\gamma_{\hat{0}}\Psi_{\hat{p}})\gamma_{\hat{p}}\Psi_{\hat{q}} - \frac{i}{2}\sum_{\hat{p},\hat{q},\hat{r},\hat{s}}\eta^{\hat{0}\hat{p}\hat{q}\hat{r}\hat{s}}(\bar{\Psi}_{\hat{p}}\gamma_{\hat{q}}\Psi_{\hat{r}})\Psi_{\hat{s}}$$
$$+\frac{i}{2}\sum_{\hat{p},\hat{q},\hat{s},\hat{r}>\hat{s}}\eta^{\hat{0}\hat{p}\hat{q}\hat{r}\hat{s}}(\bar{\Psi}_{\hat{p}}\Psi_{\hat{q}})\gamma_{\hat{s}}\Psi_{\hat{r}} + \frac{i}{4}\sum_{\hat{q},\hat{r},\hat{s},\hat{k},\hat{p}>\hat{k}}\eta^{\hat{0}\hat{p}\hat{q}\hat{r}\hat{s}}(\bar{\Psi}_{\hat{k}}\gamma_{\hat{q}\hat{r}}\Psi_{\hat{s}} + \bar{\Psi}_{\hat{s}}\gamma_{\hat{q}\hat{r}}\Psi_{\hat{k}})(\gamma_{\hat{k}}\Psi_{\hat{p}} + \gamma_{\hat{p}}\Psi_{\hat{k}})$$

Appearance of G2^++ and K(G2^++)

Four dominant potential walls= simple roots of G2^++

$$\alpha^{(1)}(\beta) = \beta^1,$$
$$\alpha^{(12)}(\beta) = \beta^2 - \beta^1,$$
$$\alpha^{(23)}(\beta) = \beta^3 - \beta^2,$$
$$\alpha^{(34)}(\beta) = \beta^4 - \beta^3,$$

$$A_{ij} = 2\frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle}$$

$$(A_{ij}) = \begin{pmatrix} 2 & -3 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

$$\mathrm{ad}(e_i)^{1-A_{ij}}(e_j) = 0; \qquad \mathrm{ad}(f_i)^{1-A_{ij}}(f_j) = 0$$

$$[h_i, h_j] = 0; \qquad [e_i, f_j] = \delta_{ij} h_j;$$
$$[h_i, e_j] = A_{ij} e_j; \qquad [h_i, f_j] = -A_{ij} f_j$$

$$K(G_2^{++}) : \ x_i \equiv e_i - f_i$$

Serre-Berman relations for K(G2^++):

simple root of G_2^{++}, as listed in Eq. (4.4). Therefore, we will denote them simply as J_1, J_{12}, J_{23}, and J_{34}, respectively associated with $\alpha^{(1)}(\beta)$, $\alpha^{(12)}(\beta)$, $\alpha^{(23)}(\beta)$, and $\alpha^{(34)}(\beta)$.

$$\mathrm{ad}^4(J_1)J_{12} - 10\ \mathrm{ad}^2(J_1)J_{12} + 9J_{12} = 0,$$
$$\mathrm{ad}^2(J_{12})J_1 - J_1 = 0,$$
$$\mathrm{ad}^2(J_{12})J_{23} - J_{23} = 0,$$
$$\mathrm{ad}^2(J_{23})J_{12} - J_{12} = 0,$$
$$\mathrm{ad}^2(J_{23})J_{34} - J_{34} = 0,$$
$$\mathrm{ad}^2(J_{34})J_{23} - J_{23} = 0,$$
$$[J_1, J_{23}] = [J_1, J_{34}] = [J_{12}, J_{34}] = 0.$$

Consistency of quantum constraints

$$\{\hat{\Phi}^{aA},\hat{\Phi}^{bB}\}=0, \qquad \{\hat{\Phi}^{\dagger aA},\hat{\Phi}^{\dagger bB}\}=0$$

$$\{\hat{S}_A,\hat{S}_B^\dagger\}=i\hbar L(\hat{\Phi})^{\dagger C}_{AB}\hat{S}_C-i\hbar L(\hat{\Phi})^{C}_{AB}\hat{S}^\dagger_C+\hbar\delta_{AB}\hat{H}_1 \qquad \{\hat{\Phi}^{aA},\hat{\Phi}^{\dagger bB}\}=\hbar G^{ab}\delta^{AB},$$

$$\hat{H}^{(0)}=G^{ab}\hat{\pi}_a\hat{\pi}_b+2\sum_{a<b}e^{-2(\beta^b-\beta^a)}(\hat{P}_{ab})^2+\frac{1}{2}\sum_a e^{-2\beta^a}(\hat{P}^a)^2,$$

$$\hat{H}^{(2)}=+2\sum_{a<b}e^{-(\beta^b-\beta^a)}\hat{P}_{ab}\hat{J}_{ab}(\Psi)-\frac{1}{\sqrt{3}}\sum_a e^{-\beta^a}\hat{P}^a\hat{J}_a(\Psi),$$

$$\hat{J}_{ab}(\Psi)=(G_{cd}-2\alpha_c^{(ab)}\alpha_d^{(ab)})\hat{\Phi}^{\dagger c}\left(\frac{i\gamma^{ab}}{2}\right)\hat{\Phi}^d,$$

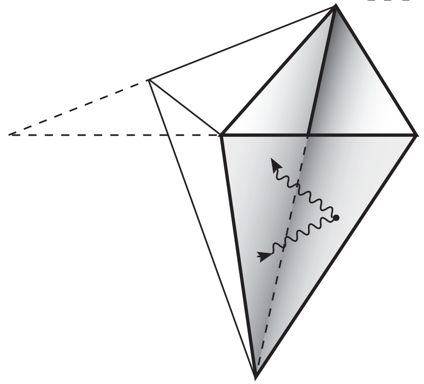
$$\hat{J}_a(\Psi)=(G_{cd}-2\alpha_c^{(a)}\alpha_d^{(a)})\hat{\Phi}^{\dagger c}\left(\frac{3\gamma^a}{2}\right)\hat{\Phi}^d,$$

generate a 2^16-dim rep. of K(G2^++)

$$[\hat{J}_{\alpha_I},\hat{\mu}^2]=0; \quad \text{for } I=(ab),(a).$$

$$H^{(4)}=\frac{1}{2}\sum_{a<b}(J_{ab}(\Psi))^2+\frac{1}{6}\sum_a(J_a(\Psi))^2-L^{\rm cg}_{\Psi^4}$$

invariance of quartic term in H wrt K(G2^++)



quantum reflection operators

$$\hat{\mathcal{R}}_\alpha=e^{i\frac{\pi}{2}\hat{J}_\alpha}$$

Conclusions

There is an array of striking, tantalizing facts suggesting the presence of hyperbolic Kac-Moody structures (E10, K(E10),…) in supergravity.

However, the precise role, and extent, of these structures remains unclear (sound of symmetry?)

Thank you Hermann for your crucial role in pointing towards, and deciphering, the possible role of these structures, and for your frienship