Michela Mapelli

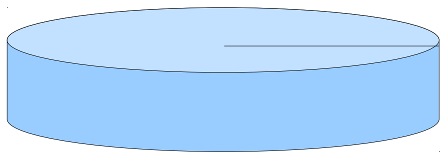
LECTURES on COLLISIONAL DYNAMICS:

1. RELEVANT TIMESCALES

COLLISIONAL/COLLISIONLESS?

Collisional systems are systems where interactions between particles are EFFICIENT with respect to the lifetime of the system

Collisionless systems are systems where interactions are negligible



When is a system collisional/collisionless?

RELAXATION TIMESCALE

Gravity is a LONG-RANGE force → cumulative influence on each star/body of distant stars/bodies is important: often more important than influence of close stars/bodies

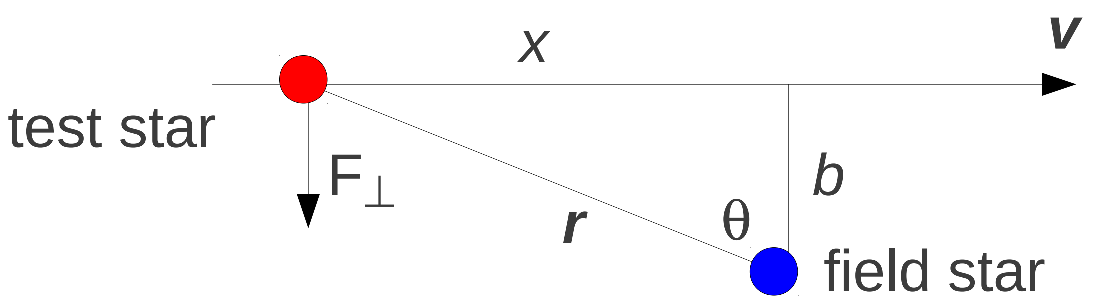
Let us consider a IDEALIZED galaxy of N identical stars with mass *m*, size *R* and uniform density

Let us focus on a single star that crosses the system

How long does it take for this star to change its initial velocity completely?, i.e. by

$$\frac{\delta\vec{v}_{\perp}}{\vec{v}} \sim 1$$

Let us assume that our test star passes close to a field star at relative velocity *v* and impact parameter *b*



The test star and the perturber interact with a force

$$F_\perp = \frac{G\,m^2}{r^3}\,r_\perp = \frac{G\,m^2}{r^2}\,\cos\theta$$

$$= \frac{G\,m^2\,b}{(x^2+b^2)^{3/2}} = \frac{G\,m^2}{b^2}\,\frac{1}{\left[1+\left(\frac{v\,t}{b}\right)^2\right]^{3/2}}$$

From Newton's second law $m\,\dot{v}_\perp = F_\perp$

we get that the perturbation of the velocity integrated over one entire encounter is

$$\delta v_\perp = \int_{-\infty}^{+\infty} \frac{F_\perp}{m} dt = \frac{G\,m}{b^2} \int_{-\infty}^{+\infty} \frac{dt}{\left[1 + \left(\frac{v\,t}{b}\right)^2\right]^{3/2}}$$

$$= \frac{G\,m}{b\,v} \int_{-\infty}^{+\infty} \frac{ds}{(1+s^2)^{3/2}} = \frac{2\,G\,m}{b\,v}$$

$$dt = \frac{b}{v} d\left(\frac{v\,t}{b}\right)$$

Force at closest approach

Force duration

$$2\left(\frac{G\,m}{b^2}\right)\left(\frac{b}{v}\right)$$

Now we account for all the particles in the system

Surface density of stars in idealized galaxy: $\frac{N}{\pi R^2}$

Number of interactions per unit element:

$$\delta n = \frac{N}{\pi R^2} 2 \pi b \, db$$

We define

$$\delta v^2_{\mathrm{TOT}} = \delta n \, \delta v^2_{\perp} = \frac{2N}{R^2} \left(\frac{2 G m}{b v} \right)^2 b \, db$$

And we integrate over all the possible impact parameters...

And we integrate over all the possible impact parameters...

$$\delta v^2_{\rm TOT} = 8\,N\,\left(\frac{G\,m}{R\,v}\right)^2 \int_{b_{\rm min}}^{R} \frac{db}{b}$$

$$\delta v^2_{\rm TOT} = 8\,N\,\left(\frac{G\,m}{R\,v}\right)^2 \ln\frac{R}{b_{\rm min}}$$

* low integration limit: smallest *b* to avoid close encounter $\delta v_\perp \sim v$

$$v = \frac{2\,G\,m}{b_{\rm min}\,v} \qquad \Longrightarrow b_{\rm min} = \frac{2\,G\,m}{v^2}$$

* top integration limit: size *R* of the system

And we integrate over all the possible impact parameters...

$$\delta v^2_{\rm TOT} = 8\,N\,\left(\frac{G\,m}{R\,v}\right)^2\,\ln\left(\frac{R\,v^2}{2\,G\,m}\right)$$

Typical speed of a star in a virialized system

$$N\,m\,v^2 = \frac{G\,(N\,m)^2}{R} \qquad \Longrightarrow v^2 = \frac{G}{N\,m}R$$

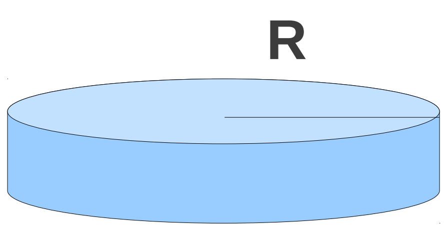
Replacing *v*

$$\boxed{\frac{\delta v^2_{\rm TOT}}{v^2} = \frac{8\,\ln N}{N}}$$

Number of crossings of the system for which $\frac{\delta v^2_{\mathrm{TOT}}}{v^2} = 1$

$$n_{\mathrm{cross}} = \frac{N}{8\,\ln N}$$

CROSSING TIME= time needed to cross the system (also named DYNAMICAL TIME)



$$t_{cross} = \frac{R}{v}$$

$$= \sqrt{\frac{R^3}{G\,M}} = \frac{1}{\sqrt{G\,\rho}}$$

RELAXATION TIME = time necessary for stars in a system to lose completely the memory of their initial velocity

$$t_{\rm rlx} = n_{\rm cross}\, t_{\rm cross} = \frac{N}{8\,\ln N}\,\frac{R}{v}$$

with more accurate calculations, based on diffusion coefficients (Spitzer & Hart 1971):

$$t_{\rm rlx} = 0.34\,\frac{\sigma^3}{G^2\, m\, \rho\, \ln\Lambda}$$

The two expressions are almost equivalent

If we put $\sigma = v = (G\, N\, m\, /\, R)^{1/2}$
and $\rho \propto N\, m/\, R^3$
and $\ln\Lambda \sim \ln N$

$$t_{\rm rlx} \propto \frac{\sigma^3}{G^2\, m\, \rho \ln \Lambda} \sim \frac{(G\, N\, m\, R^{-1})^{3/2}}{G^2\, N\, m^2\, R^{-3} \ln N}$$

$$\sim G^{-1/2}\, N^{1/2}\, m^{-1/2}\, R^{3/2} \ln N^{-1} \, \frac{v}{v}$$

multiply and divide per v

$$\sim \left(\frac{N\, R}{G\, m}\right)^{1/2} v \ln N^{-1} \, \frac{R}{v}$$

Rearrange and substitute again v

$$\sim \frac{N}{\ln N} \, \frac{R}{v}$$

RELAXATION & THERMALIZATION

*Relaxation and thermalization are almost **SYNONYMOUS**!*

* ***Thermalization**:*

- is one case of relaxation

*- is defined for gas (because needs definition of **T**), but can be used also for stellar system (kinetic extension of T)*

*- is the **process of particles reaching thermal equilibrium through mutual interactions** (involves concepts of equipartition and evolution towards maximum entropy state)*

*- has velocity distribution function: **Maxwellian** velocity*

**** Relaxation**:*

- is defined not only for gas

*- is the **process of particles reaching equilibrium through mutual interactions***

Which is the typical t_{rlx} of stellar systems?

* **globular clusters, dense young star clusters, nuclear star clusters** (far from SMBH influence radius)
  R~1-10 pc, N~10^3-10^6 stars, v~1-10 km/s

  t_{rlx}~10^{7-10} yr

→ **COLLISIONAL**

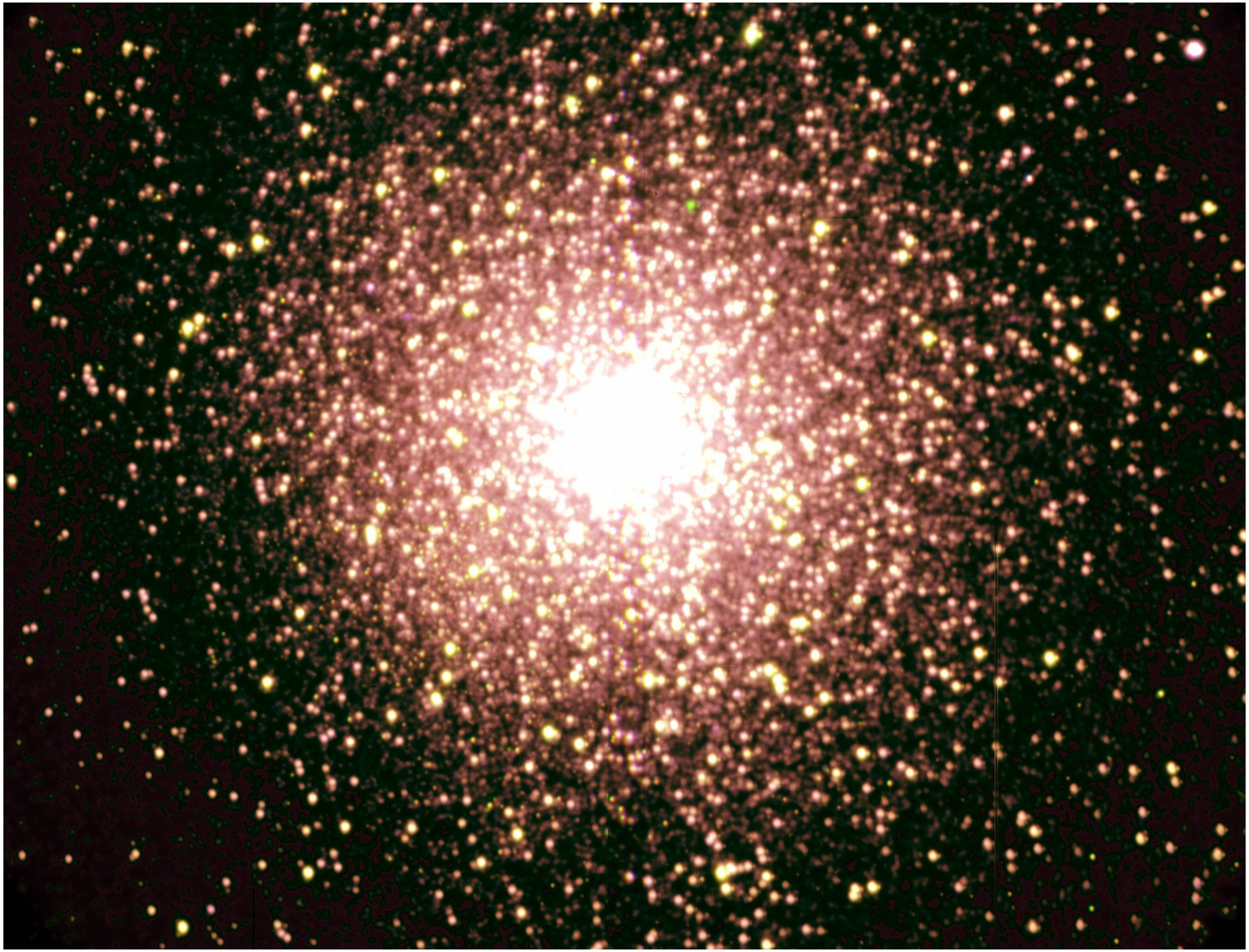
* **galaxy field/discs**
  R~10 kpc, N~10^10 stars, v~100-500 km/s

  t_{rlx} >> Hubble time

→ **COLLISIONLESS**
described by collisionless Boltzmann equation

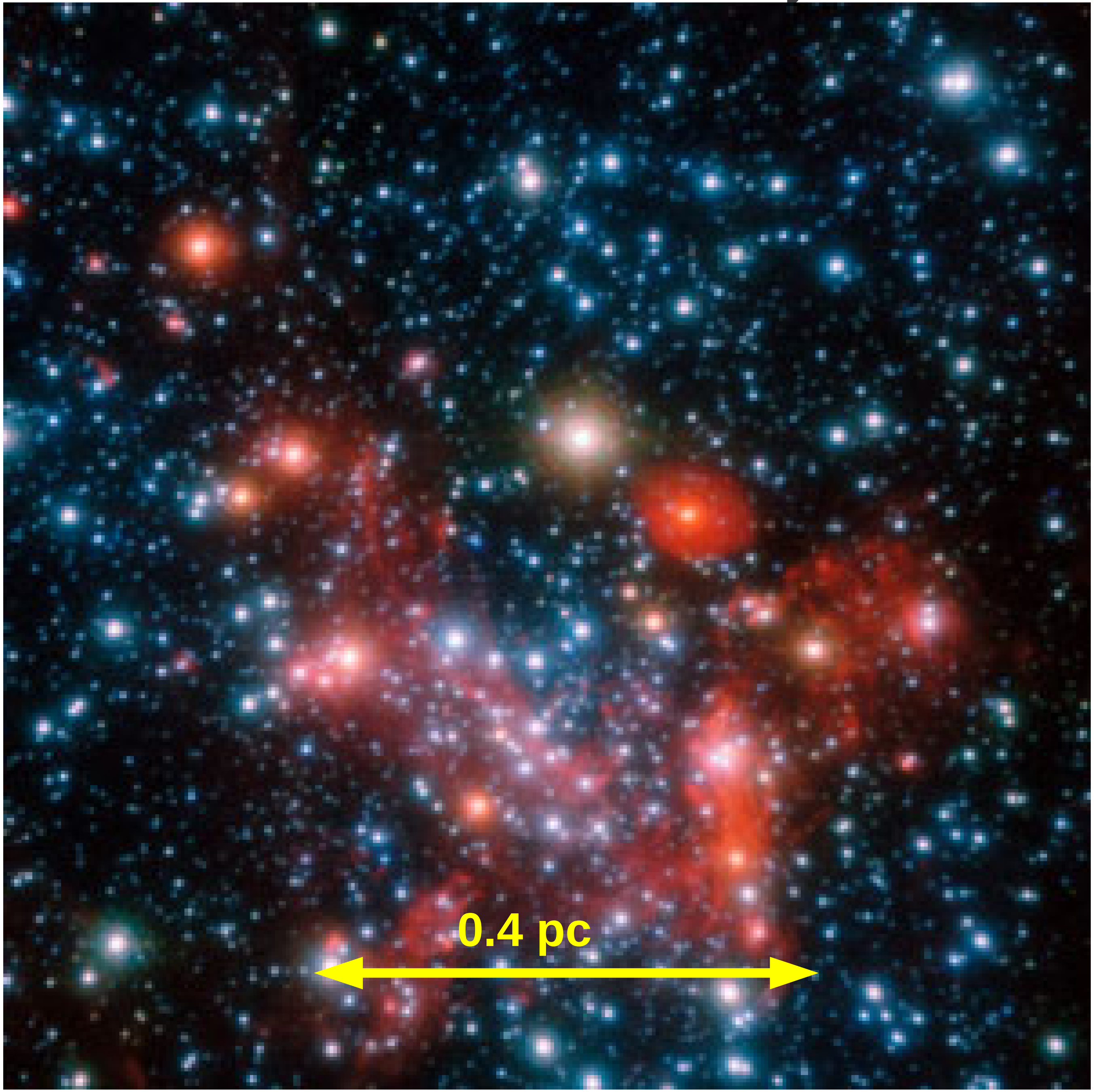
$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_{i=1,6} \frac{\partial x_i}{\partial t}\frac{\partial f}{\partial x_i} = 0$$

EXAMPLES of COLLISIONAL stellar systems



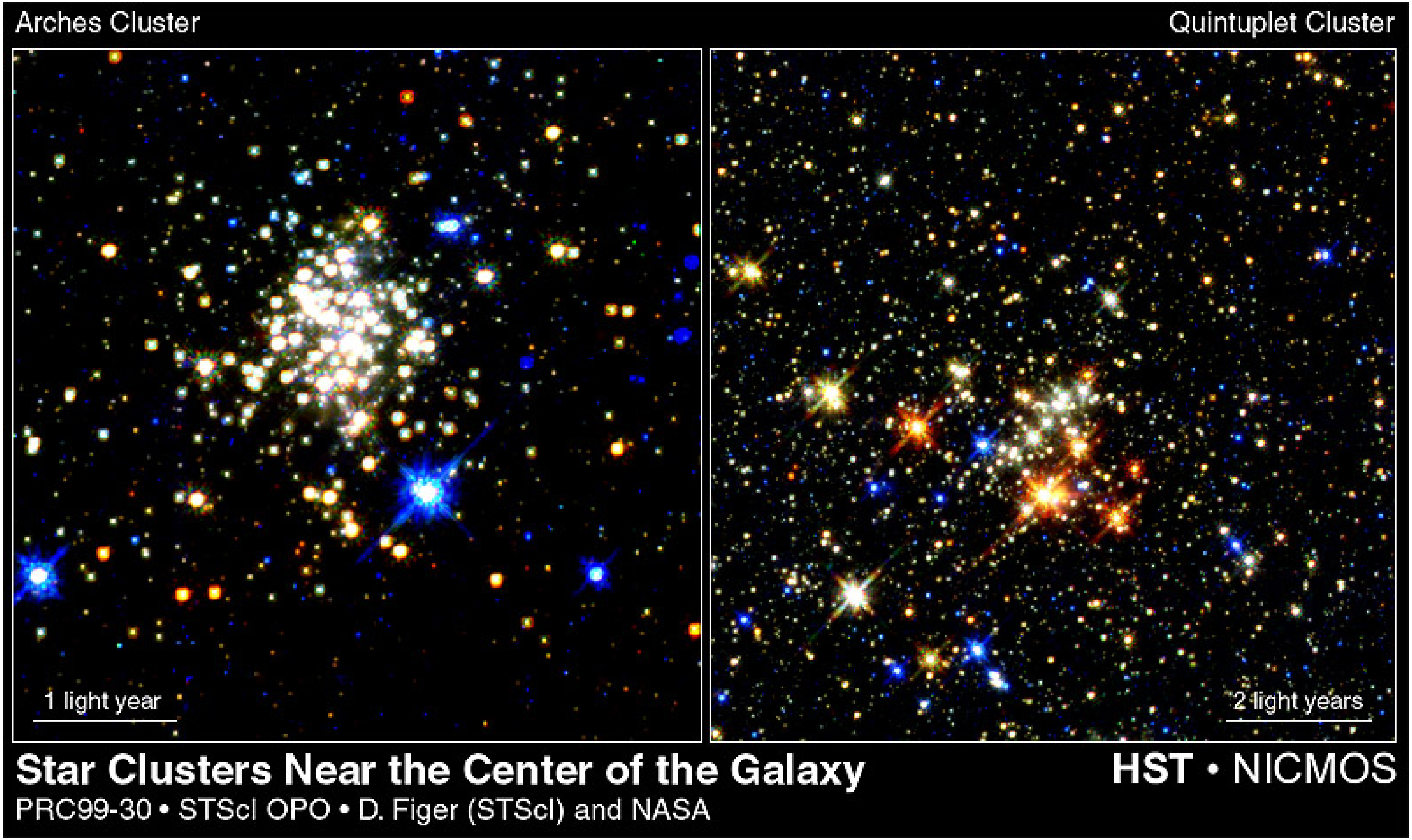
Globular clusters (47Tuc), by definition

EXAMPLES of COLLISIONAL stellar systems



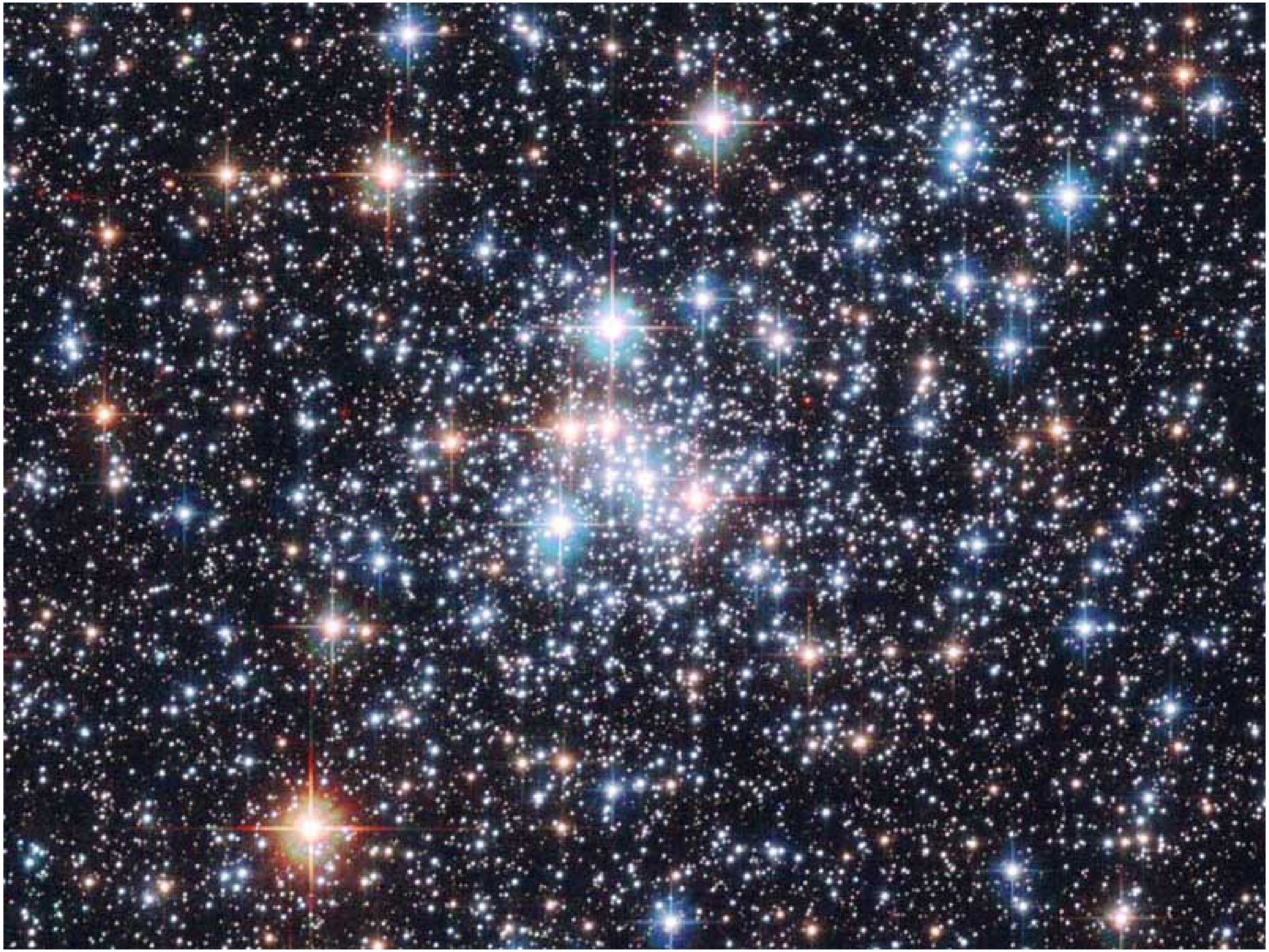
Nuclear star clusters (MW)
NaCo @ VLT
Genzel+2003

EXAMPLES of COLLISIONAL stellar systems



Young dense star clusters (Arches, Quintuplet)

EXAMPLES of COLLISIONAL stellar systems



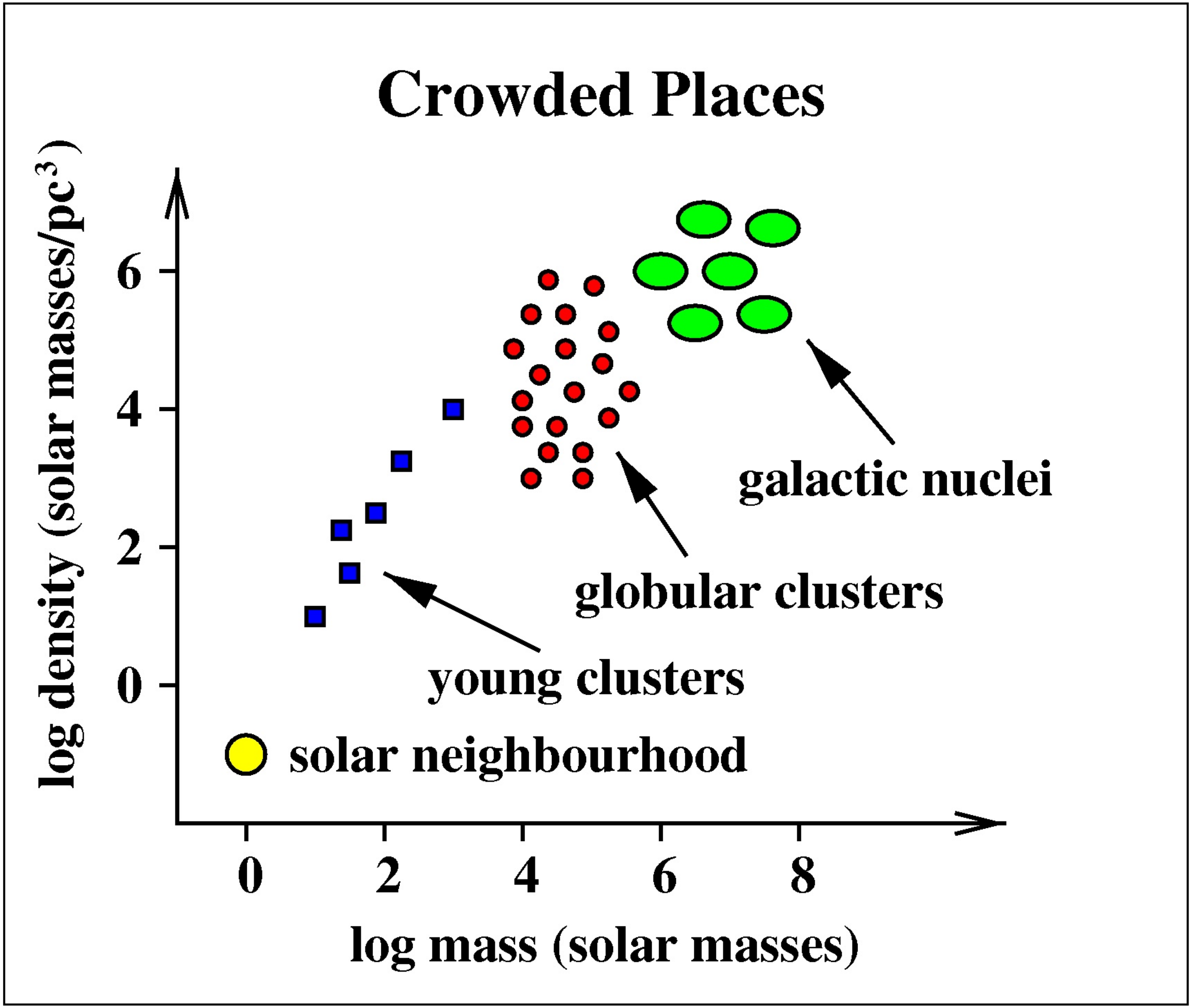
Open clusters, especially in the past (NGC290)

EXAMPLES of COLLISIONAL stellar systems

Embedded clusters, i.e. baby clusters (RCW 38)

NaCo @ VLT

DENSITY & MASS ORDER OF MAGNITUDES



M. B. Davies, 2002, astroph/0110466

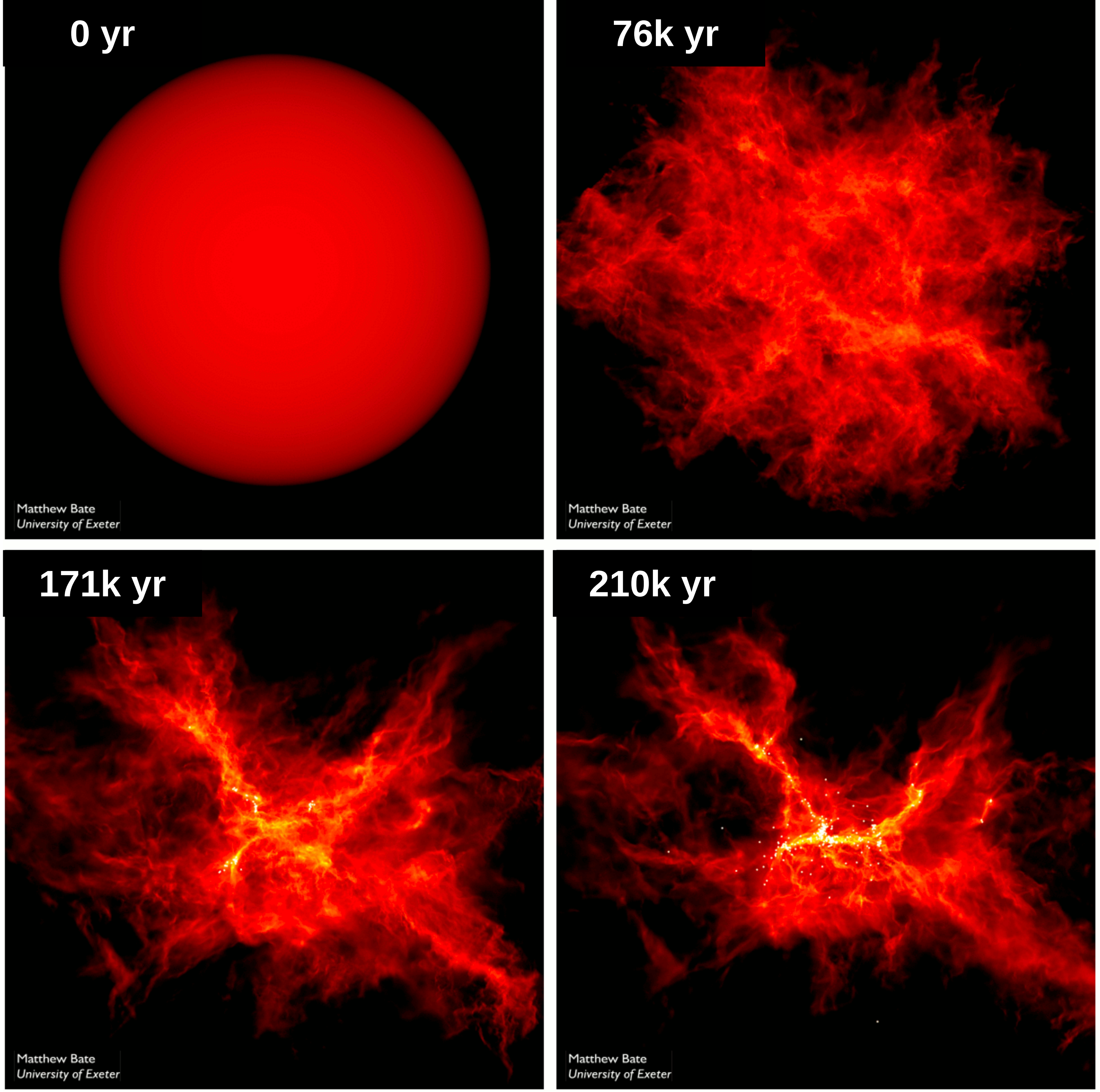
How do star clusters form? BOH

* from giant molecular clouds

* possibly from aggregation of many sub-clumps

GAS SIMULATION by Matthew Bate (Exeter):

- **gas**
- **SPH**
- **turbulence**
- **fragmentation**
- **sink particles**



How do star clusters form? BOH

* from giant molecular clouds

* possibly from aggregation of many sub-clumps

* reach first configuration by VIOLENT RELAXATION (?)

VIOLENT RELAXATION? BOH-2

- Theory by Linden-Bell in 1967 (MNRAS 136, 101)

- Starts from a problem: galaxy discs and elliptical galaxies are RELAXED (stars follow thermal distribution),
  even if $t_{rlx} >> t_{Hubble}$

- IDEAs: (1) there should be another relaxation mechanism (not two-body) efficient on **DYNAMICAL timescale** (crossing time)

  (2) by 2^{nd} law of thermodynamics: such mechanism must MAXIMIZE entropy

  (3) RELAXATION DRIVER: the **POTENTIAL** of a newly formed galaxy or star cluster **CHANGES** VIOLENTELY (on dynamical time)

  → changes in potential must redistribute stellar ENERGY in a CHAOTIC WAY (losing memory of initial conditions)

VIOLENT RELAXATION? **BOH-2**

- HERE the problems start, because
    * in simulations, it is difficult to disentangle numerical instability from true effects
    * in calculations, there are too many approximations

    Note: there is NOT a complete mathematical formulation of this problem

- Formulation by Linden-Bell distinguishes

Fine-grained distribution function in phase space

$$f(\vec{x},\vec{y},t) = \frac{dm(\vec{x},\vec{y},t)}{d^3\vec{x}\,d^3\vec{v}}$$

average of

Coarse-grained distribution function in phase space

$$F(\vec{x},\vec{y},t) = \frac{1}{\Delta^3\vec{x}\,\Delta^3\vec{v}} \int_{\Delta^3\vec{x}\,\Delta^3\vec{v}} f(\vec{x},\vec{y},t)\,d^3\vec{x}\,d^3\vec{v}$$

and demonstrates that the CORRECT coarse-grained distribution after 1 dynamical time is a thermal distribution (= Maxwellian), which maximizes entropy.

BUT: this may be wrong..there are cases where it does not work

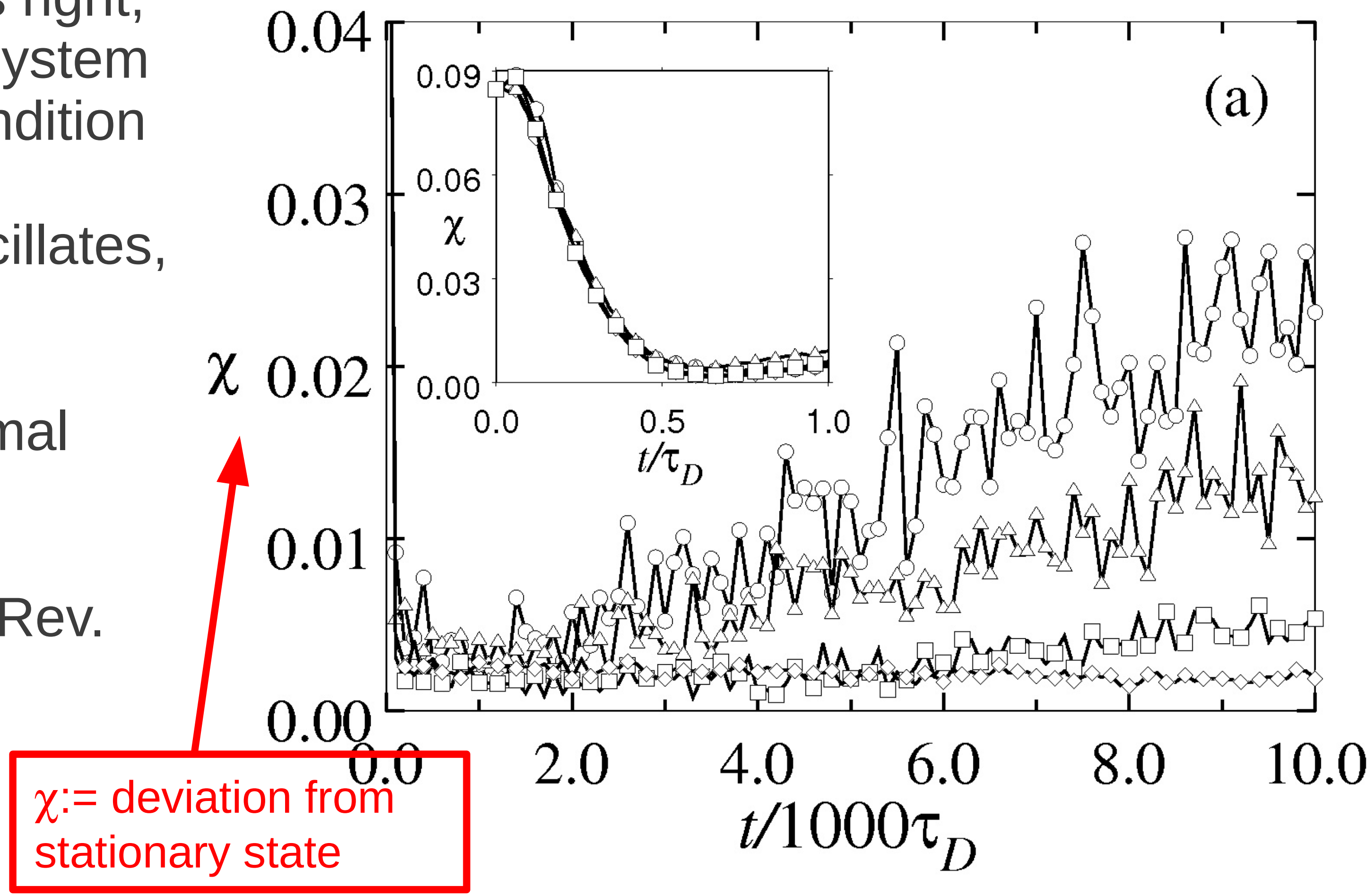
VIOLENT RELAXATION? **BOH-2**

- POSSIBLE SOLUTIONS:

(1) relaxation is too fast to maximize entropy in all intermediate states (Arad and Lynden-Bell, MNRAS, 361, 385 2005)

(2) Lynden-Bell is right, but only if the initial system satisfies the virial condition

If not, the system oscillates, ejects mass (by evaporation) and starts gravothermal collapse (by Levin, Pakter & Rizzato 2008, Phys. Rev. E 78, 021130)



See also review by Bindoni & Secco 2008, New Astronomy, 52, 1

How do star clusters form? BOH

* from giant molecular clouds

* possibly from aggregation of many sub-clumps

* reach first configuration by VIOLENT RELAXATION (?)

* after this can be modelled by
  PLUMMER SPHERE
  ISOTHERMAL SPHERE
  LOWERED ISOTHERMAL SPHERE
  KING MODEL

* after reaching first configuration, they become COLLISIONAL and relax through two-body encounters faster than their lifetime (even without mass spectrum and stellar evolution!)

* can DIE by INFANT MORTALITY!!!

INFANT MORTALITY:= clusters can die when GAS is removed

Embedded
CLUSTERS

GAS
REMOVAL

DENSE CLUSTERs
bound
<~ 10^5 pc^-3 (coll.)
~ 10^3-10^5 stars

OPEN CLUSTERS
loosely bound
< 10^4 stars

ASSOCIATIONS
unbound
<10^3 stars

OPEN and DENSE STAR CLUSTERS
as SURVIVORS of INFANT MORTALITY:
how and with which properties?

INFANT MORTALITY

-DEPENDENCE on SFE : = $M_{star}/(M_{star} + M_{gas})$

(1) Velocity dispersion from virial theorem before gas removal:

$$\sigma_0^2 = \frac{G\,(M_{star} + M_{gas})}{R_0}$$

(2) Energy after gas removal (hypothesis of instantaneous gas removal):

$$E = \frac{1}{2}\,M_{star}\,\sigma_0^2 - \frac{G\,M_{star}^2}{R_0}$$

(3) Energy after new virialization:

$$E = -\frac{G\,M_{star}^2}{2\,R}$$

New cluster size:
- From (2) = (3)

$$-\frac{G\,M_{star}^2}{2\,R} = \frac{1}{2}\,M_{star}\,\sigma_0^2 - \frac{G\,M_{star}^2}{R_0}$$

Hills 1980, ApJ, 225, 986

-DEPENDENCE on SFE : = $M_{star}/(M_{star}+ M_{gas})$

New cluster size:
- Using (1)

$$-\frac{M_{star}}{2\,R} = \frac{1}{2}\,\frac{(M_{star}+M_{gas})}{R_0} - \frac{M_{star}}{R_0}$$

-Rearranging

$$R = R_0\,\frac{M_{star}}{M_{star}+M_{gas}}\,\frac{1}{\left(2\,\frac{M_{star}}{M_{star}+M_{gas}} - 1\right)}$$

R>0 only if

$$\boxed{\frac{M_{star}}{M_{star}+M_{gas}} > 0.5}$$

Hills 1980, ApJ, 225, 986

INFANT MORTALITY

-DEPENDENCE on SFE : <30% disruption

-DEPENDENCE on t_gas:
explosive removal: t_gas<< t_cross
[smaller systems]

adiabatic removal: t_gas >~ t_cross
[dense clusters]

-DEPENDENCE on the (+/-) VIRIAL state of the embedded cluster

-DEPENDENCE on Z: metal poor clusters more compact than metal rich

Hills 1980; Lada & Lada 2003; Bastian & Goodwin 2006;
Baumgardt & Kroupa 2007; Bastian 2011;
Pelupessy & Portegies Zwart 2011

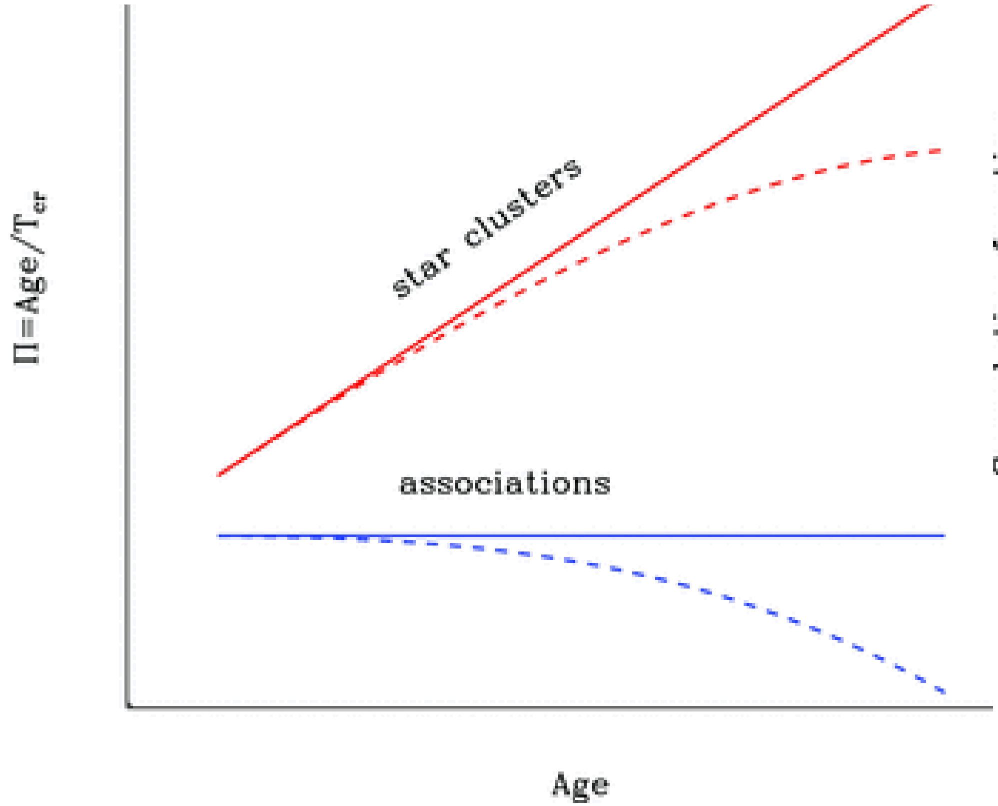
INFANT MORTALITY

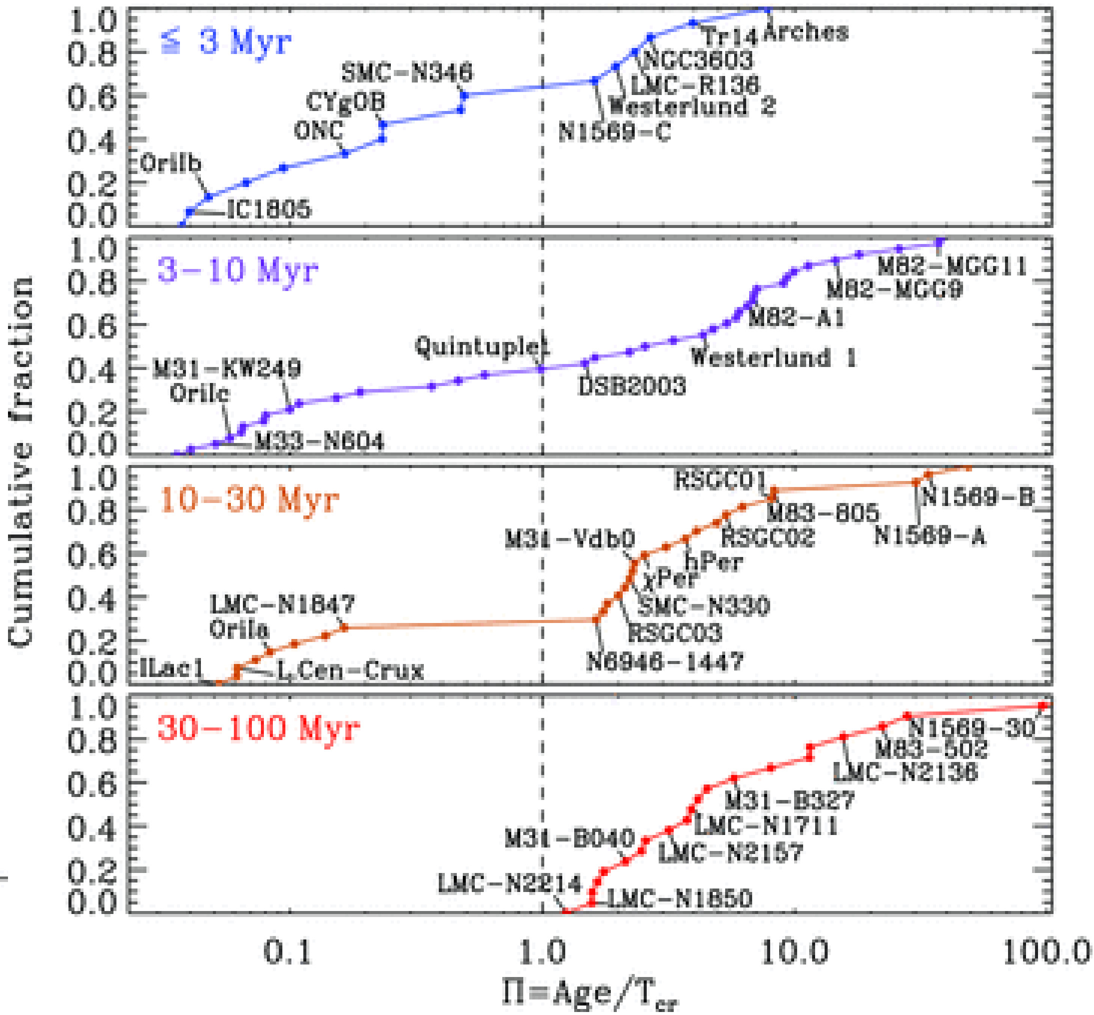
A criterion to infer whether a star cluster is dying or will survive, empirically found by Gieles & Portegies Zwart (2011, MNRAS, 410, L6)

$$\Pi \equiv \frac{Age}{t_{dyn}} \qquad t_{dyn} \equiv 10\left(\frac{R_{hl}^3}{G\,M}\right)^{1/2}$$

Π > 1 surviving star cluster

Π < 1 association (maybe)



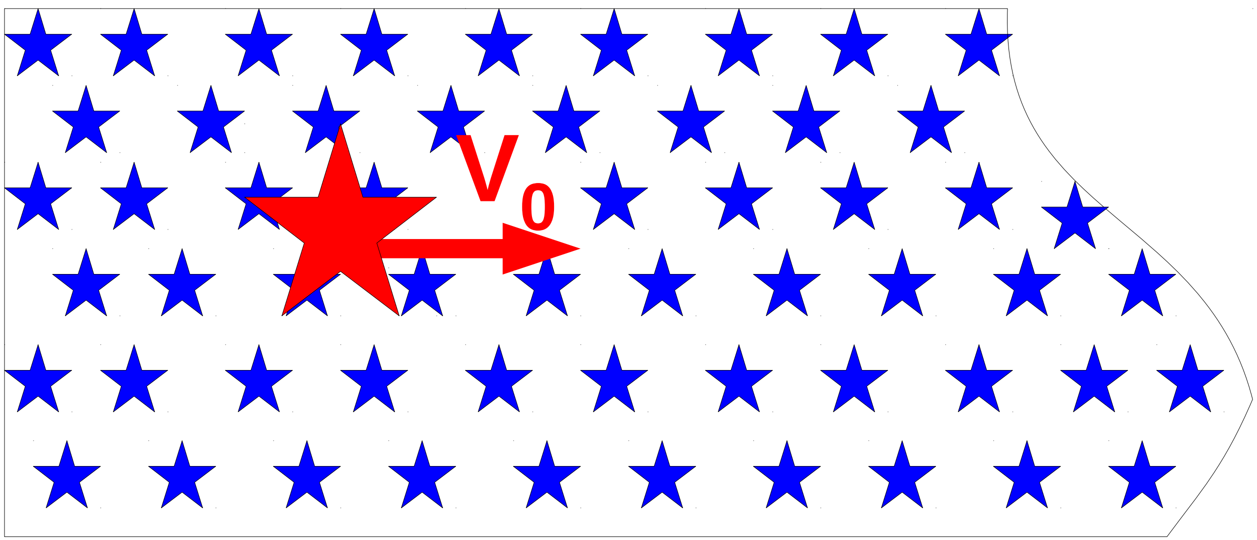


REPETITA IUVANT: DYNAMICAL FRICTION

After crossing time, relaxation time, the third important timescale for Collisional (and even collisionless) systems is

DYNAMICAL FRICTION TIMESCALE

A body of mass *M*, traveling through an infinite & homogeneous sea of bodies (mass *m*) suffers a steady deceleration: the dynamical friction



infinite & homogeneous sea: otherwise the body M would be deflected

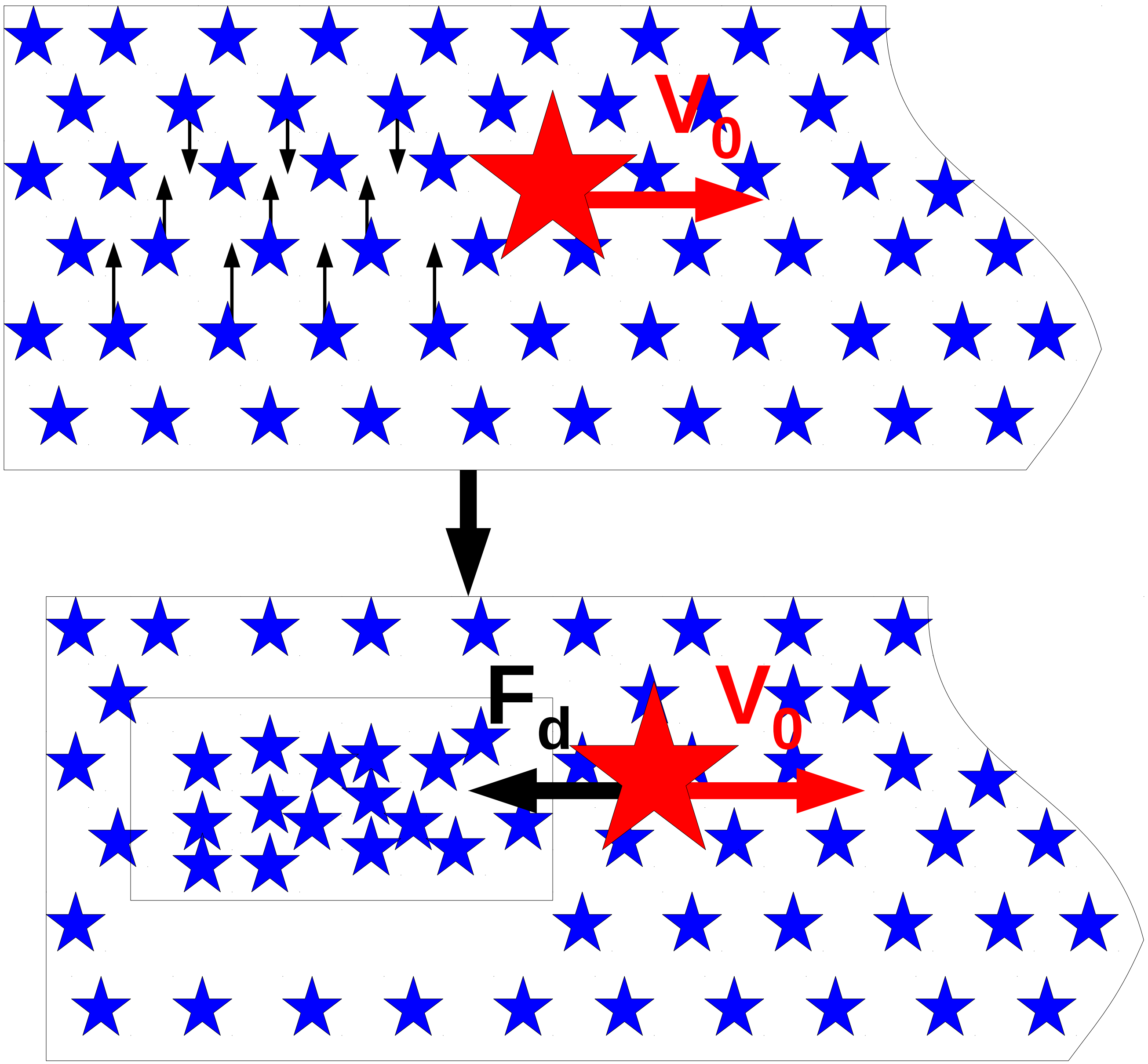
The sea exerts a force **parallel and opposite** to the velocity V_0 of the body

It can be shown that DYNAMICAL FRICTION TIMESCALE is

$$t_{df} = \frac{3}{4\,(2\,\pi)^{1/2}\,G^2\,\ln\Lambda}\,\frac{\sigma^3(r)}{M\,\rho(r)}$$

REPETITA IUVANT: DYNAMICAL FRICTION

BASIC IDEA:



The heavy body *M* attracts the lighter particles.

When lighter particles approach, the body *M* has already moved and leaves a local overdensity behind it.

The overdensity attracts the heavy body (with force F_d) and slows it down.

REPETITA IUVANT: Plummer sphere

Isotropic velocity distribution function: $$f(E) \propto \begin{cases} (-E)^p & if\, E < 0 \\ 0 & if\, E \geq 0 \end{cases}$$

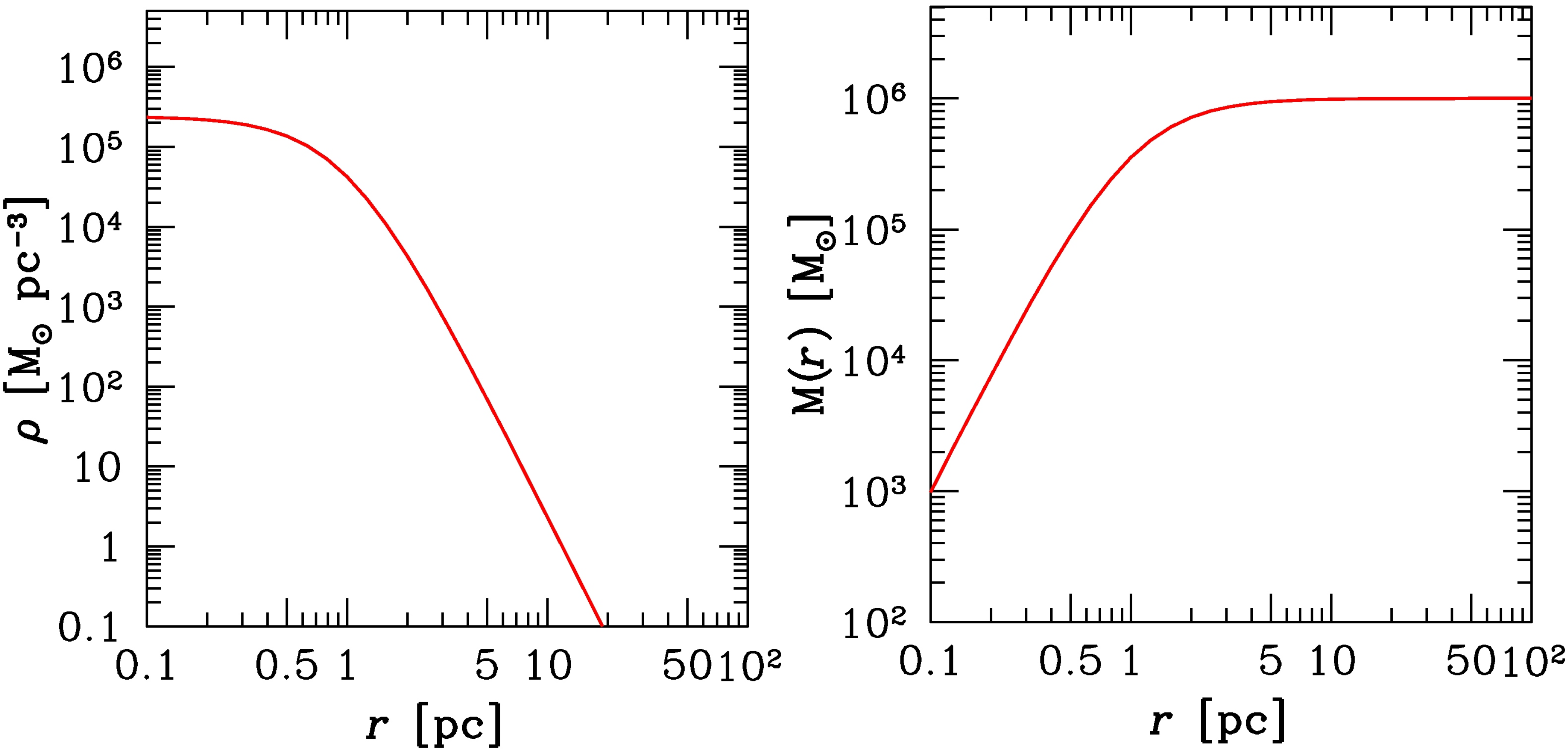
if p=1 corresponds to potential $$\phi(r) = -\frac{G\,M}{(r^2 + a^2)^{1/2}}$$

From Poisson equation $\nabla^2\phi = 4\,\pi\,G\,\rho$

We derive density $$\rho(r) = \frac{M}{\frac{4}{3}\,\pi a^3}\,\frac{1}{\left[1 + \left(\frac{r}{a}\right)^2\right]^{5/2}}$$

and corresponding mass $$M(r) = \frac{M}{a^3}\,\frac{r^3}{\left[1 + \left(\frac{r}{a}\right)^2\right]^{3/2}}$$

REPETITA IUVANT: Plummer sphere



REPETITA IUVANT: Isothermal sphere

* Why isothermal? From formalism of ideal gas $$P = \frac{\kappa_B}{\mu m_p} \rho T$$

If T= const → P = const x ρ

* For politropic equation of state is isothermal if γ = 1 $$P = \kappa \, \rho^{\gamma}$$

γ = 1

if we assume hydrostatic equilibrium

$$\frac{d\phi}{dr} = -\frac{1}{\rho}\frac{dP}{dr} = -\frac{\kappa}{\rho}\frac{d\rho}{dr}$$

we derive the potential

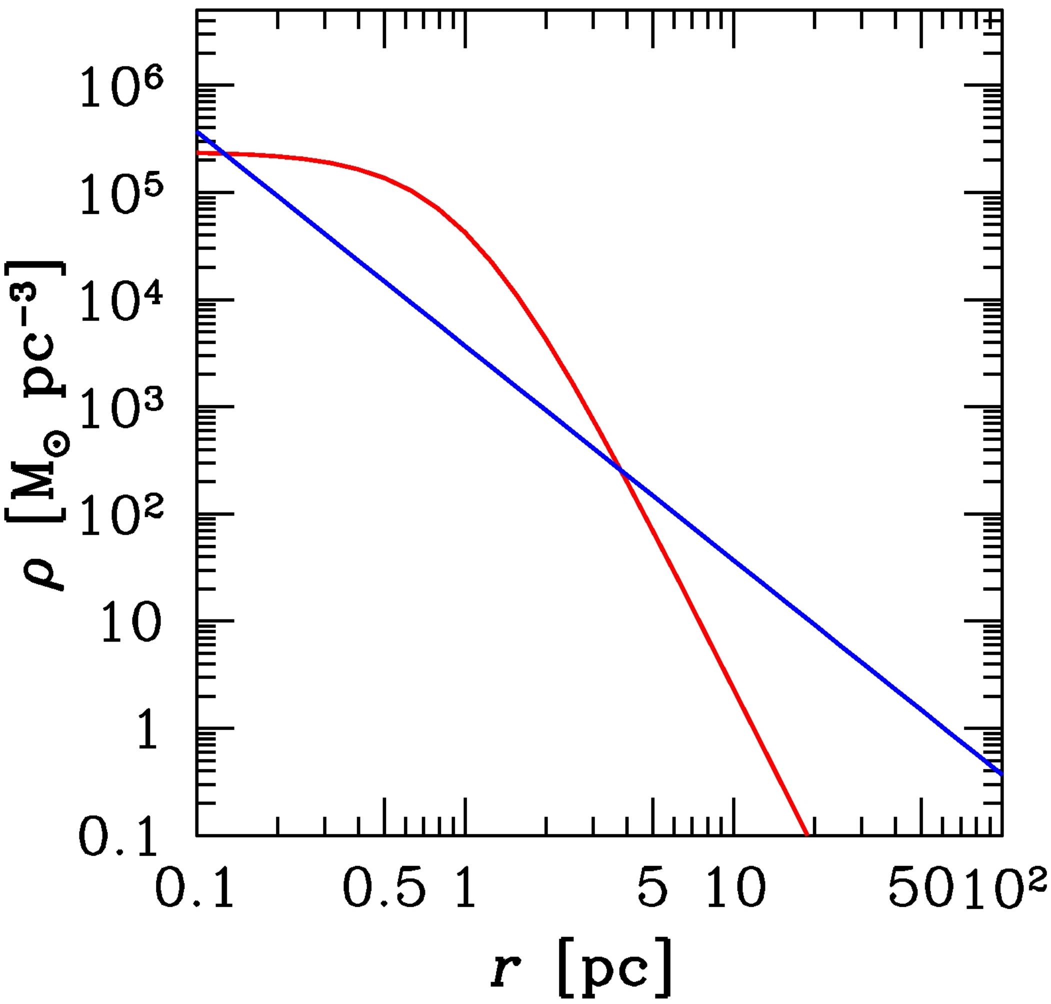
$$\phi = -\kappa \ln\left(\frac{\rho}{\rho_c}\right)$$

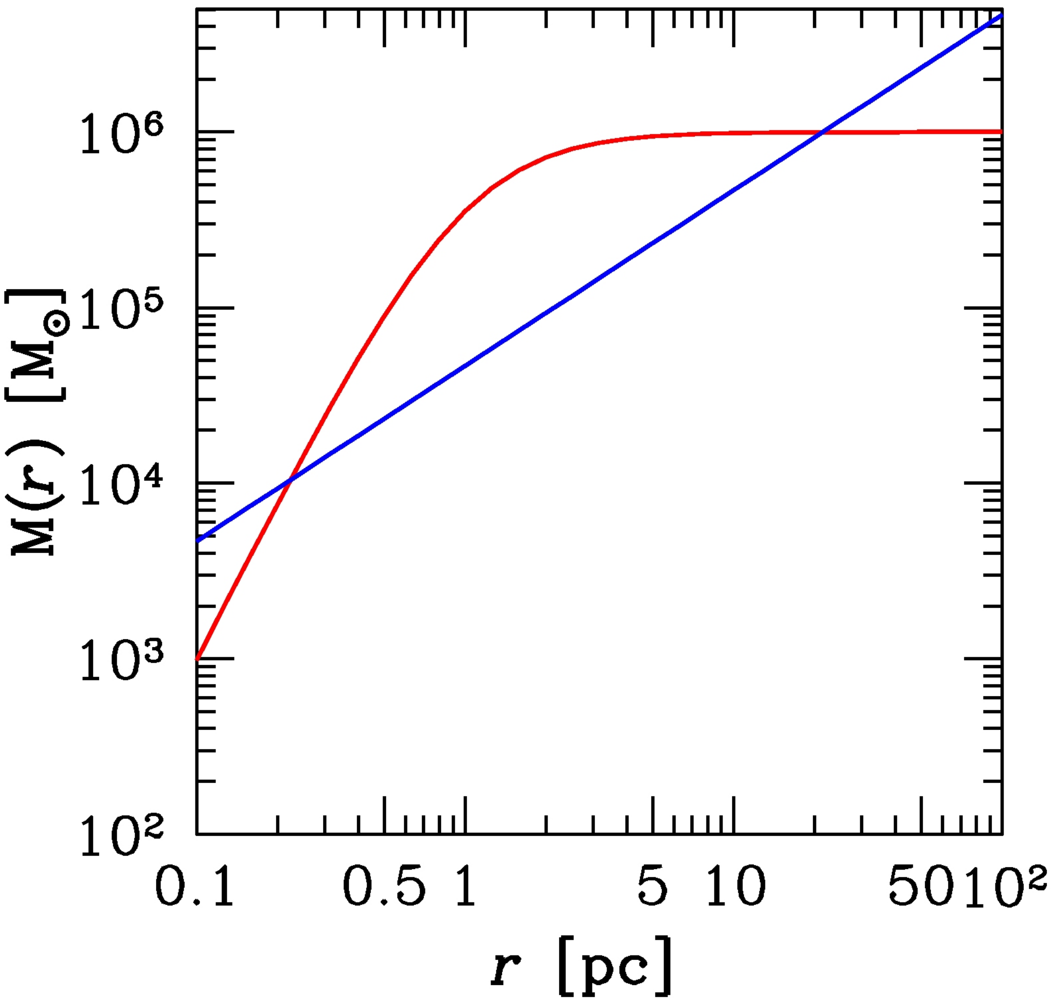
using Poisson's equation we find

$$\rho(r) = \frac{\sigma^2}{2\,\pi\,G} r^{-2}$$

expressing the constant k with some physical quantities

REPETITA IUVANT: Isothermal sphere





REPETITA IUVANT: PROBLEMS of isothermal sphere

1) DENSITY goes to infinity if radius goes to zero

$$\rho(r) = \frac{\sigma^2}{2\,\pi\,G} r^{-2}$$

2) MASS goes to infinity if radius goes to infinity

$$M(r) = 4\,\pi \int_0^r \rho(r)\, r^2\, dr = \frac{2\,\sigma^2}{G} r$$

REPETITA: Lowered isothermal sphere and King model

1) King model (also said non-singular isothermal sphere) solves the problem at centre by introducing a CORE

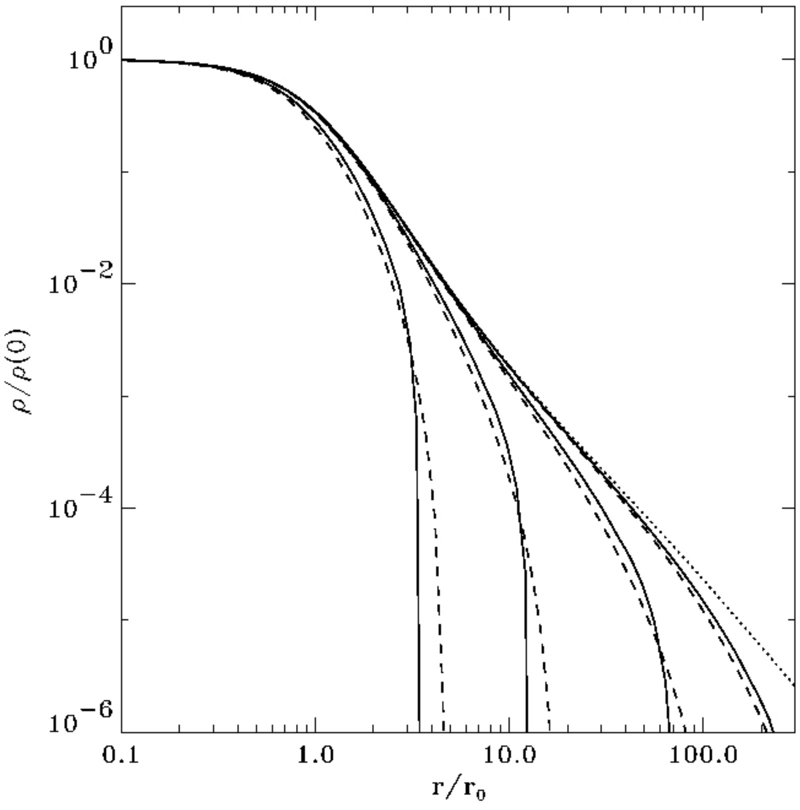
$$r_0 = \sqrt{\frac{9\,\sigma^2}{4\,\pi\,G\,\rho_0}}$$

with the core, ρ has a difficult analytical shape, but can be approximated with the singular isothermal sphere for $r \gg r_0$

and with

$$\rho(r) = \rho_0 \frac{1}{\left[1 + \left(\frac{r}{r_0}\right)^2\right]^{3/2}}$$

For $r \lesssim 2\, r_0$



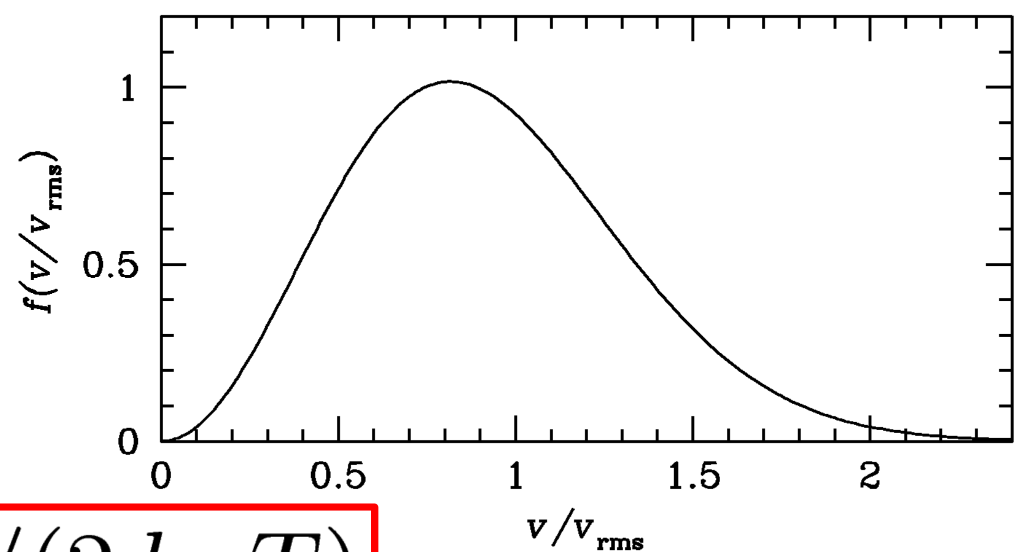
REPETITA: Lowered isothermal sphere and King model

2) Non-singular LOWERED isothermal sphere := all King models where mass is truncated at a certain radius (does not go to infinity)

VELOCITY DISTRIBUTION FUNCTION

$$f(E) \propto \begin{cases} \kappa \left(e^{-B\,E} - e^{-B\,E_e} \right) & if\, E < E_e \\ 0 & if\, E \geq E_e \end{cases}$$

NOTE: **VELOCITY DISTRIBUTION FUNCTION is the MAXWELLIAN for isothermal sphere and truncated Maxwellian for lowered non-singular isothermal sphere!!!**



$$f(v) \propto v^2\, e^{-m\,v^2/(2\,k_b\,T)}$$

References:

* Binney & Tremaine, Galactic Dynamics, First edition, 1987, Princeton University Press

* Elson R. & Hut P., Dynamical evolution of globular clusters, Ann. Rev. Astron. Astrophys., 1987, 25, 565

* Spitzer L., Dynamical evolution of globular clusters, 1987, Princeton University Press

Michela Mapelli

LECTURES on COLLISIONAL DYNAMICS:

2. OVERALL STAR CLUSTER EVOLUTION

GRANULARITY of the GRAVITATIONAL FIELD

Interactions between 2 stars (**two-body encounters**)
PRODUCE LOCAL FLUCTUATIONS of ENERGY BALANCE i.e.
CHANGE LOCALLY THE MAGNITUDE of STELLAR VELOCITIES

On the RELAXATION TIMESCALE, this induces
GLOBAL CHANGES in the CLUSTER EQUILIBRIUM

HOW?

Very simple approach based on probability distribution:

Relaxation leads to a Maxwellian (thermal) velocity distribution

$\wp$:= probability of finding a star within an energy interval ΔE at a given energy E (for Maxwellian velocity distribution)

$$\wp \propto \mathrm{EXP}(-\mathrm{B}E)\,\Delta E$$

$\Rightarrow$ stars tend to have $E \ll 0$ (because $\wp$ is higher)

⇒ most bound stars (central core) become more bound

⇒ central CORE CONTRACTS

BUT the TOTAL ENERGY of cluster must remain ~ CONSTANT (cluster self-bound and isolated)

⇒ less bound stars (halo stars) absorb kinetic energy released by most bound stars (core stars)

⇒ less bound stars become less bound

⇒ HALO EXPANDS (stars move to higher E or become unbound) to keep E_{TOT}=constant

With virial theorem:

$$\mathbf{0 = \mathit{W} + 2\,\mathit{K}}$$

$$\text{if } W \downarrow \;\Rightarrow\; K \uparrow$$

IN DETAIL: there are at least 3 physical processes that determine cluster evolution

(1) EVAPORATION:

escape of stars in the high velocity tail of the Maxwellian

$$f\,(v \rightarrow \infty) > 0$$

(2) GRAVOTHERMAL INSTABILITY:

Instability which occurs in a small core confined in outer ISOTHERMAL halo

(3) EQUIPARTITION:

stars in a cluster tend to have the same average kinetic energy

(1) if equal mass stars → the same average velocity

(2) if bodies in the system have different mass

$$m_i \langle v_i^2 \rangle = m_j \langle v_j^2 \rangle$$

$$\text{if} \quad m_i > m_j \Rightarrow \langle v_i^2 \rangle < \langle v_j^2 \rangle$$

(1) EVAPORATION:

Escape velocity of a star from a cluster: $\frac{1}{2} v_e^2 = |\phi|$

where ϕ= potential
as the kinetic energy of
the star must overcome its potential energy

MEAN SQUARE escape velocity of a star from a cluster:

$$\langle v_e^2 \rangle = \frac{\int \rho(r)\, v_e^2(r)\, dV}{\int \rho(r)\, dV} = \frac{\int \rho(r)\, 2\, |\phi(r)|\, dV}{M} = -4 \frac{W}{M}$$

$$W = \frac{1}{2} \int \rho(r)\, \phi(r) dV$$

from virial theorem:

$$\langle v_e^2 \rangle = 4 \frac{2\, K}{M} = 4 \langle v^2 \rangle$$

a star can escape if its velocity is higher than 2 times the root mean square velocity

(1) EVAPORATION:

The concept of evaporation is simple: if $v>v_e \Rightarrow$ the star escapes

We add a mathematical model to understand the evolution of the system induced by evaporation in the case of

CONSTANT RATE OF MASS LOSS PER UNIT MASS PER TIME INTERVAL dt / t_{rlx}

This assumption implies self-similarity, as the radial variation of density, potential and other quantities are time-invariant except for TIME DEPENDENT SCALE FACTORS

Example: a contracting uniform sphere which remains uniform (:= density independent of radius) during contraction

MASS LOSS RATE:

$$\frac{dM}{dt} = \frac{-\xi_e M(t)}{t_{rlx}(t)} = -\frac{\xi_e M(0)}{t_{rlx}(0)} \left[\frac{R(t)}{R(0)}\right]^{-3/2} \left[\frac{M(t)}{M(0)}\right]^{1/2}$$

CONSTANT RATE OF MASS LOSS

where we used the fact that

$$M(t) = \frac{M(0)}{M(0)} M(t)$$

$$t_{rlx}(t) = t_{rlx}(0) \left(\frac{R(t)}{R(0)}\right)^{3/2} \left(\frac{M(t)}{M(0)}\right)^{1/2}$$

(1) EVAPORATION:

Previous equation has two unknowns (*M(r)*, *R(t)*) → we need another equation:
Change of total cluster energy, as each escaping star carries away a certain **kinetic energy per unit mass (=$\zeta\, E_m$, where E_m is the mean energy per unit mass of the cluster)**

$$\frac{dE_{TOT}}{dt} = \zeta\, E_m \frac{dM}{dt} = \frac{\zeta\, E_{TOT}}{M}\frac{dM}{dt}$$

Since $E_{TOT} \propto - M^2/R$

$$\frac{dE_{TOT}}{dt} = -\frac{d}{dt}\left(\frac{M^2}{R}\right) = -\frac{2\,M}{R}\frac{dM}{dt} + \frac{M^2}{R^2}\frac{dR}{dt}$$

$$\frac{\zeta\, E_{TOT}}{M}\frac{dM}{dt} = -\zeta\frac{M}{R}\frac{dM}{dt}$$

$$(2-\zeta)\frac{dM}{M} = \frac{dR}{R}$$

$$\frac{R}{R(0)} = \left[\frac{M}{M(0)}\right]^{2-\zeta}$$

(1) EVAPORATION:

Inserting

$$\frac{R}{R(0)} = \left[\frac{M}{M(0)}\right]^{2-\zeta} \qquad (@)$$

into the equation for mass loss rate, i.e.

$$\frac{dM}{dt} = \frac{-\xi_e\, M(t)}{t_{rlx}(t)} = -\frac{\xi_e\, M(0)}{t_{rlx}(0)} \left[\frac{R(t)}{R(0)}\right]^{-3/2} \left[\frac{M(t)}{M(0)}\right]^{1/2}$$

we find:

$$\frac{dM}{dt} = -\frac{\xi_e\, M(0)}{t_{rlx}(0)} \left[\frac{M}{M(0)}\right]^{(-5+\zeta)/2}$$

Integrating the above equation:

$$\frac{M}{M(0)} = \left[1 - \frac{\xi_e\,(7-3\,\zeta)}{2}\,\frac{t}{t_{rlx}(0)}\right]^{2/(7-3\,\zeta)} \equiv \left(1 - \frac{t}{t_{coll}}\right)^{2/(7-3\,\zeta)}$$

t_{coll} := collapse time, time at which *M* and *R* vanish.

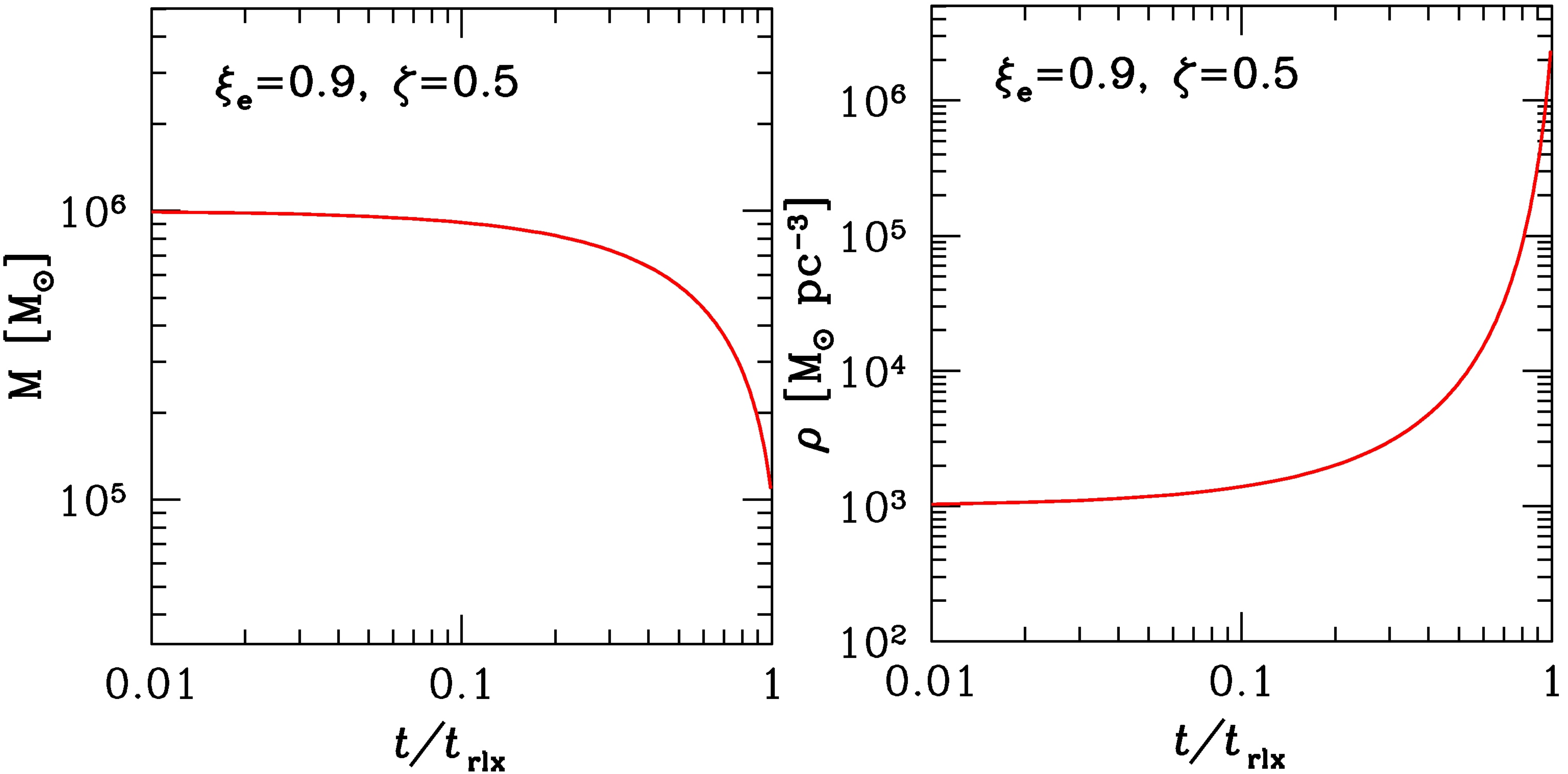
(1) EVAPORATION:

Note: from (@), using the fact that $\rho = 3\,M\,/\,(\,4\,\pi\,R^3\,)$

$$\frac{\rho}{\rho(0)} \propto \left[\frac{M}{M(0)}\right]^{-(5-3\,\zeta)}$$

Since $\zeta < 1$ (for realistic clusters), when M decreases for evaporation, ρ increases:

Collapse! Evaporation may induce collapse!!



(2) GRAVOTHERMAL INSTABILITY or CORE COLLAPSE:

Instability which occurs in a small core
confined in outer ISOTHERMAL halo
GRAV. INST. even if **STARS ARE EQUAL MASS!!!!!!**

1. MATHEMATICAL APPROACH
2. PHYSICAL APPROACH

1. MATHEMATICAL APPROACH:

analogy with IDEAL GAS

We define the temperature T of a self-gravitating system

$$\frac{1}{2}\, m \left\langle v^2 \right\rangle = \frac{3}{2}\, \kappa_B\, T$$

Total kinetic energy of a system

$$K = \sum \frac{1}{2}\, m \left\langle v^2 \right\rangle = \frac{3}{2}\, \kappa_B\, N\, \frac{\int_V \rho(\mathbf{x})\, T(\mathbf{x})\, d\mathbf{x}}{\int_V \rho(\mathbf{x})\, d\mathbf{x}}$$

(2) GRAVOTHERMAL INSTABILITY or CORE COLLAPSE:

Virial theorem: $$E_{\mathrm{TOT}} = -K = -\frac{3}{2} N \kappa_B \langle T \rangle$$

Definition of heat capacity: $$C \equiv \frac{dE}{d\langle T \rangle} = -\frac{3}{2} N \kappa_B$$

always negative

MEANS THAT by LOSING ENERGY THE SYSTEM BECOMES HOTTER

→ **system contracts more and becomes hotter in runaway sense**

How?

If we put a negative heat capacity system in a bath and heat is transferred to the bath (**d*Q* = *dE* > 0**)

→ temperature of the system changes by $T - \frac{dQ}{C} > T$

→ system becomes hotter and heat keeps flowing from system to bath: *T* rises without limits!!

Note: Any bound finite system in which dominant force is gravity exhibits *C*<0

(2) GRAVOTHERMAL INSTABILITY or CORE COLLAPSE:

2. PHYSICAL APPROACH:

INITIAL CONDITIONS:

* SMALL HIGH-DENSITY CORE in a very LARGE ISOTHERMAL HALO (the bath)
* MAXWELLIAN VELOCITY DISTRIBUTION (or, in general, velocity distribution where stars can evaporate)

$$f(v) \propto v^2 \, e^{-v^2} \qquad f(v) > 0 \quad \text{if } v \to \infty$$

IF Maxwellian velocity distribution

⇒ high velocity tail of stars ESCAPE from the core into halo

$$\Rightarrow \quad K = \sum_i \frac{1}{2} m_i \, v_i^2 \quad \downarrow$$

because high velocity stars escape

$$W = - \sum_{i,j,i\neq j} \frac{G \, m_i \, m_j}{r_{ij}} \quad \uparrow$$

because the mass of escaping stars is lost

BUT DECREASE in *K* is more important than increase in *W* since the FASTER STARS LEAVE the cluster!

$$2\,K_f + W_f < 2K_i + W_i$$

⇒ GRAVITY is NO longer supported by *K*, by random motions

⇒ SYSTEM CONTRACTS (*)

⇒ TO REACH NEW VIRIAL EQUILIBRIUM AVERAGE VELOCITY MUST INCREASE

Or, to say it in a different way (more physical?)

(*) IF the system contracts, it becomes DENSER

⇒ higher density implies MORE two-body encounters (higher two-body encounter rate)

⇒ stars exchange more energy and become dynamically hotter

⇒ faster stars tend to EVAPORATE even more than before

⇒ *K* decreases faster than *W* increases

⇒ system contracts even more

⇒ CATASTROPHE!!!

Note: CONDITION that HALO is LARGE with respect to the core is crucial!
so that K continuously injected into the halo does not imply the heating of the halo

Otherwise, if the *K* of the halo overcomes the *K* of the core,
The energy injected into the halo FLOWS BACK to the core
and stops contraction!!!

WHAT DOES REVERSE THE CATASTROPHE??

WHAT DOES REVERSE THE CORE COLLAPSE??

ONLY SWITCHING ON A NEW SOURCE OF K = K_{ext}

THIS SOURCE CAN OPERATE IN TWO WAYS

⇒ (1) $$2\,K_f + W_f = 2\,(K_{ext} + K_i) + W_i > 2\,K_i + W_i$$

Kinetic energy increases not from gravitational contraction but from an **EXTERNAL** SOURCE (breaks virial equilibrium and negative heat capacity)

→ CORE EXPANDS (lasts only till energy source is on)

⇒ (2) **THE NEW KINETIC ENERGY TRANSFERRED TO CORE STARS INDUCES THE EJECTION OF STARS THAT WERE NOT NECESSARILY THE FASTER STARS BEFORE RECEIVING THE NEW KINETIC ENERGY**:

$$K = \sum_i \frac{1}{2} m_i\, v_i^2$$ ↓ because stars which received external kinetic energy escape

$$W = -\sum_{i,j,i\neq j} \frac{G\, m_i\, m_j}{r_{ij}}$$ ↑ because the mass of escaping stars is lost

BUT INCREASE in *W* (DECREASE OF |*W*|) is more important than decrease in *K* since

(I) STARS that LEAVE the cluster were not the faster before receiving the kick and

(II) since K_f is the sum of K_i and K_{ext}!

$$2 K_f + W_f > 2K_i + W_i$$

⇒ **POTENTIAL WELL BECOMES PERMANENTLY SHALLOWER**

⇒ **AND SYSTEM EXPANDS** (*)

⇒ TO REACH NEW VIRIAL EQUILIBRIUM AVERAGE VELOCITY MUST DECREASE

Or, to say it in a different way (more physical?)

(*) IF the system expands, it becomes LESS DENSE

⇒ lower density implies LESS two-body encounters (lower two-body encounter rate)

⇒ stars exchange less energy and become dynamically cooler

⇒ gravitational CATASTROPHE is reversed !!!

Even if sources of heating (partially) switch off, the ejection of stars and the lowering of potential well ensures reversal of catastrophe (but see gravothermal oscillations at end of lecture)

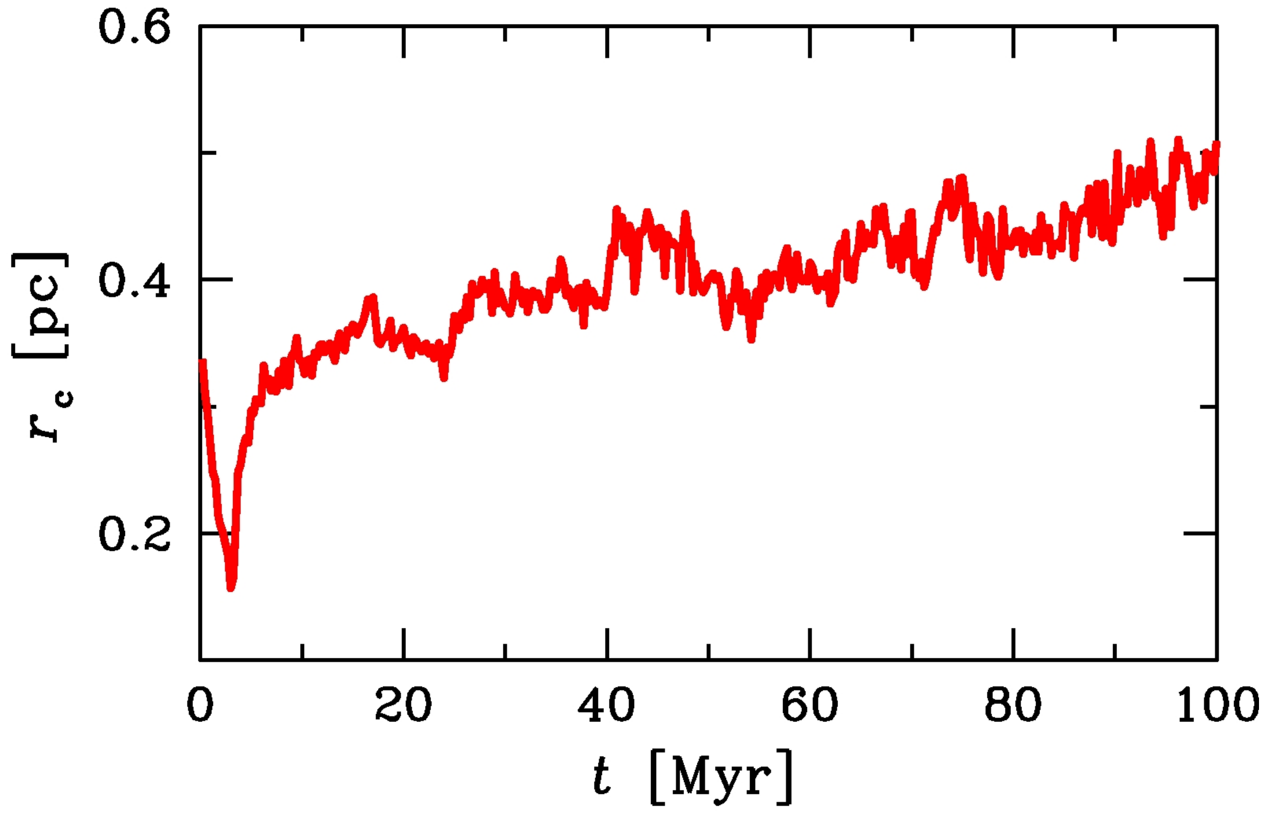
BUT WHICH IS THE NEW SOURCE OF K ENERGY WHICH SWITCHES ON?

(1) **MASS LOSSES by STELLAR WINDS and SUPERNOVAE which remove mass without changing *K* of other stars**

$$2\,K_i + W_f > 2\,K_i + W_i$$

IMPORTANT only if massive star evolution lifetime is similar to core collapse timescale (see last lecture)

(2) **BINARIES as ENERGY RESERVOIR (see next lecture)**



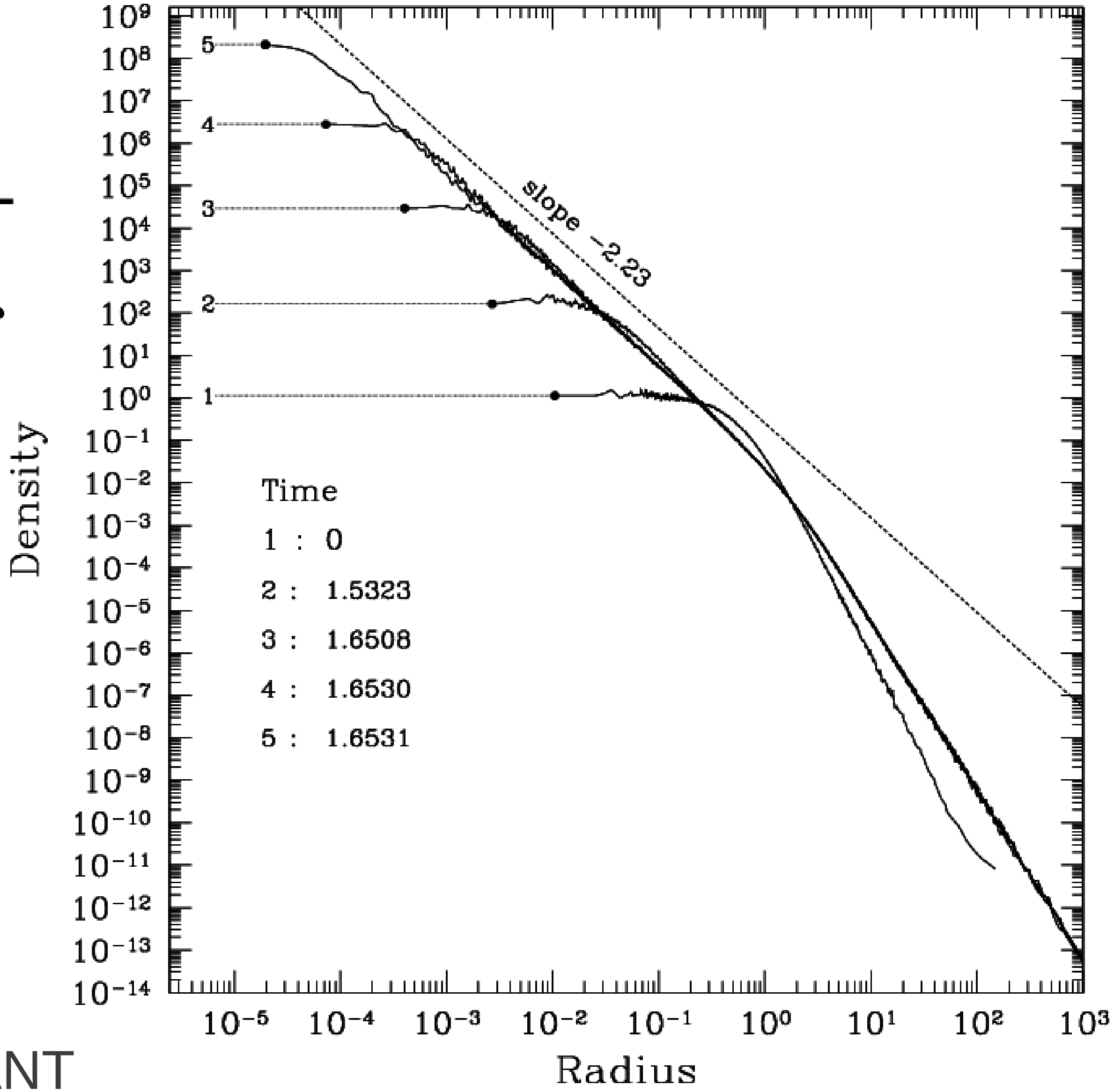
CORE COLLAPSE properties:

CORE COLLAPSE is SELF-SIMILAR (cfr. model of evaporation in slides 5-7: self-similarity is correct!)

$$\frac{d\rho_c}{dt} \rightarrow \mathrm{const}\, \frac{\rho_c}{t_{cr}}$$

central relaxation time

const ~ 3.6x10^-3 – 6x10^-3 from N-body simulations



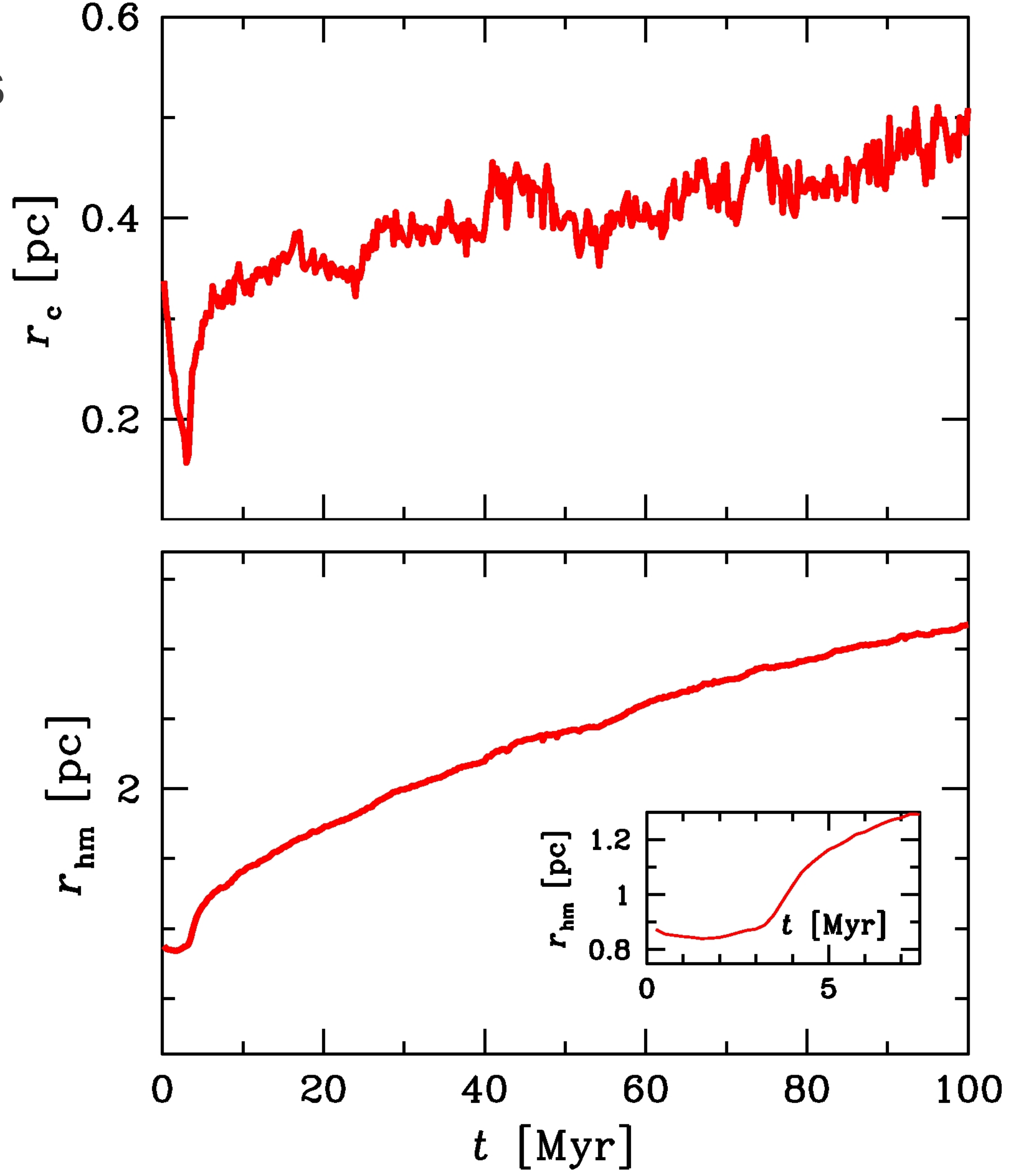
Freitag & Benz 2001, A&A, 375, 711

* DURING CORE COLLAPSE HALF-MASS RADIUS ~CONSTANT

POST CORE COLLAPSE PHASE:

CORE EXPANDS → **INJECTS ENERGY IN THE HALO IN FORM OF HIGH VELOCITY STARS** and matter

HALO is a good bath but not an ideal (i.e. perfect) bath: **HALO EXPANDS due to energy injection and also half-mass radius expands** (Note: when speaking of half-mass radius, we refer mostly to the halo as core generally is << 1/10 of total mass)



POST CORE COLLAPSE PHASE:

HOW does halo expand?

(1) core collapse is self-similar
half-mass relaxation time

$$t_{hm} \propto t$$

(2) from 1st lecture

$$t_{hm} \propto \frac{N}{\ln N} t_{cross} \sim N\, t_{cross}$$

(3) VIRIAL theorem

$$\frac{1}{2} M \langle v^2 \rangle = \frac{1}{2} \frac{G M^2}{r_{hm}}$$

(4)

$$t_{cross} = \frac{r_{hm}}{\langle v \rangle} \Rightarrow r_{hm}^3 \sim G\, M\, t_{cross}^2$$

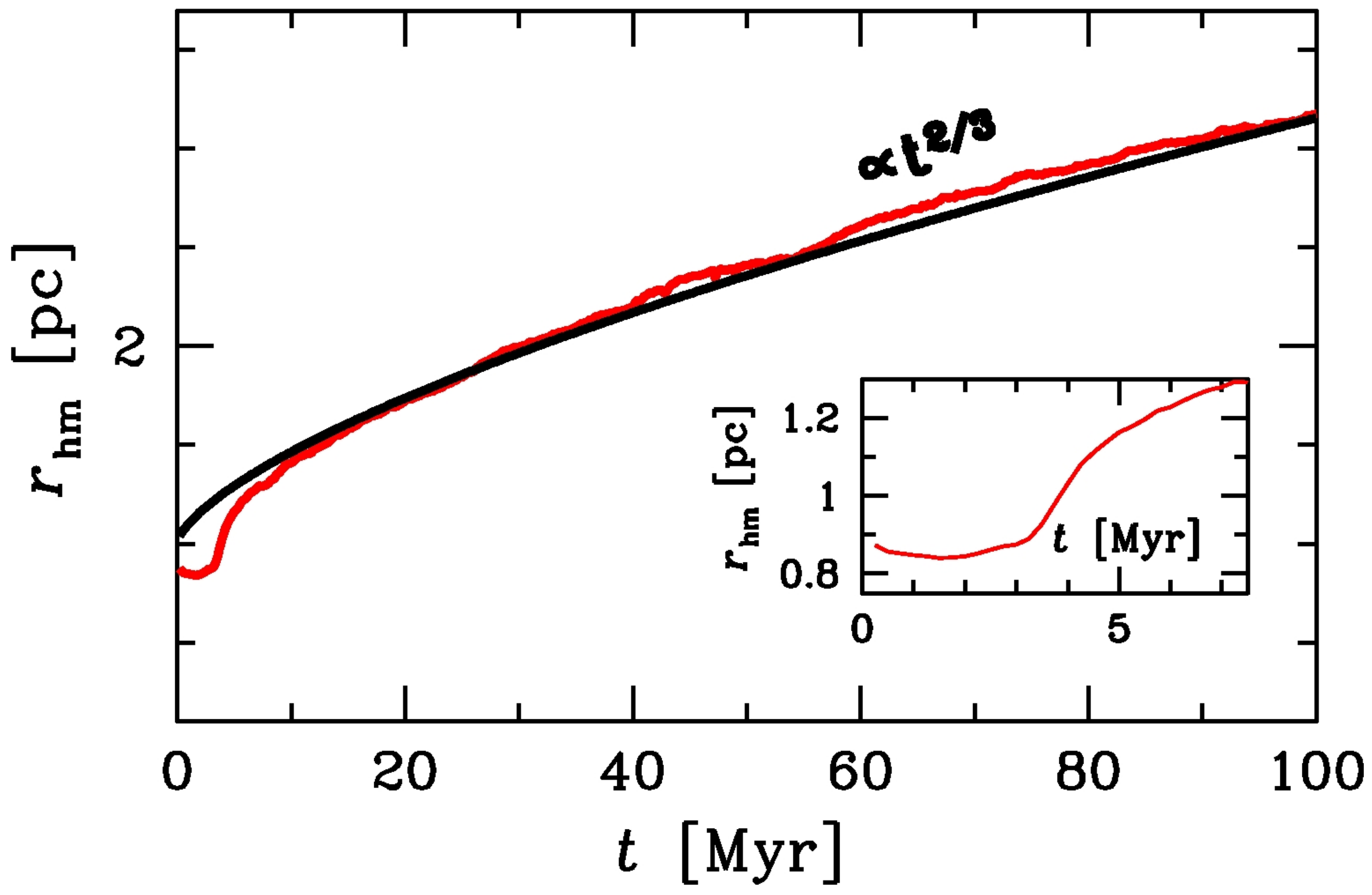
$$\Rightarrow t_{cross} \propto r_{hm}^{3/2} \quad \text{assuming } M \sim const$$

$$\Rightarrow r_{hm} \propto t_{cross}^{2/3} \propto t_{hm}^{2/3} \propto t^{2/3}$$

POST CORE COLLAPSE PHASE:

HOW does halo expand?

$$\Rightarrow r_{hm} \propto t_{cross}^{2/3} \propto t_{hm}^{2/3} \propto t^{2/3}$$



GRAVOTHERMAL OSCILLATIONS:

After first core collapse there may be a series of contractions/re-expansions of the core

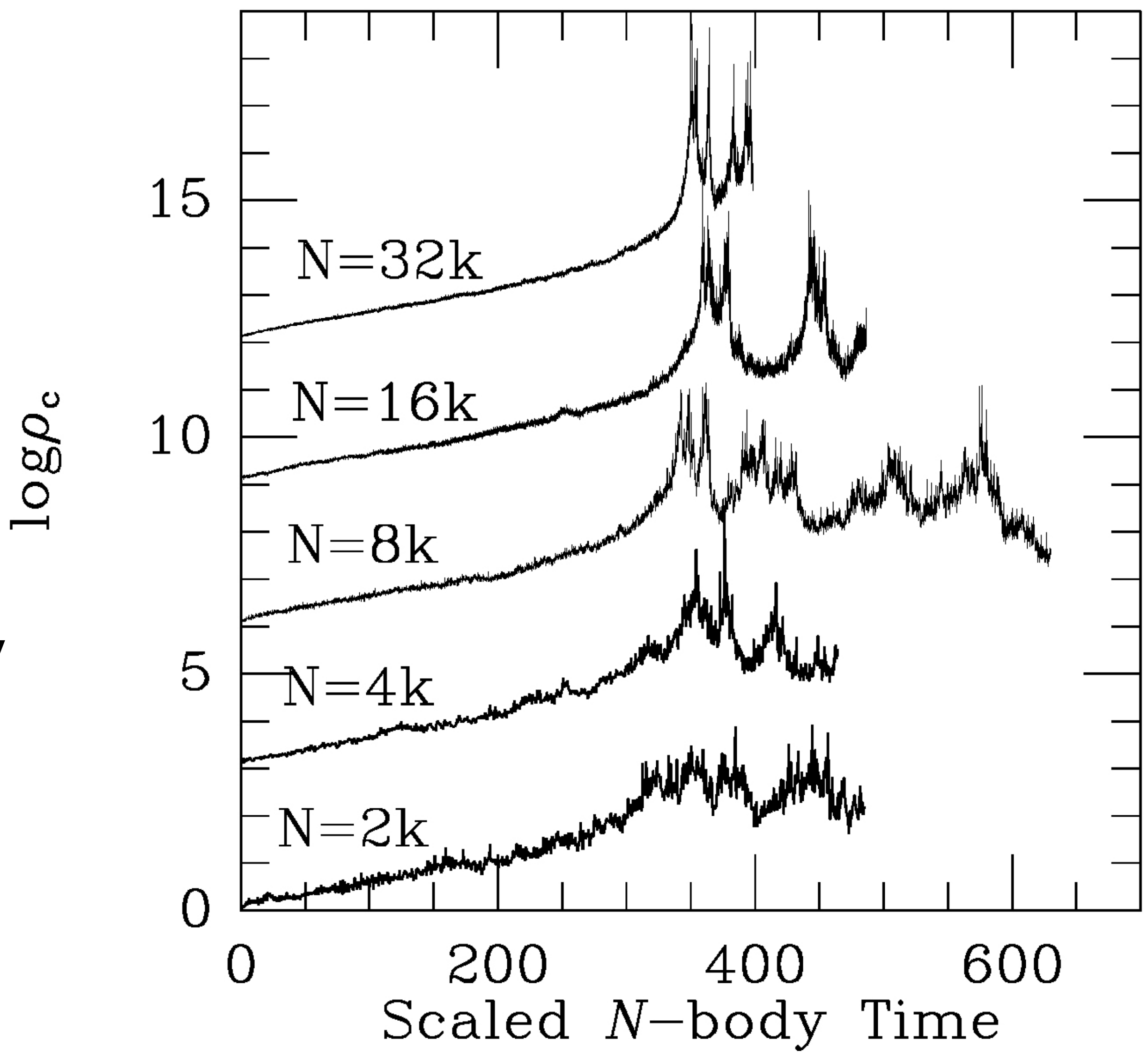
These are consequences of the fact that HEAT CAPACITY can be still negative

Note: only when *N*>10 000
why? Boh..

Hut 1997 (astro-ph/9704286) gives good idea:

For *N*<10 000, binaries are a steady engine.

For *N*>10000 the central density reached after 1st collapse is so high that no engine is sufficient to keep system stable after first bounce



(3) EQUIPARTITION, MASS SEGREGATION AND SPITZER'S INSTABILITY

Processes described up to now (two-body relaxation, evaporation, gravothermal instability, core collapse and reversal) OCCUR EVEN IF STARS ARE EQUAL MASS

BUT stars form with a mass spectrum

The most important effects of unequal-mass system are MASS SEGREGATION and SPITZER'S INSTABILITY

EQUIPARTITION: even collisional systems (i.e. where two-body relaxation is efficient) subject to gravity evolve to satisfy equipartition theorem of statistical mechanics, i.e.
PARTICLES TEND TO HAVE THE SAME AVERAGE KINETIC ENERGY

Thus, equipartition occurs EVEN if stars are equal mass.

If stars are equal mass → equipartition implies that have the same average VELOCITY

$$k_i = \frac{1}{2} m \, v_i^2$$

(3) EQUIPARTITION, MASS SEGREGATION AND SPITZER'S INSTABILITY

If stars are equal mass → equipartition implies that have the same average VELOCITY

$$k_i = \frac{1}{2} m\, v_i^2$$

If particles have different masses, this has a relevant consequence:

$$m_i \langle v_i^2 \rangle = m_j \langle v_j^2 \rangle$$

$$\text{if} \quad m_i > m_j \Rightarrow \langle v_i^2 \rangle < \langle v_j^2 \rangle$$

During two-body encounters, massive stars transfer kinetic energy to light stars. Massive stars slow down, light stars move to higher velocities.

Equipartition in multi-mass systems is reached via ***dynamical friction***

(3) EQUIPARTITION, MASS SEGREGATION AND SPITZER'S INSTABILITY

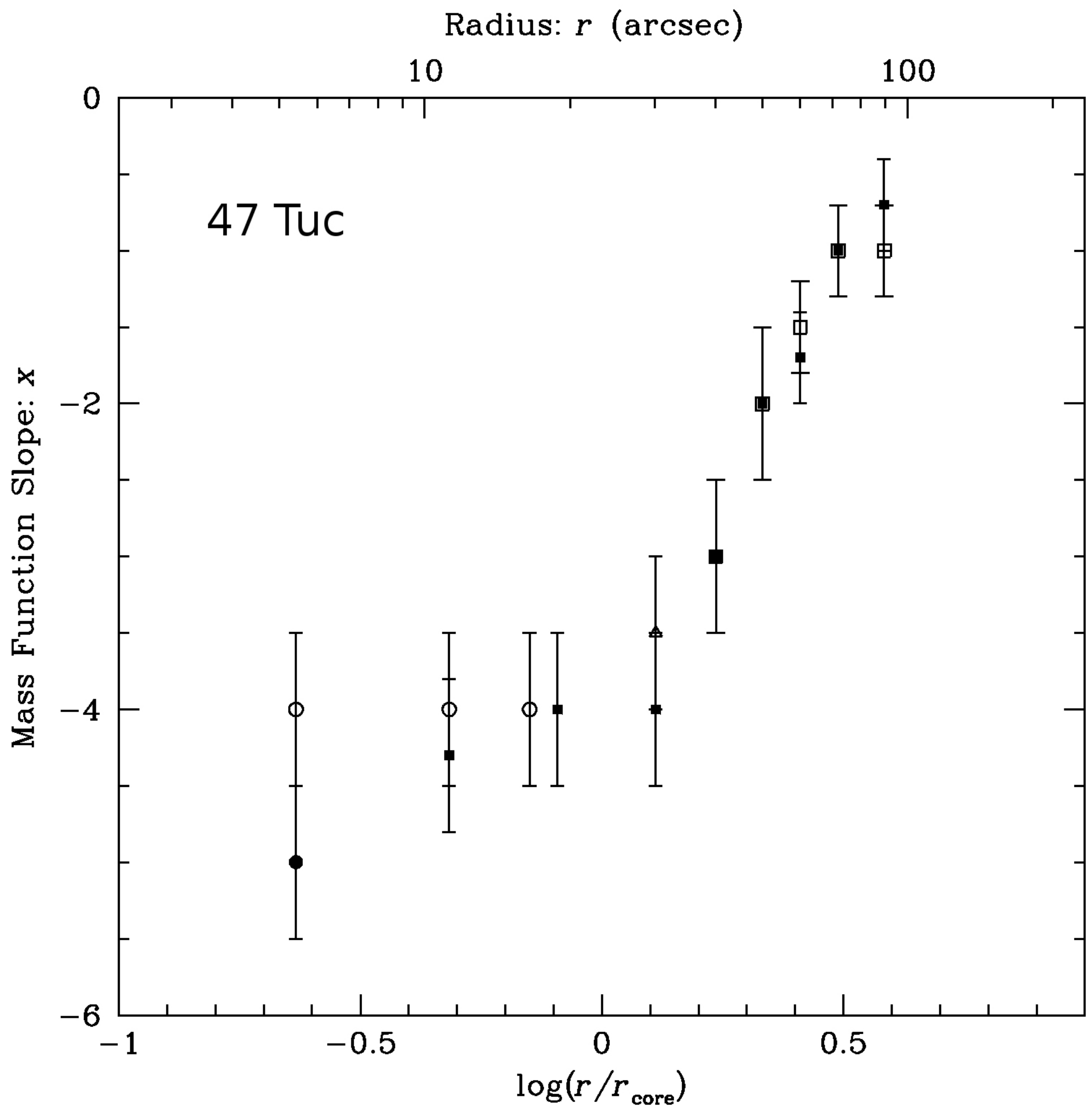
This means that **heavier stars drift to the centre of the cluster**, producing **MASS SEGREGATION** (i.e. local mass function different from IMF)

→ MASS SEGREGATION increases the instability of the system and induces a **FASTER COLLAPSE (t_{coll}~0.2 t_{rlx} rather than t_{coll}~15 t_{rlx})**.

e.g. core of 47 Tucanae
(Monkman et al. 2006, ApJ, 650, 195)
x:= mass function slope

$$\mathbf{dN/dm = m^{-(1+x)}}$$

where Salpeter x=+1.35



(3) SPITZER'S INSTABILITY

SPITZER'S INSTABILITY (or mass stratification instability):
It is not always possible to reach equipartition in a multi-mass.

Let us suppose that there are two populations with two different masses: m_1 *(total mass* M_1*) and* m_2 *(total mass* M_2*), with* $m_1 < m_2$.

We explore 2 limit cases where equipartition is impossible.

1) $M_2 >> M_1 \Rightarrow$ *potential is dominated by massive stars*

$\Rightarrow \langle v^2 \rangle$ *of the massive stars is ~* $\frac{1}{4} \langle v_{esc}^2 \rangle$

$\Rightarrow$ *if* $m_2/m_1 > 4$, *the* $\langle v^2 \rangle$ *of light stars is higher than* $\langle v_{esc}^2 \rangle$

$\Rightarrow$ ***ALL LIGHT STARS EVAPORATE FROM THE CLUSTER!!!***

Not very important in practice because IMF is not sufficiently top-heavy

(3) SPITZER'S INSTABILITY

SPITZER'S INSTABILITY:

2) $M_2 \sim M_1$ (the case of the so called Spitzer's instability)

If the total mass of the heavy population is similar to the total mass of the light population, equipartition is not possible:
the heavy population forms a cluster within the cluster,
i.e. a sub-cluster at the centre of the cluster,
dynamically decoupled from the rest of the cluster.
The sub-cluster of the heavy population tends to contract.

DEMONSTRATION:
(Note that I did not put numerical coefficients & simplified!)
(a) Assume that there are two populations (1 and 2) with $m_2 >> m_1$
(b) assume total mass $M_2 < M_1$
(c) assume $M_1(r) \sim \rho_{01}\, r^3$ (ρ_{01}:= initial density of population 1)

$\rho_{m2} \sim M_2/r_2^3$ (ρ_{m2}:= average density of population 2, r_2:=half mass radius of population 2)

$\rho_{m1} \sim M_1/r_1^3$ (ρ_{m1}:= average density of population 1, r_1:=half mass radius of population 1)

(3) SPITZER'S INSTABILITY

From equipartition: $$m_i \langle v_i^2 \rangle = m_j \langle v_j^2 \rangle \quad (1)$$

From virial theorem: $$\langle v_2^2 \rangle \sim \frac{G\,M_2}{r_2} + \frac{G}{M_2} \int_0^\infty \frac{\rho_2\, M_1(\vec{r})}{\vec{r}} dV \quad (2)$$

$$\langle v_1^2 \rangle \sim \frac{G\,M_1}{r_1} + \frac{G}{M_1} \int_0^\infty \frac{\rho_1\, M_2(\vec{r})}{\vec{r}} dV \quad (3)$$

$M_1 \gg M_2(\vec{r})$ → 0

Substituting (2) and (3) into (1) and using the assumptions a, b and c:

$$m_2 \left(\frac{G\,M_2}{r_2} + \frac{G\,\rho_{m2}}{M_2} \int \rho_{01} \frac{\vec{r}^3}{\vec{r}} dV \right) = m_1 \frac{G\,M_1}{r_1}$$

$$m_2 \left[\frac{M_2}{\left(\frac{M_2}{\rho_{m2}}\right)^{1/3}} + \frac{\rho_{m2}}{M_2} \int \rho_{01}\, \vec{r}^2\, dV \right] = m_1 \frac{M_1}{\left(\frac{M_1}{\rho_{m1}}\right)^{1/3}}$$

(3) SPITZER'S INSTABILITY

$$m_2\, M_2^{2/3}\, \rho_{m2}^{1/3} \left(1 + \frac{\rho_{m2} \int \rho_{01}\, \vec{r}^2\, dV}{M_2^{5/3}\, \rho_{m2}^{1/3}} \right) = m_1\, M_1^{2/3}\, \rho_{m1}^{1/3}$$

$$\frac{m_2}{m_1} \left(\frac{M_2}{M_1} \right)^{2/3} = \frac{\left(\rho_{m1}/\rho_{m2}\right)^{1/3}}{\left[1 + \frac{\rho_{m2} \int \rho_{01}\, \vec{r}^2\, dV}{\left(\rho_{m2}\, r_2^3\right)^{5/3}\, \rho_{m2}^{1/3}} \right]}$$

$$\frac{m_2}{m_1} \left(\frac{M_2}{M_1} \right)^{2/3} = \frac{\left(\rho_{m1}/\rho_{m2}\right)^{1/3}}{\left(1 + \frac{\rho_{m2} \int \rho_{01}\, \vec{r}^2\, dV}{\rho_{m2}^2\, r_2^5} \right)}$$

$$\frac{m_2}{m_1} \left(\frac{M_2}{M_1} \right)^{2/3} = \frac{\left(\rho_{m1}/\rho_{m2}\right)^{1/3}}{\left[1 + \frac{\rho_{01}}{\rho_{m2}} \left(\frac{r_2}{r_2} \right)^5 \right]}$$

(3) SPITZER'S INSTABILITY

$$\left(\frac{m_2}{m_1}\right)^{3/2} \left(\frac{M_2}{M_1}\right) = \frac{(\rho_{m1}/\rho_{m2})^{1/2}}{\left[1 + \frac{\rho_{01}}{\rho_{m2}}\right]^{3/2}}$$

$$\left(\frac{m_2}{m_1}\right)^{3/2} \left(\frac{M_2}{M_1}\right) = \frac{(\rho_{m1}/\rho_{m2})^{1/2}}{\left[1 + \frac{\rho_{m1}}{\rho_{m1}} \frac{\rho_{01}}{\rho_{m2}}\right]^{3/2}}$$

$$\left(\frac{m_2}{m_1}\right)^{3/2} \left(\frac{M_2}{M_1}\right) = \frac{(\rho_{m1}/\rho_{m2})^{1/2}}{\left[1 + \frac{\rho_{01}}{\rho_{m1}} \left(\frac{\rho_{m1}}{\rho_{m2}}\right)\right]^{3/2}}$$

(3) SPITZER'S INSTABILITY

$$\left(\frac{m_2}{m_1}\right)^{3/2} \left(\frac{M_2}{M_1}\right) = \frac{\left(\rho_{m1}/\rho_{m2}\right)^{1/2}}{\left(1+\alpha\,\frac{\rho_{m1}}{\rho_{m2}}\right)^{3/2}}$$

Our simplified α :

$$\alpha \equiv \frac{\rho_{01}}{\rho_{m1}}$$

Spitzer's α :

$$\alpha \equiv \frac{5\,\rho_{01}}{4\,\rho_{m1}} \left(\frac{r_{2s}}{r_2}\right)^2 \sim 5.6$$

where r_{2s}^2 is the mean value of r^2 for the population 2

Maximum possible value for the right-hand term ~ 0.16

$$\Rightarrow \frac{M_2}{M_1} < 0.16 \left(\frac{m_2}{m_1}\right)^{3/2}$$

OTHERWISE EQUIPARTITION CANNOT BE REACHED!

(3) SPITZER'S INSTABILITY

SPITZER'S INSTABILITY:
It is not possible to reach equipartition if $M_2/M_1 < 0.16\ (m_2/m_1)^{3/2}$.

If the total mass of the heavy population is similar to the total mass of the light population, equipartition is not possible:

the heavy population forms a cluster within the cluster,
i.e. a sub-cluster at the centre of the cluster,
dynamically decoupled from the rest of the cluster.

The massive stars in the sub-cluster keep transferring kinetic energy to the lighter stars but cannot reach equipartition: the core of massive stars continues to contract till infinite density!

The contraction stops when most of the massive stars eject each-other from the cluster by 3-body encounters (see next lecture) or when most of the massive stars collapse into a single object (see last lecture).

TIMESCALES FOR RELAXATION and CORE COLLAPSE in different SCs

Table 1.3. Time scales

Time scale	symbol	bulge	globular	YoDeC	Open cluster
Star	$t_{\rm ms}$	10Gyr	10Gyr	10Myr	10Myr
size	R	100pc	10pc	$\lesssim$ 1pc	10pc
mass	M	$10^9 M_\odot$	$10^6 M_\odot$	$10^5 M_\odot$	$1000 M_\odot$
velocity	$\langle v \rangle$	100km s^{-1}	10km s^{-1}	10km s^{-1}	1km s^{-1}
relaxation	$t_{\rm rt}$	10^{15}yr	3 Gyr	50Myr	100Myr
crossing	$t_{\rm hm}$	100Myr	10 Myr	100Kyr	1Myr
$t_{\rm rt}/t_{\rm ms}$		10^5	3	5	10
$t_{\rm hm}/t_{\rm ms}$		0.01	1	10^{-4}	0.1

From Portegies Zwart 2004, astro-ph/0406550

***Note:* t_{coll}~0.2 t_{rlx}**

Young dense star clusters (YoDeC) are the only clusters with relaxation and core collapse time of the same order of magnitude as massive star evolution

References:

*** Spitzer L., Dynamical evolution of globular clusters, 1987, Princeton University Press**

* Elson R. & Hut P., Dynamical evolution of globular clusters, Ann. Rev. Astron. Astrophys., 1987, 25, 565

* Hut P., Gravitational Thermodynamics, astroph/9704286

* Spitzer L. Jr., Equipartition and the formation of compact nuclei in spherical stellar systems, 1969, ApJ, 158, L139

* Binney & Tremaine, Galactic Dynamics, First edition, 1987, Princeton University Press

Michela Mapelli

LECTURES on COLLISIONAL DYNAMICS:

3. BINARIES and 3-BODY ENCOUNTERS

BINARIES as ENERGY RESERVOIR

Binaries have a energy reservoir (their internal energy) that can be exchanged with stars.

INTERNAL ENERGY: total energy of the binary – kinetic energy of the centre-of-mass

$$E_{int} = \frac{1}{2}\,\mu\,v^2 - \frac{G\,m_1\,m_2}{r}$$

where m_1 and m_2 are the mass of the primary and secondary member of the binary, μ is the reduced mass (:= $m_1\,m_2/(m_1+m_2)$).
r and v are the relative separation and velocity.

$E_{int}<0$ if the binary is bound

Note that E_{int} can be interpreted as the energy of the 'reduced particle': a fictitious particle of mass μ orbiting in the potential $-G\,m_1\,m_2/r$

BINARIES as ENERGY RESERVOIR

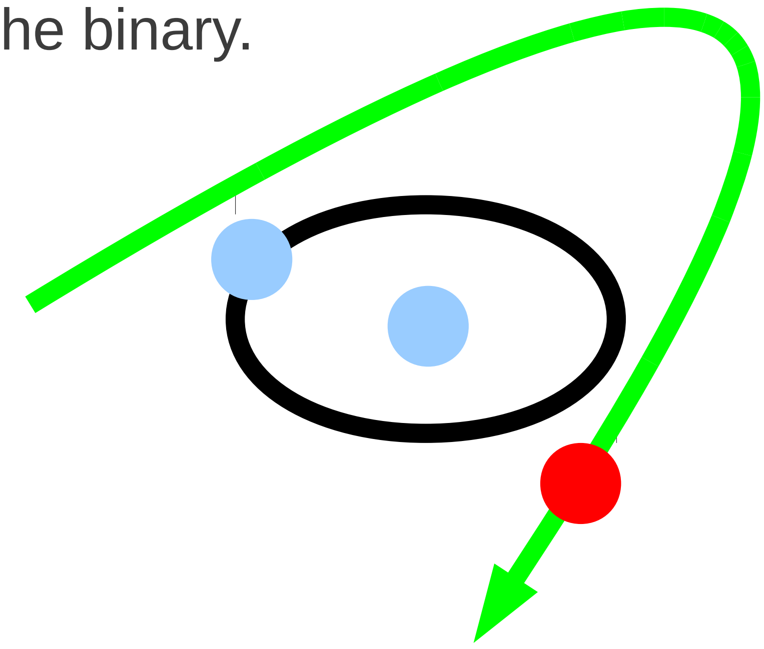
As far as the binary is bound, the orbit of the reduced particle is a Kepler ellipse with semi-major axis *a*. Thus, the energy integral of motion is

$$E_{int} = -\frac{G\, m_1\, m_2}{2\, a} = -E_b$$

where E_b is the **BINDING ENERGY** of the binary.

THE ENERGY RESERVOIR of BINARIES can be EXCHANGED with stars:

during a **3-BODY INTERACTION**, i.e. an interaction between a binary and a single star, the single star can either EXTRACT INTERNAL ENERGY from the binary or lose a fraction of its kinetic energy, which is converted into internal energy of the binary.



BINARIES as ENERGY RESERVOIR

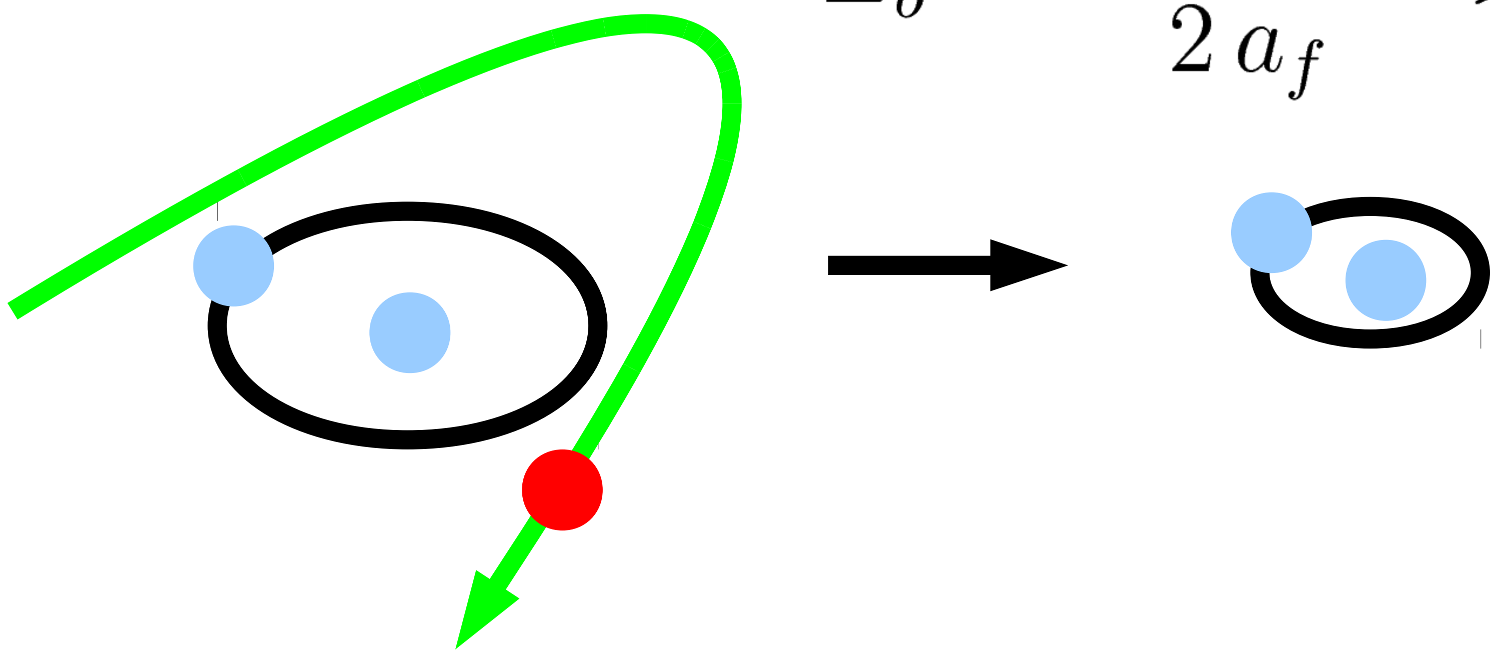
If the star extracts E_{int} from the binary, its final kinetic energy (K_f) is higher than the initial kinetic energy (K_i). To better say: K_f of the centres-of-mass of the single star and of the binary is higher than their K_i.

We say that the STAR and the BINARY acquire RECOIL VELOCITY.

E_{int} becomes more negative, i.e. E_b higher: the binary becomes more bound (e.g. a decreases or m_1 and m_2 change).

$a_f < a_i$

$$E_b = \frac{G\, m_1\, m_2}{2\, a_f} > \frac{G\, m_1\, m_2}{2\, a_i}$$



CARTOON of a FLYBY ENCOUNTER where $a_f < a_i \rightarrow E_b$ increases

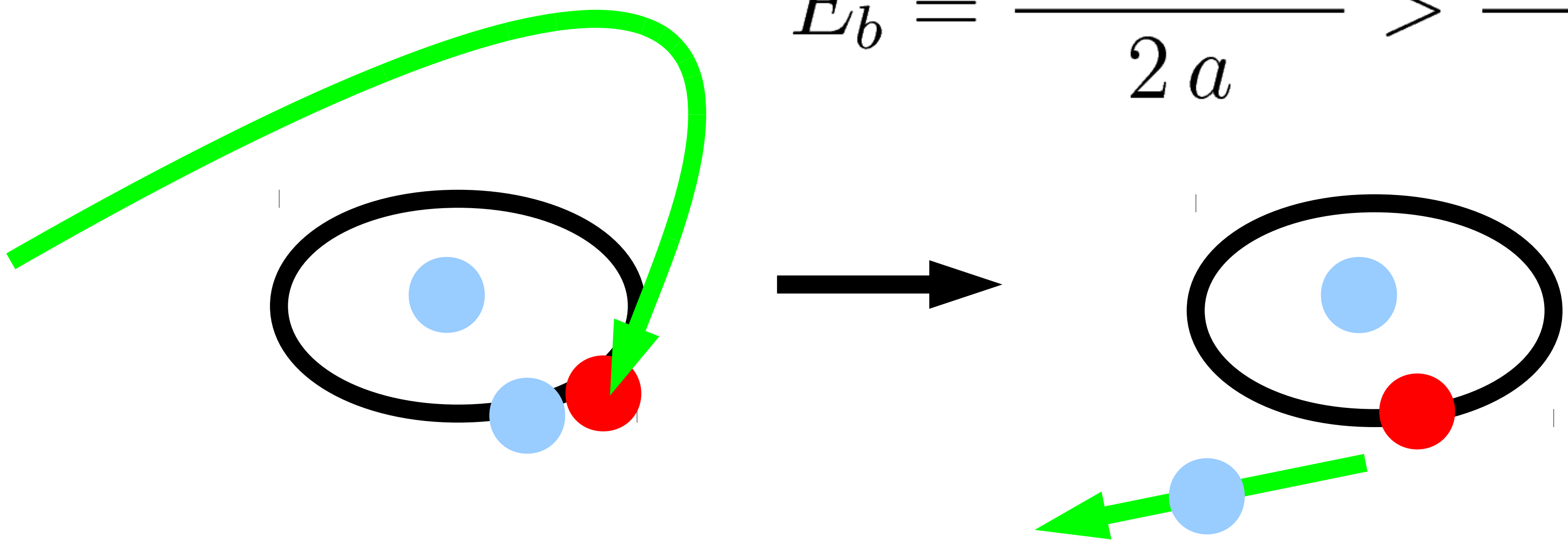
BINARIES as ENERGY RESERVOIR

An alternative way for a binary to transfer internal energy to field stars and increase its binding energy E_b is an **EXCHANGE**:
the single star replaces one of the former members of the binary.

An exchange interaction is favoured when the mass of the single star m_3 is HIGHER than the mass of one of the members of the binary so that the new E_b of the binary is higher than the former:

$m_3 > m_2$

$$E_b = \frac{G\, m_1\, m_3}{2\, a} > \frac{G\, m_1\, m_2}{2\, a}$$



CARTOON of a EXCHANGE ENCOUNTER where $m_3 > m_2 \rightarrow$ E_b increases

BINARIES as ENERGY RESERVOIR

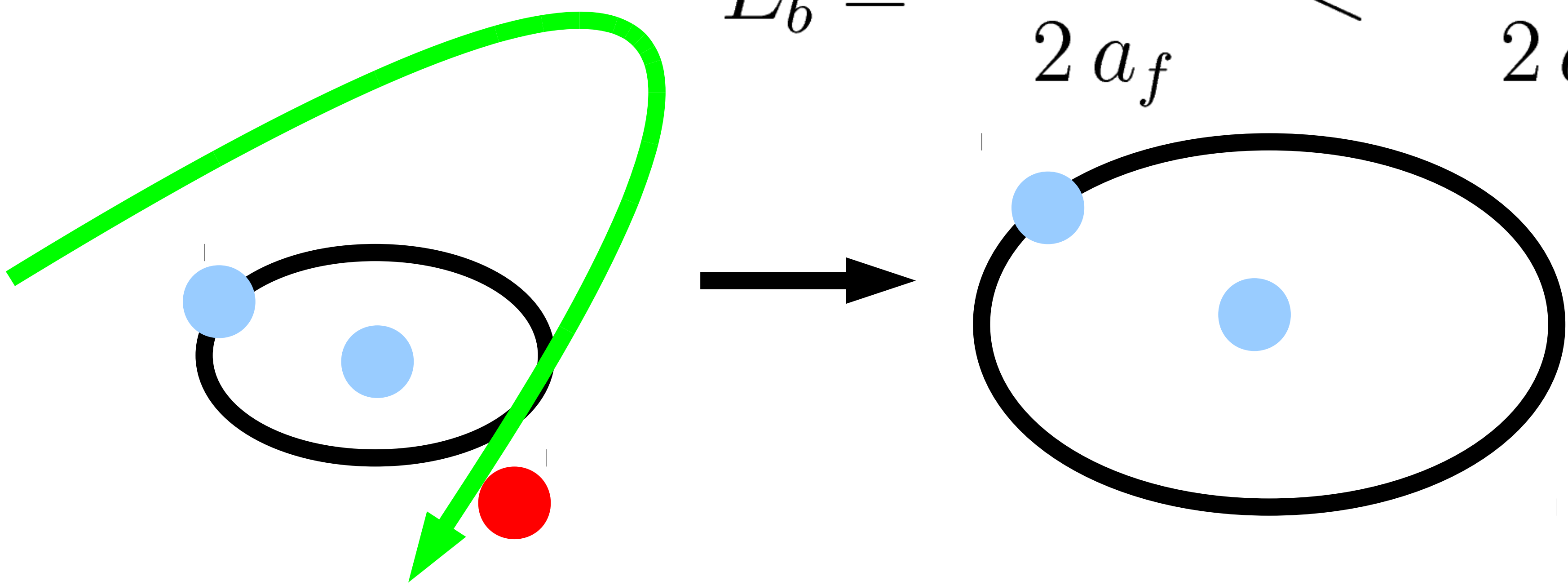
If the star transfers kinetic energy to the binary, its final kinetic energy (K_f) is obviously lower than the initial kinetic energy (K_i). To better say:

K_f of the centres-of-mass of the single star and of the binary is lower than their K_i.

E_{int} becomes less negative, i.e. E_b smaller: the binary becomes less bound (e.g. a increases) or is even IONIZED (:= becomes UNBOUND).

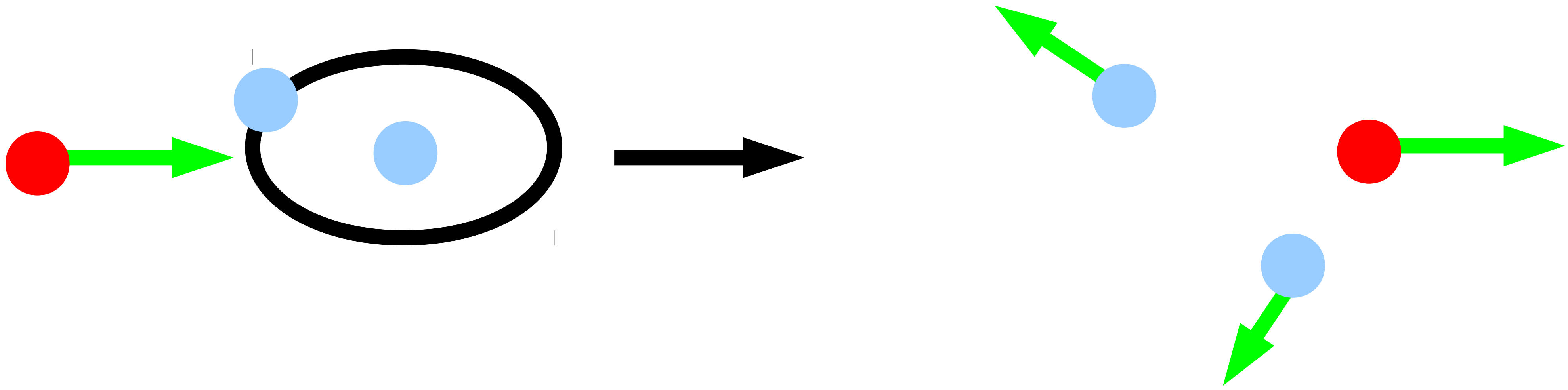
$a_f > a_i$

$$E_b = \frac{G\, m_1\, m_2}{2\, a_f} < \frac{G\, m_1\, m_2}{2\, a_i}$$



CARTOON of a FLYBY ENCOUNTER where $a_f > a_i \rightarrow E_b$ decreases

BINARIES as ENERGY RESERVOIR



A single star can IONIZE the binary only if its velocity at infinity (=when it is far from the binary, thus unperturbed by the binary) exceeds the CRITICAL VELOCITY (Hut & Bahcall 1983, ApJ, 268, 319)

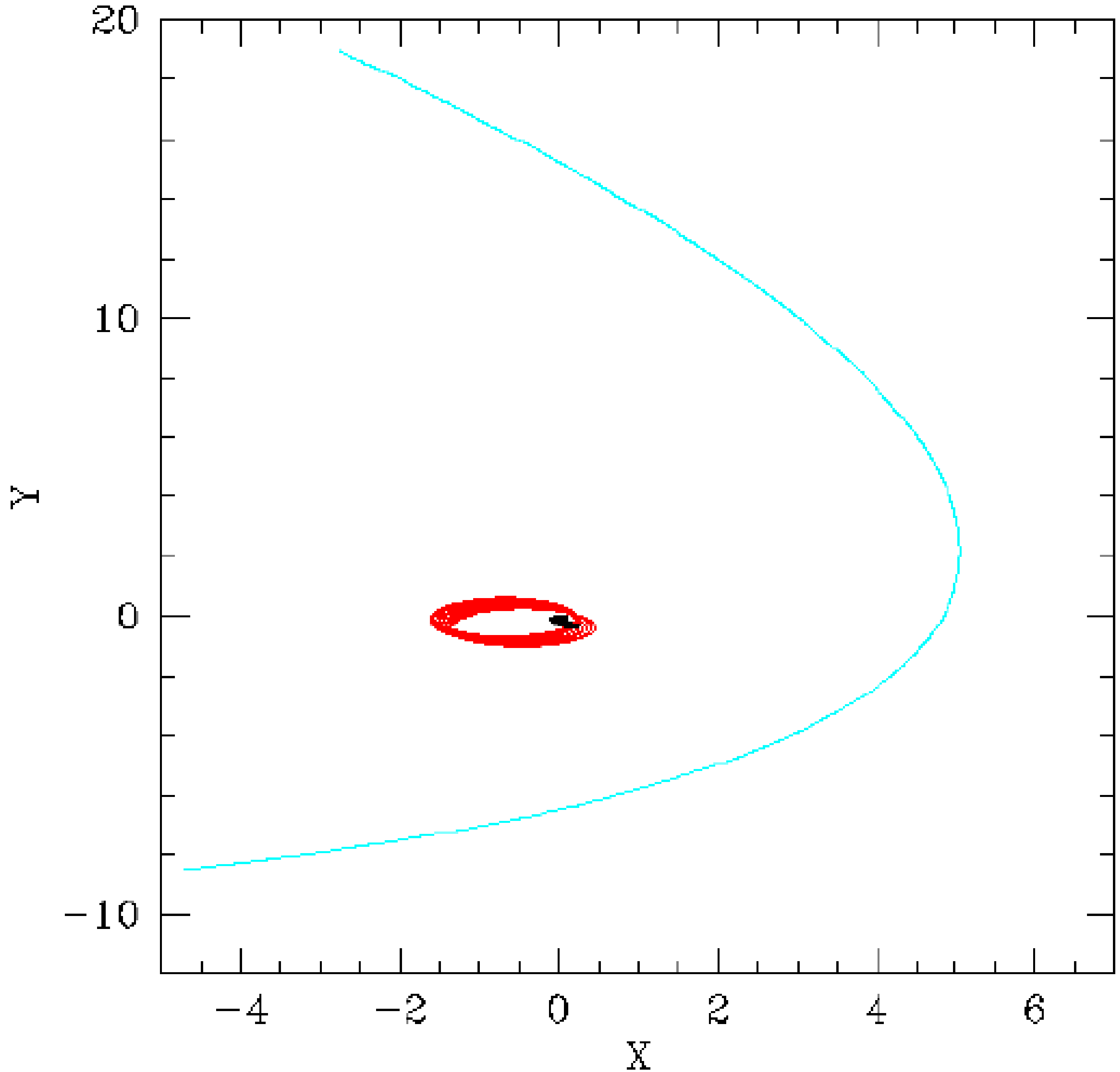
$$v_c = \sqrt{\frac{G\, m_1\, m_2\, (m_1 + m_2 + m_3)}{m_3\, (m_1 + m_2)\, a}}$$

This critical velocity was derived by imposing that the K of the reduced particle of the 3-body system is equal to E_b:

$$\frac{1}{2}\,\frac{m_3\,(m_1 + m_2)}{(m_1 + m_2 + m_3)}\,v_c^2 = \frac{G\, m_1\, m_2}{2\, a}$$

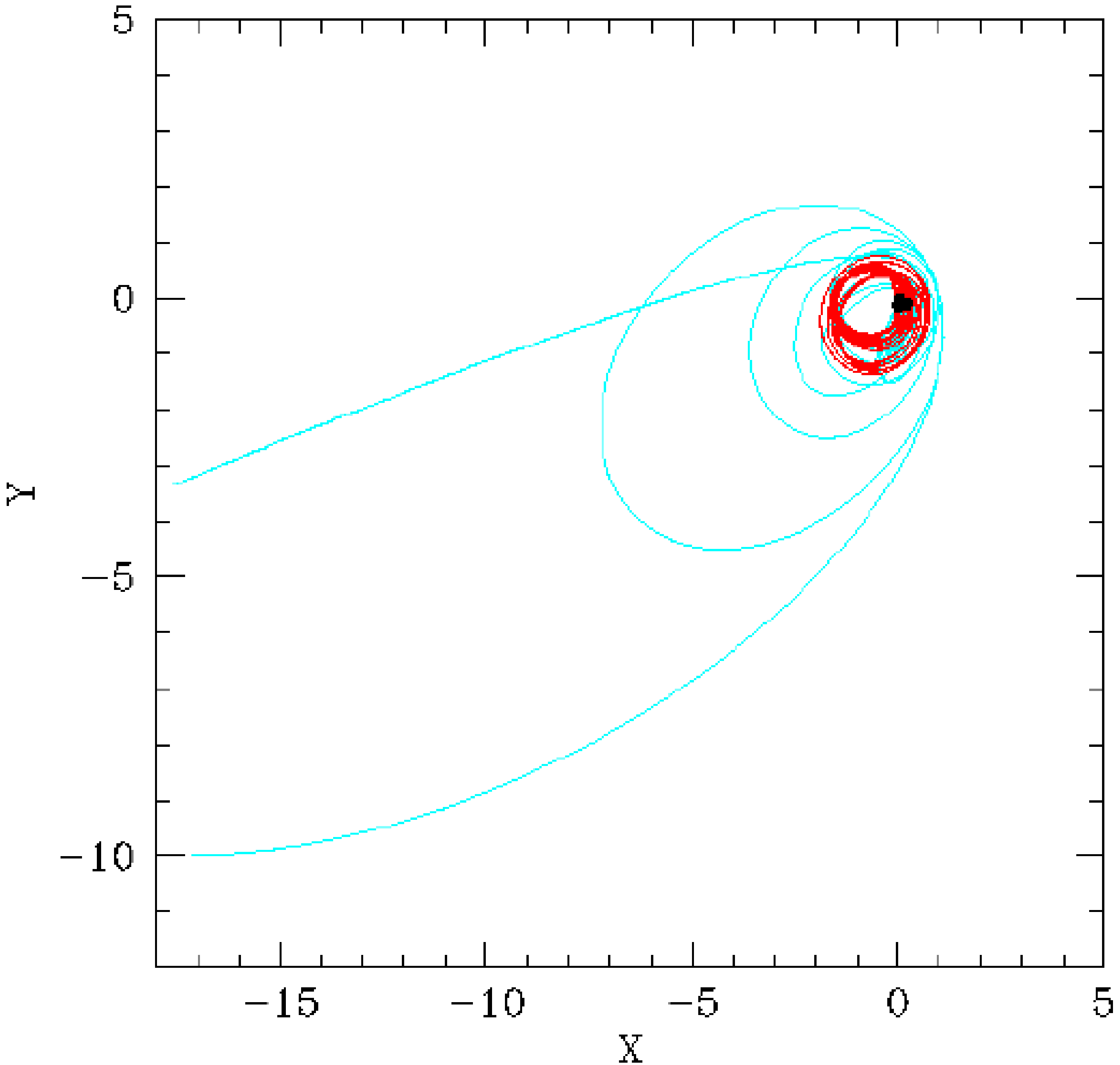
EXAMPLES of SIMULATED 3-BODY ENCOUNTERS

PROMPT
FLYBY:



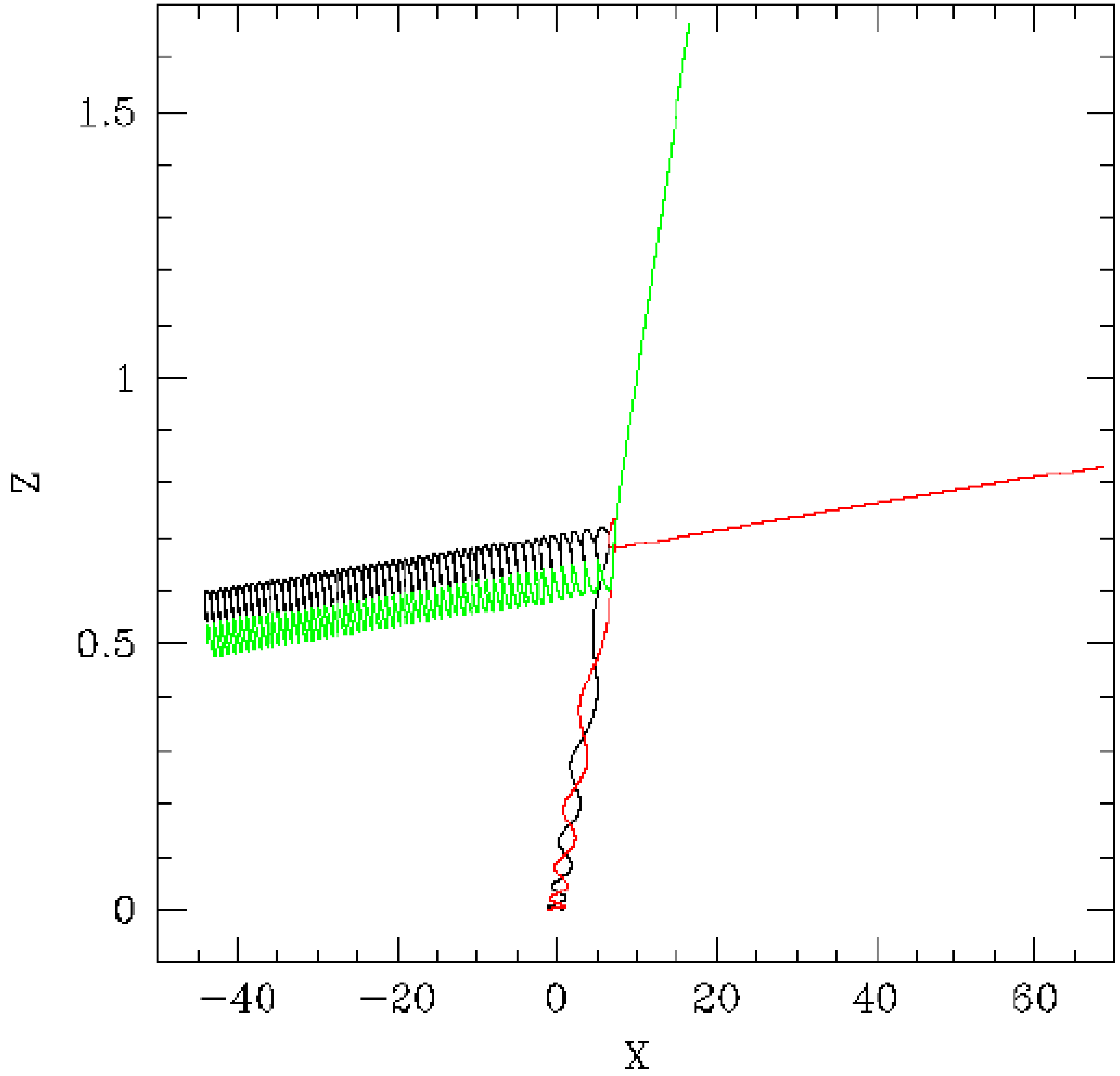
EXAMPLES of SIMULATED 3-BODY ENCOUNTERS

RESONANT
FLYBY:



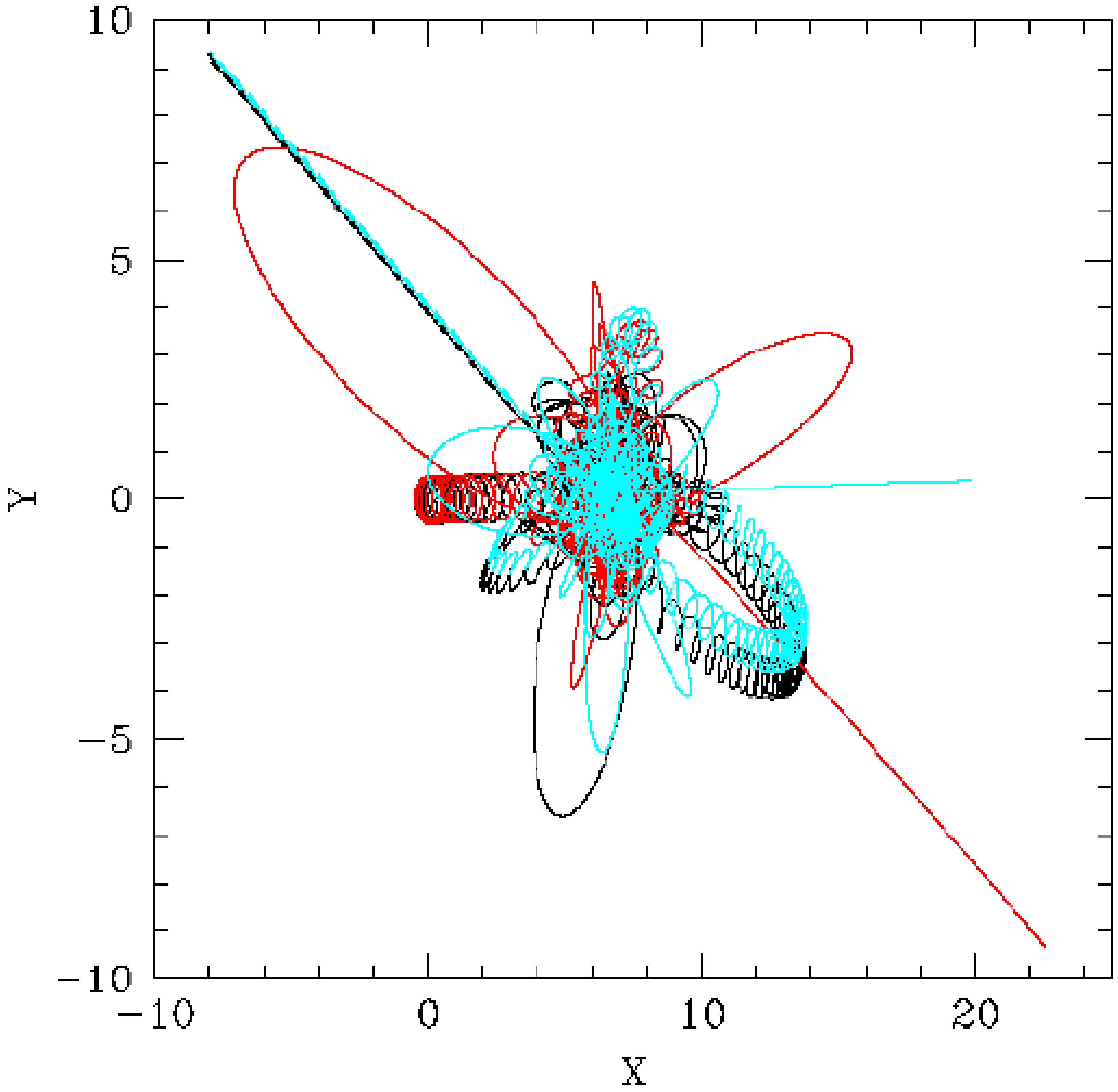
EXAMPLES of SIMULATED 3-BODY ENCOUNTERS

PROMPT
EXCHANGE:



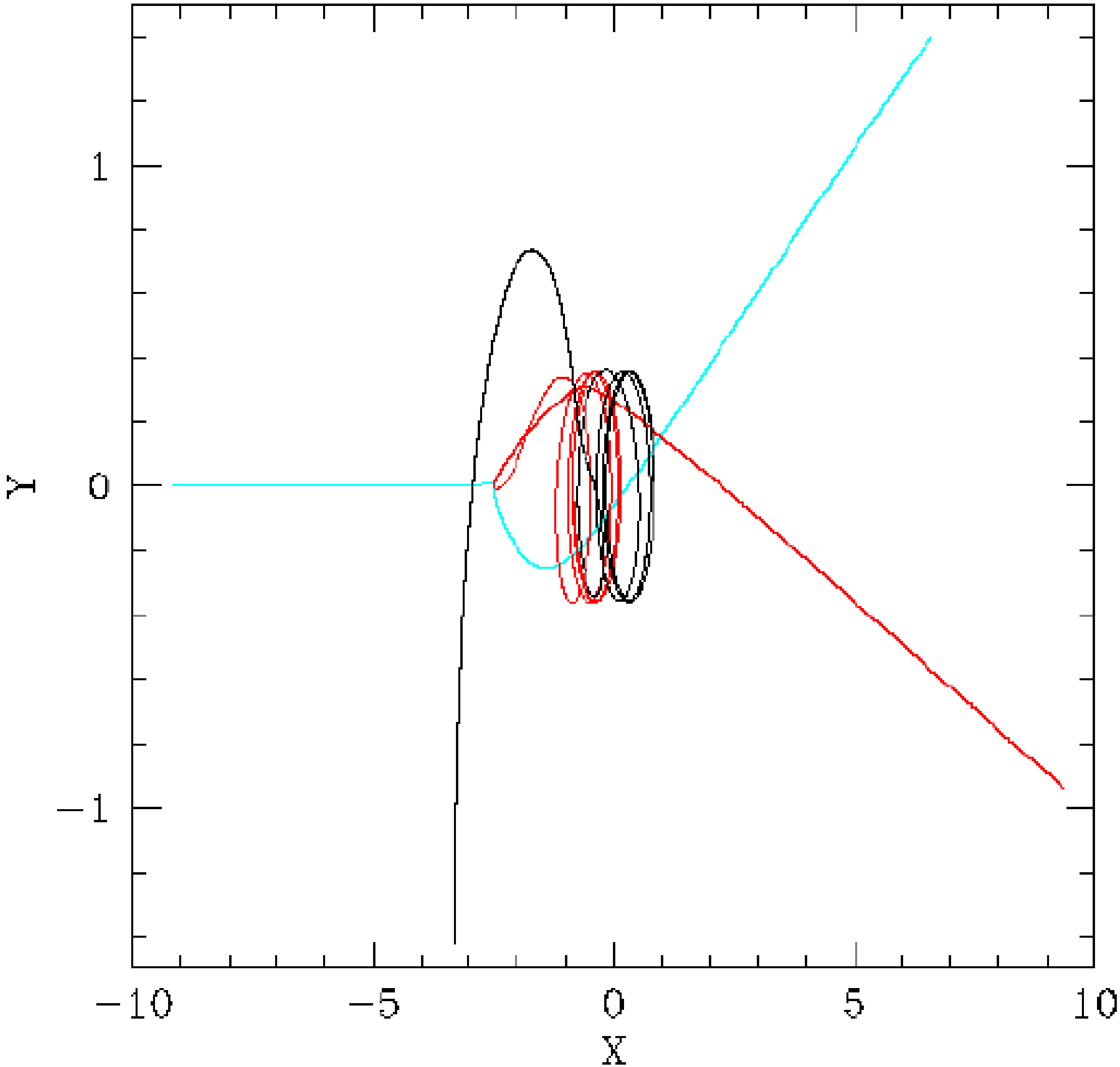
EXAMPLES of SIMULATED 3-BODY ENCOUNTERS

RESONANT
EXCHANGE:



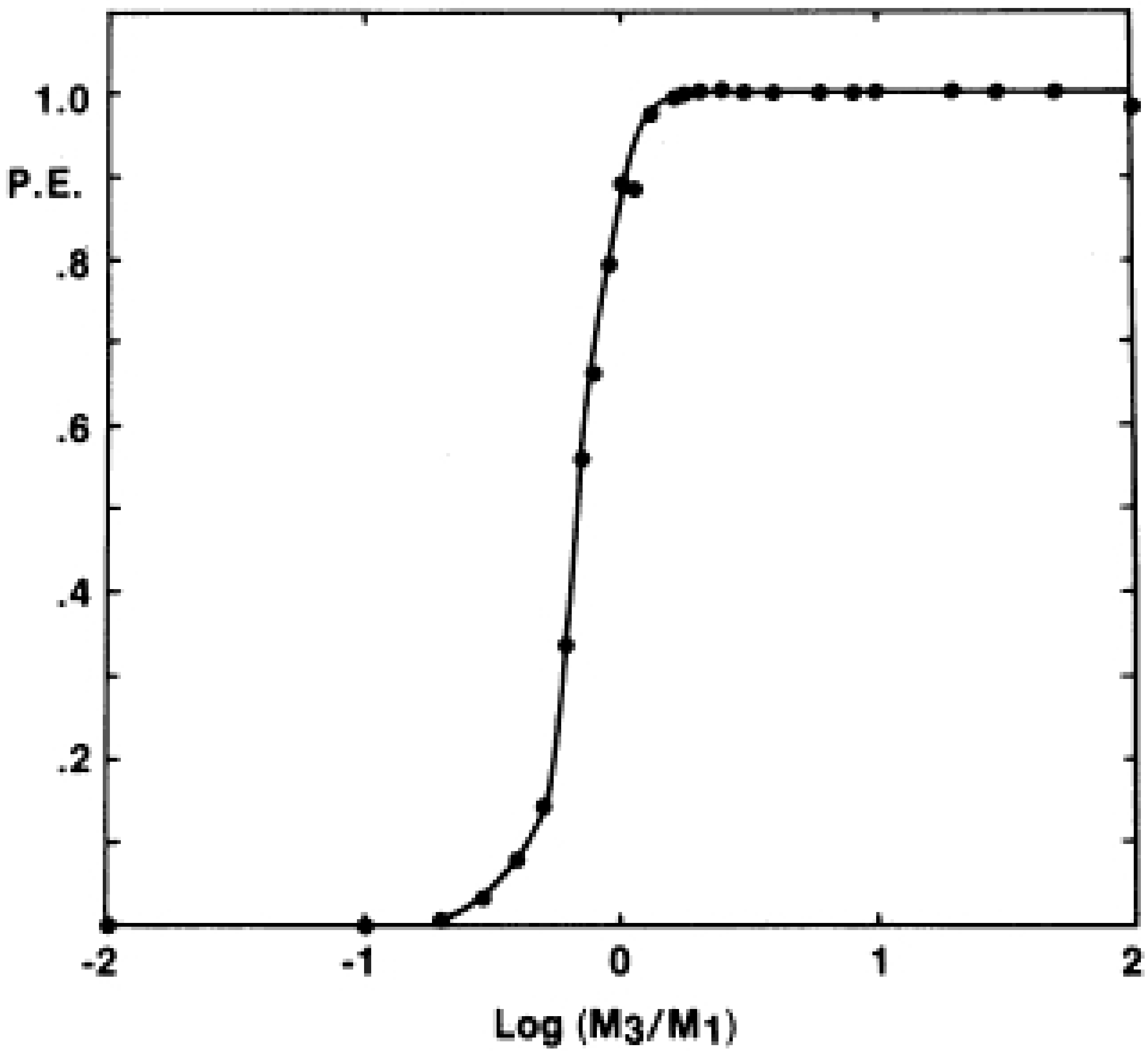
EXAMPLES of SIMULATED 3-BODY ENCOUNTERS

IONIZATION:



EXCHANGE PROBABILITY

Probability increases dramatically if $m_3 \geq m_1$



Hills & Fullerton 1980, AJ, 85, 1281

Can we understand whether a binary will lose or acquire E_b?

YES, but ONLY in a STATISTICAL SENSE

We define **HARD BINARIES:** binaries with binding energy higher than the average kinetic energy of a star in the cluster

$$\frac{G\, m_1\, m_2}{2\, a} > \frac{1}{2}\, \langle m \rangle\, \sigma^2$$

$$\frac{G\, m_1\, m_2}{2\, a} < \frac{1}{2}\, \langle m \rangle\, \sigma^2$$

SOFT BINARIES: binaries with binding energy lower than the average kinetic energy of a star in the cluster

HEGGIE'S LAW (1975):

Hard binaries tend to become harder (i.e. increase E_b)

Soft binaries tend to become softer (i.e. decrease E_b)

as effect of three-body encounters

Cross Section for 3-body encounters

Importance of binaries for dynamical encounters also due to the LARGER CROSS SECTION with respect to single stars

Simplest formalism for the cross section of stars and binaries:
GEOMETRICAL CROSS SECTION

$$\Sigma_{GE} = \pi\, a^2 \qquad\qquad \Sigma_{GE} = \pi\, {R_*}^2$$

For binaries (scales with a^2) For stars (scales with star radius R_*)

$a > 10^{13}$ cm, $R_* \sim 10^{10-13}$ cm $\rightarrow$ $a >> R_*$

It is sufficient that a fraction <~ 0.1 of stars in a cluster are binaries for 3-body encounters to be more important than two-body encounters.
The difference is even larger if we take a more realistic definition of **CROSS SECTION for 3-BODY ENCOUNTERS:**

$$\Sigma = \pi\, b^2_{max}$$

b_{max} is the maximum impact parameter for a non-zero energy exchange between star and binary

Cross Section for 3-body encounters

How do we estimate b_{max}? It depends on which energy exchange we are interested in.
If we are interested only in particularly energetic exchanges, we can derive b_{max} from GRAVITATIONAL FOCUSING.

GRAVITATIONAL FOCUSING:
If the binary is significantly more massive than the single star, the TRAJECTORY of the single star is deflected by the binary, when approaching the pericentre.
The link between the impact parameter b and the effective pericentre p can be derived by the conservation of energy and angular momentum in the system of the reduced particle (see demonstration in the next slide):

$$p = \frac{G\, m_T}{{v_\infty}^2}\left[\sqrt{1+\left(\frac{{v_\infty}^2}{G\, m_T}\right)^2 b^2}-1\right]$$

where $m_T \equiv m_1 + m_2 + m_3$

Note: in most calculations v_∞ and σ (velocity dispersion) will be used as synonymous

$$\text{if } \frac{G\, m_T}{{v_\infty}^2\, b} >> 1 \Rightarrow p \sim b^2\, \frac{v_\infty^2}{2\, G\, m_T}$$

DEMONSTRATION of GRAVITATIONAL FOCUSING FORMULA:

1) Energy conservation

$$\frac{1}{2}\,\frac{m_3\,(m_1+m_2)}{m_T}\,v_\infty^2-\frac{G\,m_1\,m_2}{a}-\frac{G\,m_3\,(m_1+m_2)}{D}=\frac{1}{2}\,\frac{m_3\,(m_1+m_2)}{m_T}\,v_p^2-\frac{G\,m_1\,m_2}{a}-\frac{G\,m_3\,(m_1+m_2)}{p}$$

∞ $\quad 0$

$$\frac{1}{2}\,\frac{v_\infty^2}{m_T}=\frac{1}{2}\,\frac{v_p^2}{m_T}-\frac{G}{p}$$

2) Angular momentum conservation

$$\vec{J}_\infty=(m_1\,m_2)\sqrt{\frac{a_{in}\,G}{m_1+m_2}}\,\hat{z}+b\,v_\infty\left[\frac{m_3\,(m_1+m_2)}{m_T}\right]\hat{z}'$$

parallel to *b* perpendicular to *b*

$$\vec{J}_p=(m_1\,m_2)\sqrt{\frac{a_{in}\,G}{m_1+m_2}}\,\hat{z}+p\,v_p\left[\frac{m_3\,(m_1+m_2)}{m_T}\right]\hat{z}'$$

$$\Rightarrow b\,v_\infty\sim p\,v_p$$

DEMONSTRATION of GRAVITATIONAL FOCUSING FORMULA:

1) and 2) together: $$\frac{1}{2}\frac{v_\infty^2}{m_T} - \frac{1}{2}\left(\frac{b\,v_\infty}{p}\right)^2\frac{1}{m_T} + \frac{G}{p} = 0$$

Multiplying and dividing by p^2

$$\frac{1}{2}\frac{v_\infty^2}{m_T}\,p^2 + G\,p - \frac{1}{2}\frac{(b\,v_\infty)^2}{m_T} = 0$$

Finally:

$$p = \frac{-G \pm \sqrt{G^2 + \frac{b^2 {v_\infty}^4}{m_T^2}}}{{v_\infty}^2/m_T} = \frac{G\,m_T}{{v_\infty}^2}\left[\sqrt{1 + \left(\frac{{v_\infty}^2}{G\,m_T}\right)^2 b^2} - 1\right]$$

$$\text{if } \frac{G\,m_T}{{v_\infty}^2\,b} >> 1 \Rightarrow p \sim b^2\,\frac{v_\infty^2}{2\,G\,m_T}$$

Cross Section for 3-body encounters

We now express b_{max} in terms of the pericentre distance p to obtain a useful formalism for the 3-body cross section (especially for massive binaries).

$$\Sigma = \pi \left(\frac{2\, G\, m_T}{{v_\infty}^2} \right) p_{max}$$

A good choice for p_{max} (if we are interested only in the most energetic encounters) is $p_{max}=a$

$$\boxed{\Sigma = 2\, \pi\, G\, \frac{m_T\, a}{{v_\infty}^2}} \qquad (*)$$

There are many other formalisms for the cross section. We remind the one in Davies (2002):

$$\Sigma = \kappa(q_1, q_2)\, \pi a^2 \left[1 + \left(\frac{v_c}{v_\infty} \right)^2 \right]$$

3-body interaction rate

As usual, an interaction rate has the form

$$R = \frac{dN}{dt} = n\,\Sigma\,v_{\infty}$$

where *n* is the local density of stars.
For the cross section in (*), the rate becomes

$$R = 2\,\pi\,G\,\frac{m_T\,n\,a}{v_{\infty}}$$

Note: in most calculations v_{∞} and σ (velocity dispersion) will be used as synonymous

Note: the rate depends

(1) on the total mass of the interacting objects (more massive objects interact more),
(2) on the semi-major axis of the binary (wider binaries have a larger cross section),
(3) on the local density (denser environments have higher interaction rate),
(4) on the local velocity field (systems with smaller velocity dispersion have higher interaction rate).

Energy exchanges

Most general formalism (from Energy conservation):

$$-\frac{G\,m_1\,m_2}{a_i}+\frac{1}{2}\,\frac{m_3\,(m_1+m_2)}{m_T}\,v_\infty^2=-\frac{G\,m_a\,m_b}{a_f}+\frac{1}{2}\,\frac{m_e\,(m_a+m_b)}{m_T}\,v_{fin}^2 \qquad (+)$$

m_a, m_b and m_e are the final mass of the primary binary member, the final mass of the secondary binary member and the final mass of the single star, respectively (these may be different from the initial ones in the case of an exchange).

Change in binding energy:

$$\Delta E_b=\frac{1}{2}\,G\left(\frac{m_a\,m_b}{a_f}-\frac{m_1\,m_2}{a_i}\right)$$

$$\frac{\Delta E_b}{E_{b,\,in}}=\left(\frac{m_a\,m_b}{m_1\,m_2}\,\frac{a_i}{a_f}-1\right)$$

If NO EXCHANGE ($m_a=m_1$, $m_b=m_2$)

$$\frac{\Delta E_b}{E_{b,\,in}}=\left(\frac{a_i}{a_f}-1\right)$$

If $a_i/\,a_f > 1$ the quantity ΔE_b is transferred to the kinetic energy of the involved centres of mass

SUPERELASTIC ENCOUNTERS: kinetic energy increases after interaction, because the binary is source of additional energy

Energy exchanges

Energy exchanges can be approximately QUANTIFIED if

1) binary is hard

2) p is <~ 2a

3) mass of the single star is small respect to binary mass
(exchanges are unlikely)

If 1), 2) and 3) hold, simulations show that

$$\frac{\Delta E_b}{E_b} \propto \frac{m_3}{m_1 + m_2}$$

Hills (1983, AJ, 88, 1269) defines the post-encounter energy parameter

$$\xi \equiv \frac{m_1 + m_2}{m_3} \frac{\langle \Delta E_b \rangle}{E_b}$$

Where $<\Delta E_b>$ is the average binding energy variation per encounter.

ξ can be extracted from N-body simulations:

*depends slightly on binary eccentricity (ξ~2 if e=0, ξ~6 if e=0.99)

*depends slightly on binary mass ratio (ξ~2 if m_1=m_2, ξ~4 if m_1/m_2=10-30)

* depends strongly on impact parameter (a factor of >200 between b=0 and b=20a)

by averaging over relevant impact parameters **ξ=0.2 – 1**

From E_b definition

$$\langle \Delta E_b \rangle = \xi \, \frac{m_3}{m_1 + m_2} \, \frac{G\, m_1\, m_2}{2\, a}$$

Hardening rate

Rate of binding energy exchange for a hard binary

$$\frac{dE_b}{dt} = \langle \Delta E_b \rangle \frac{dN}{dt} = \xi \frac{m_3}{m_1 + m_2} E_b \frac{dN}{dt}$$

Where *dN/dt* is the collision rate. Using formalism in slide 19

$$\frac{dE_b}{dt} = \xi \frac{\langle m \rangle}{m_1 + m_2} E_b \frac{2\,\pi\,G\,(m_1 + m_2)\,n\,a}{\sigma}$$

Average star mass (because average energy exchange)

Note: **$\langle m \rangle$ n = ρ** (local mass density of stars)

$$\frac{dE_b}{dt} = \pi\,\xi\,G^2\,\frac{\rho}{\sigma}\,m_1\,m_2$$

Note: in most calculations v_∞ and σ (velocity dispersion) will be used as synonymous

**Depends only on cluster environment and binary mass!*

**Constant in time, if the cluster properties do not change and if the binary members do not exchange:*

–> *'A hard binary hardens at a constant rate'* (Heggie 1975, 3-body Bible)

Hardening rate

Expressing a in terms of E_b (assuming m_1 and m_2 constant, i.e. no exchange)

$$\frac{d}{dt}\left(\frac{1}{a}\right) = \frac{2}{G\, m_1\, m_2}\,\frac{dE_b}{dt} = 2\,\pi\, G\,\xi\,\frac{\rho}{\sigma}$$

Also called **HARDENING RATE**

From it we can derive the average time evolution of the semi-major axis of a hard binary:

$$\frac{da}{dt} = -2\,\pi\, G\,\xi\,\frac{\rho}{\sigma}\,a^2$$

It means that the smaller a, the more difficult is for the binary to shrink further (because cross section becomes smaller).

When binary is very hard, three body encounters are no longer efficient: further evolution of the binary is affected by tidal forces and merger (if it is composed of soft bodies) or by gravitational wave emission (if the two binary members are compact objects). When? See further discussion on timescales.

Relevant timescales

1) HARDENING TIMESCALE

$$t_h = \left|\frac{a}{\dot{a}}\right| = \frac{\sigma}{2\,\pi\,G\,\xi}\,\frac{1}{a}$$

2) GRAVITATIONAL WAVE (GW) TIMESCALE

For a binary of compact objects it is important to know whether the main driver of orbital evolution is hardening or GW decay.
From Peters (1964, Gravitational radiation and the motion of two point masses, Phys. Rev. B136, 1224) the timescale of orbital decay by GWs is

$$t_{GW} = \frac{5}{256}\,\frac{c^5\,a^4\,\left(1-e^2\right)^{7/2}}{G^3\,m_1\,m_2\,(m_1+m_2)}$$

Combining 1) and 2) we can find the maximum semi-major axis for GWs to dominate evolution

$$a_{GW} = \left[\frac{256}{5}\,\frac{G^2\,m_1\,m_2\,(m_1+m_2)\,\sigma}{2\,\pi\xi\,\left(1-e^2\right)^{7/2}\,c^5\,\rho}\right]^{1/5}$$

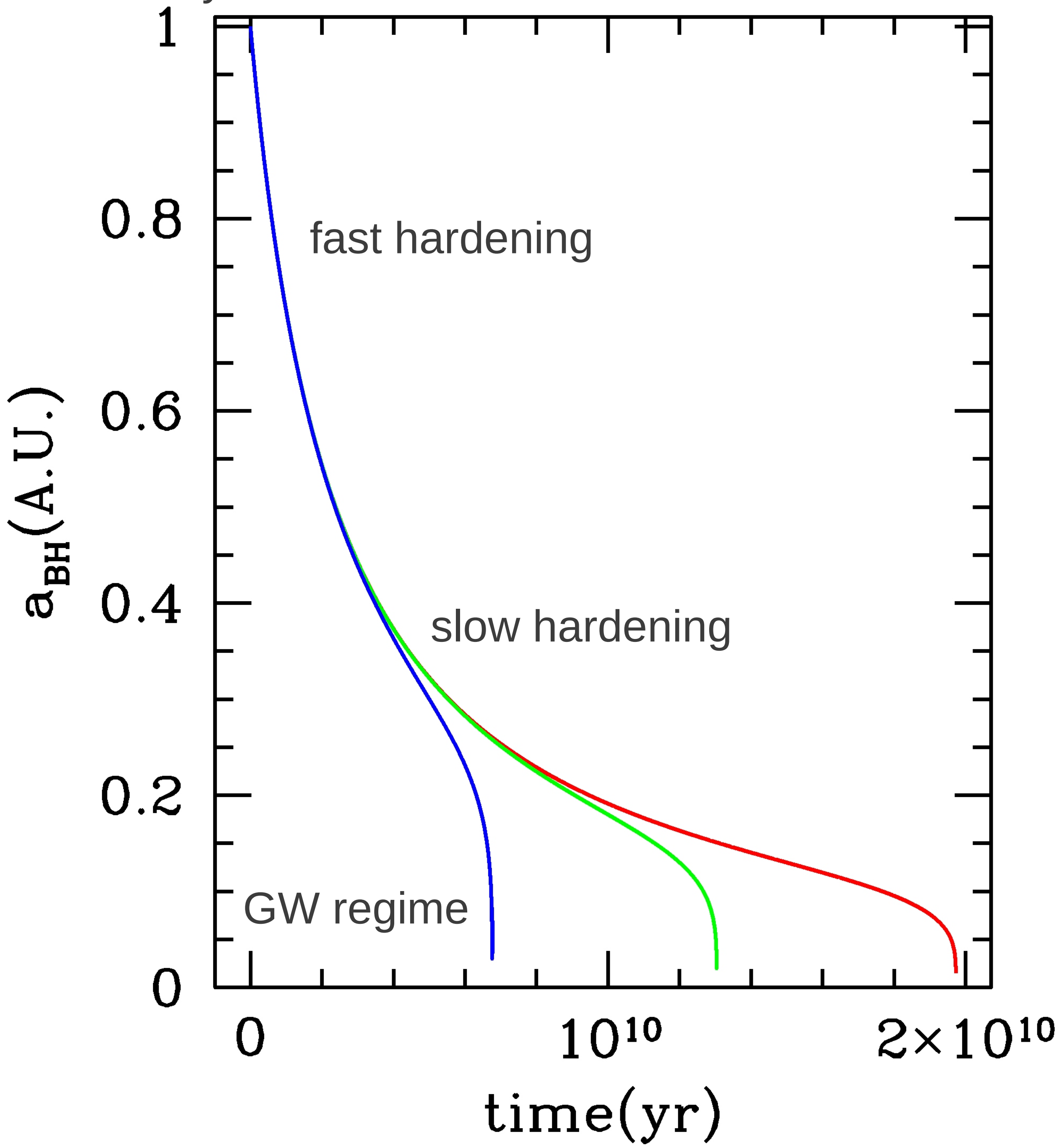
Relevant timescales

Example: black hole – black hole binary

* blue
  m_1=200 $M_\odot$
  m_2=10 $M_\odot$

* green
  m_1=50 $M_\odot$
  m_2=10 $M_\odot$

* red
  m_1=30 $M_\odot$
  m_2=3 $M_\odot$



Number of encounters before GW regime

$$N_{int} = \int_0^t R\, dt = \int_0^t \frac{2\,\pi\, G\, m_T\, n\, a}{\sigma}\, dt$$

$$= \int_{a_0}^{a(t)} \frac{2\,\pi\, G\, m_T\, n\, a}{\sigma}\, \frac{\sigma\, da}{-2\,\pi G\,\xi\rho\, a^2} = \int_{a(t)}^{a_0} \frac{1}{\xi}\, \frac{m_T}{\langle m\rangle}\, \frac{da}{a}$$

↑

$$\frac{da}{dt} = -2\,\pi\, G\,\xi\, \frac{\rho}{\sigma}\, a^2$$

$$= \frac{1}{\xi}\, \frac{m_T}{\langle m\rangle}\, \ln\left(\frac{a_0}{a(t)}\right)$$

Relevant timescales

1) INTERACTION TIMESCALE (from interaction rate)

$$t_{3b} = \frac{\sigma}{2\,\pi\,G\,m_T\,n\,a}$$

2) DYNAMICAL FRICTION TIMESCALE

$$t_{\rm df} = \frac{3}{4\,(2\,\pi)^{1/2}\,G^2\,\ln\Lambda}\,\frac{\sigma^3}{m_T\,\rho(r)}$$

If $t_{df} \ll t_{3b}$, dynamical friction washes velocity changes induced by 3-body encounters (especially for very small *a*) and especially out of core (where σ drops)

If $t_{df} \gg t_{3b}$, 3-body encounters dominate the binary velocity (especially for large *a*)

Recoil velocities

Most general expression of recoil velocity for the reduced particle (Sigurdsson & Phinney 1993)

$$v_{fin} = \sqrt{\frac{m_3\,(m_1+m_2)}{m_e\,(m_a+m_b)}\,v_\infty^2 + \frac{2\,m_T}{m_e\,(m_a+m_b)}\,\Delta E_b}$$

m_a, m_b and m_e are the final mass of the primary binary member, the final mass of the secondary binary member and the final mass of the single star, respectively (these may be different from the initial ones in the case of an exchange).
This equation comes from (+) at slide 20:

$$\frac{1}{2}\,\frac{m_3\,(m_1+m_2)}{m_T}\,v_\infty^2 + \Delta E_b = \frac{1}{2}\,\frac{m_e\,(m_a+m_b)}{m_T}\,v_{fin}^2$$

What happens to the binary, then?

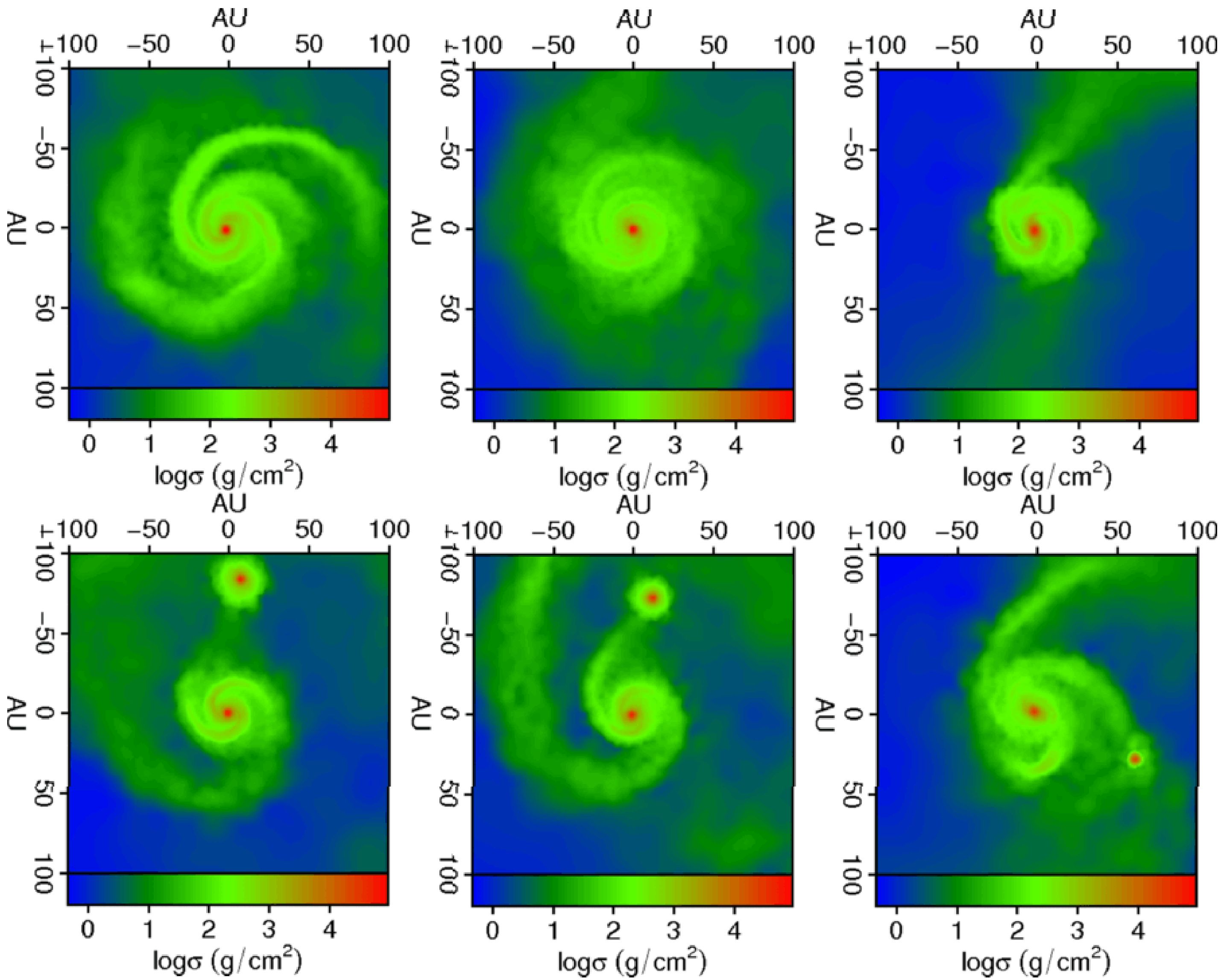
The recoil of the binary (if the binary is more massive than the single star -i.e. the motion of the single star coincides almost with that of the reduced particles) follows from conservation of linear momentum

$$v_{rec} = \frac{m_e}{m_T}\,v_{fin}$$

Origin of binaries

Which are the formation pathways of binaries?

1) primordial binaries: binaries form from the same accreting clump in the parent molecular cloud (very difficult to understand with simulations)



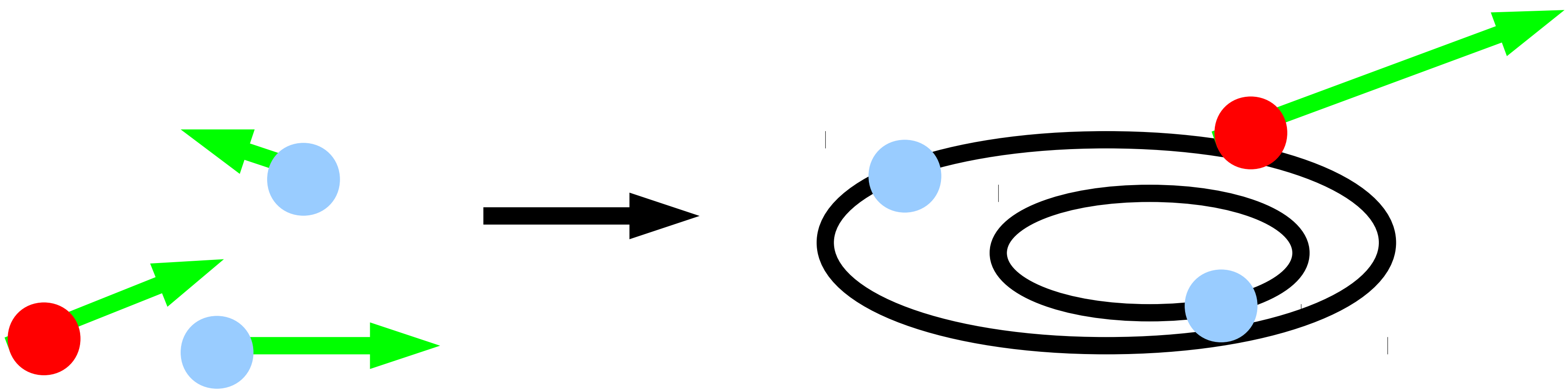
SPH simulation of a single star and a binary formed from a molecular cloud

Hayfield et al. 2011, MNRAS, 2011, 417, 1839

Origin of binaries

Which are the formation pathways of binaries?

2) three-body induced binaries: 3 single stars pass close to each other And one of the three brings away sufficient energy to leave the others bound (only soft-ish binaries with high eccentricity). Unlikely unless high density (core collapse)



$$\frac{dn_b}{dt} = 1.91 \times 10^{-13} \left(\frac{n}{10^4\,\mathrm{pc}^3}\right)^3 \left(\frac{m}{m_\odot}\right)^5 \left(\frac{10\,\mathrm{km\ s}^{1}}{\langle v \rangle}\right)^9 \mathrm{pc}^{-3}\,\mathrm{yr}^{-1}$$

Origin of binaries

Which are the formation pathways of binaries?

3) tidally induced binaries: two stars pass very close to each other so that they feel each other tidal field → energy dissipation produces a bound couple (very hard binaries with ~0 eccentricity for dissipation – merge in most cases). Unlikely unless high density (core collapse)

Tidal radius (from Roche limit):

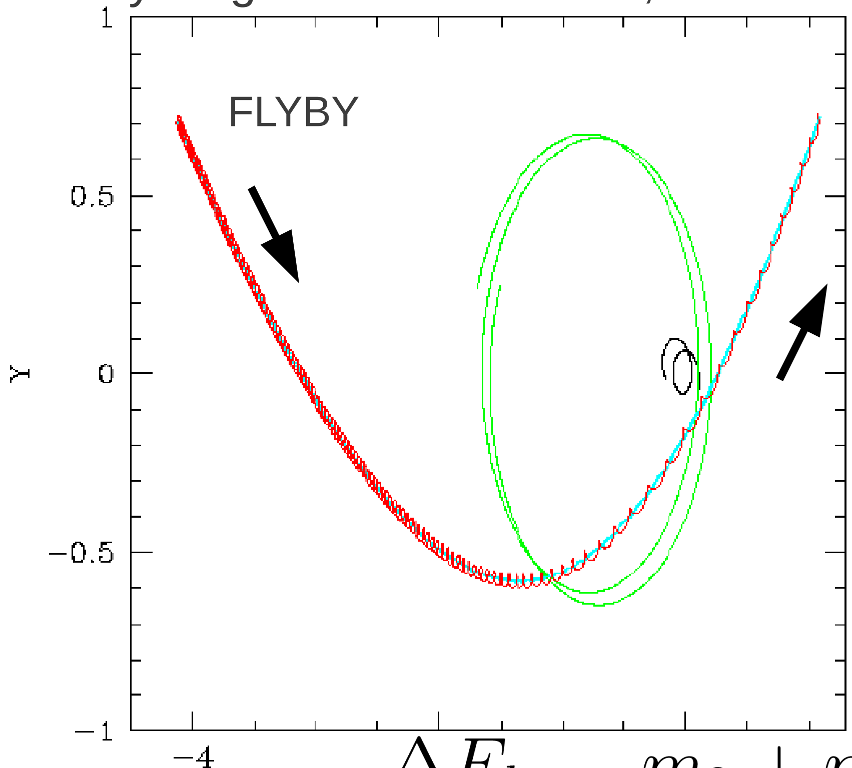
$$r_t = r_2 \left(\frac{2\, m_1}{m_2} \right)^{1/3}$$

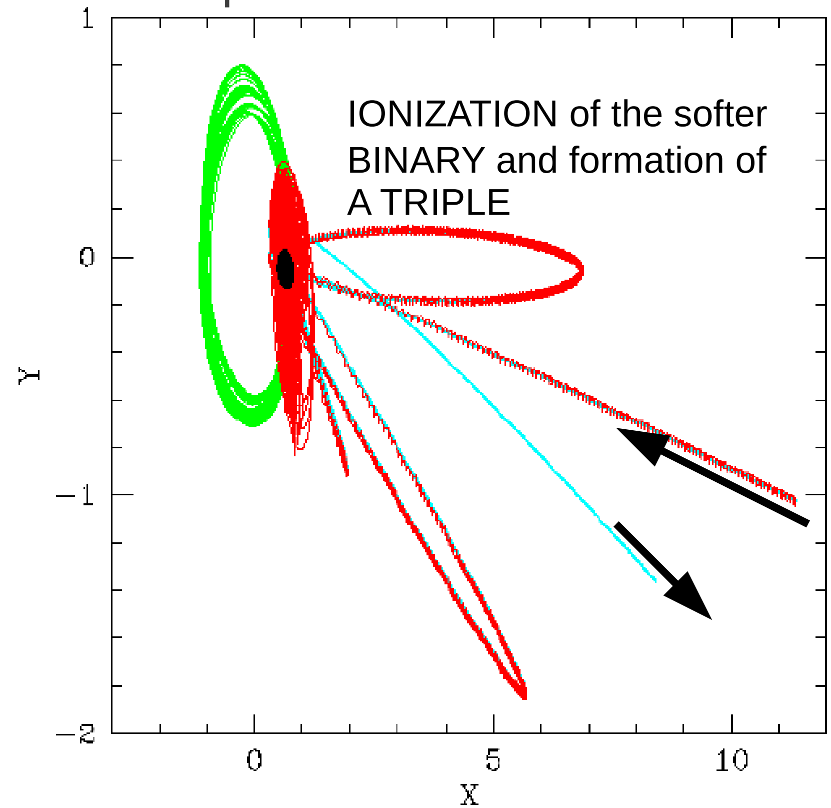
valid if $m_1 > m_2$

4) binaries formed by exchange: in general, they are not considered new binaries as they come from a pre-existing binary after exchange of members

4- (5-, 6-, …) body encounters

Everything we said still holds, but with higher level of complications





Energy exchanges: $$\frac{\Delta E_b}{E_b} \propto \frac{m_3 + m_4}{m_1 + m_2}$$

Critical velocity for ioniz. $$v_c = \sqrt{\frac{G\, m_T}{(m_1 + m_2)\,(m_3 + m_4)} \left(\frac{m_1\, m_2}{a_1} + \frac{m_3\, m_4}{a_2}\right)}$$

Even RARE STABLE TRIPLEs can form

(Mardling & Aarseth 1999 stability criterion)

r_p, e_{ou}= pericentre and eccentricity of the outer binary

a_{in}= semi-major axis of the inner binary

q=m_{ou}/m_{in} mass ratio (outer to inner binary)

$$\frac{r_p}{a_{in}} \geq 2.8 \left[(1+q)\,\frac{1+e_{ou}}{\sqrt{1-e_{ou}^2}}\right]^{2/5}$$

Role of binaries in cluster evolution

1*Density increase during core collapse HARDENS primordial binaries and ENHANCES FORMATION of binaries by encounters of three single stars and by tidal captures

2*Tidally formed binaries are too small: only a few interactions or merger → negligible for energy transfer

3*HARD primordial binaries and new 3-body formed binaries REVERSE core collapse by transferring their internal energy to *K* of stars in core VIA 3-BODY ENCOUNTERS

4*stars that undergo 3-body encounters with hard binaries are ejected (|*W*| decreases) or remain in the core and transfer *K* to other stars

5**Note: HARD binaries transfer K to the core but SOFT binaries extract K from the core. Why are hard binaries predominant?*
TOTAL $|E_i|$ *of HARD BINARIES >> TOTAL* $|E_i|$ *of SOFT BINARIES!!!!*

6*Mass losses by stellar winds and supernovae can help, but only if the timescale for massive star evolution is ~ core collapse time!!! (see last lecture)

7*If there are >1 very hard binaries might eject each other by 4-body encounters

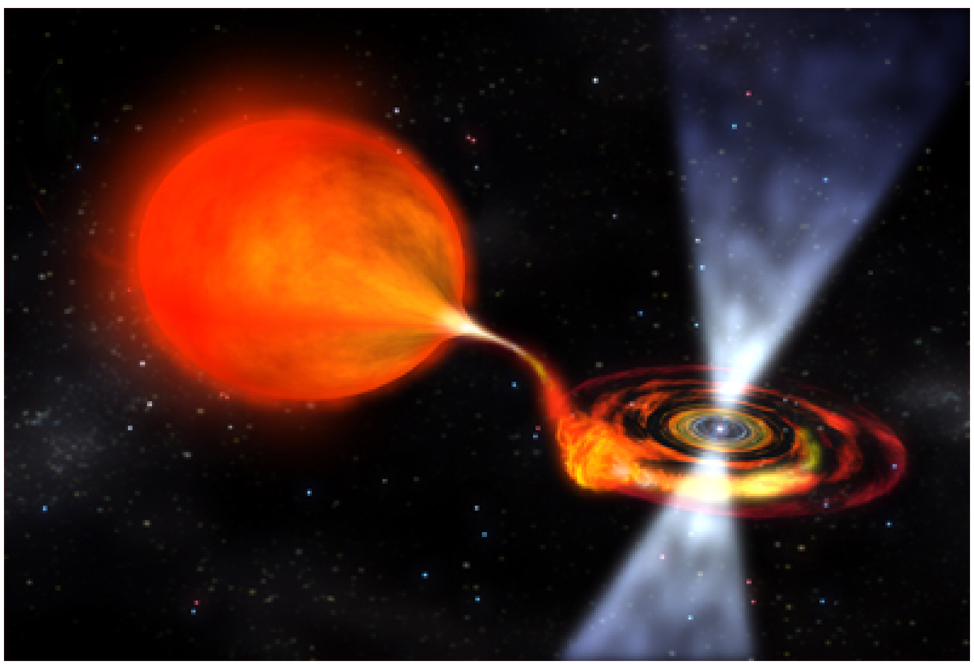
Role of binaries in formation of exotica

Binaries and three-body encounters are the main suspects for the formation of STELLAR EXOTICA, such as

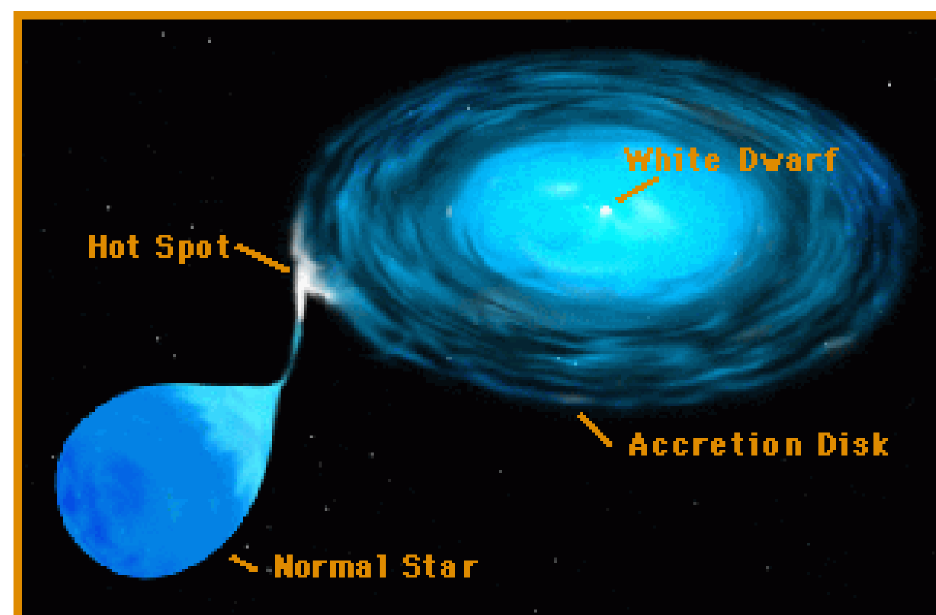
* blue straggler stars

* massive BHs and intermediate-mass BHs

* millisecond pulsars



* cataclismic variables



References:

*** Mapelli M., Master thesis, 2003, in Italian, http://web.pd.astro.it/mapelli/images/tesi.ps.gz**

* Binney & Tremaine, Galactic Dynamics, First edition, 1987, Princeton University Press

* Hut & Bahcall, Binary-single star scattering. I - Numerical experiments for equal masses, 1983, ApJ, 268, 319

* Sigurdsson S., Phinney S., Binary-single star interactions in globular clusters, 1993, ApJ, 415, 631

* Davies M. B., Stellar exotica produced from stellar encounters, 2002, ASP

* Spitzer L., Dynamical evolution of globular clusters, 1987, Princeton University Press

* Elson R. & Hut P., Dynamical evolution of globular clusters, Ann. Rev. Astron. Astrophys., 1987, 25, 565

Michela Mapelli

LECTURES on COLLISIONAL DYNAMICS:

4. HOT TOPICS on COLLISIONAL DYNAMICS

Part 1

1) IMBHs: runaway collapse, repeated mergers, ...

2) BHs eject each other?

3) Effects of 3-body on X-ray binaries (formation and escape)

3b) Gravitational waves

4) Effect of metallicity on cluster evolution

5) Formation of blue straggler stars

6) Tools for numerical simulations of collisional systems

7) Three-body and planets

8) Nuclear star clusters

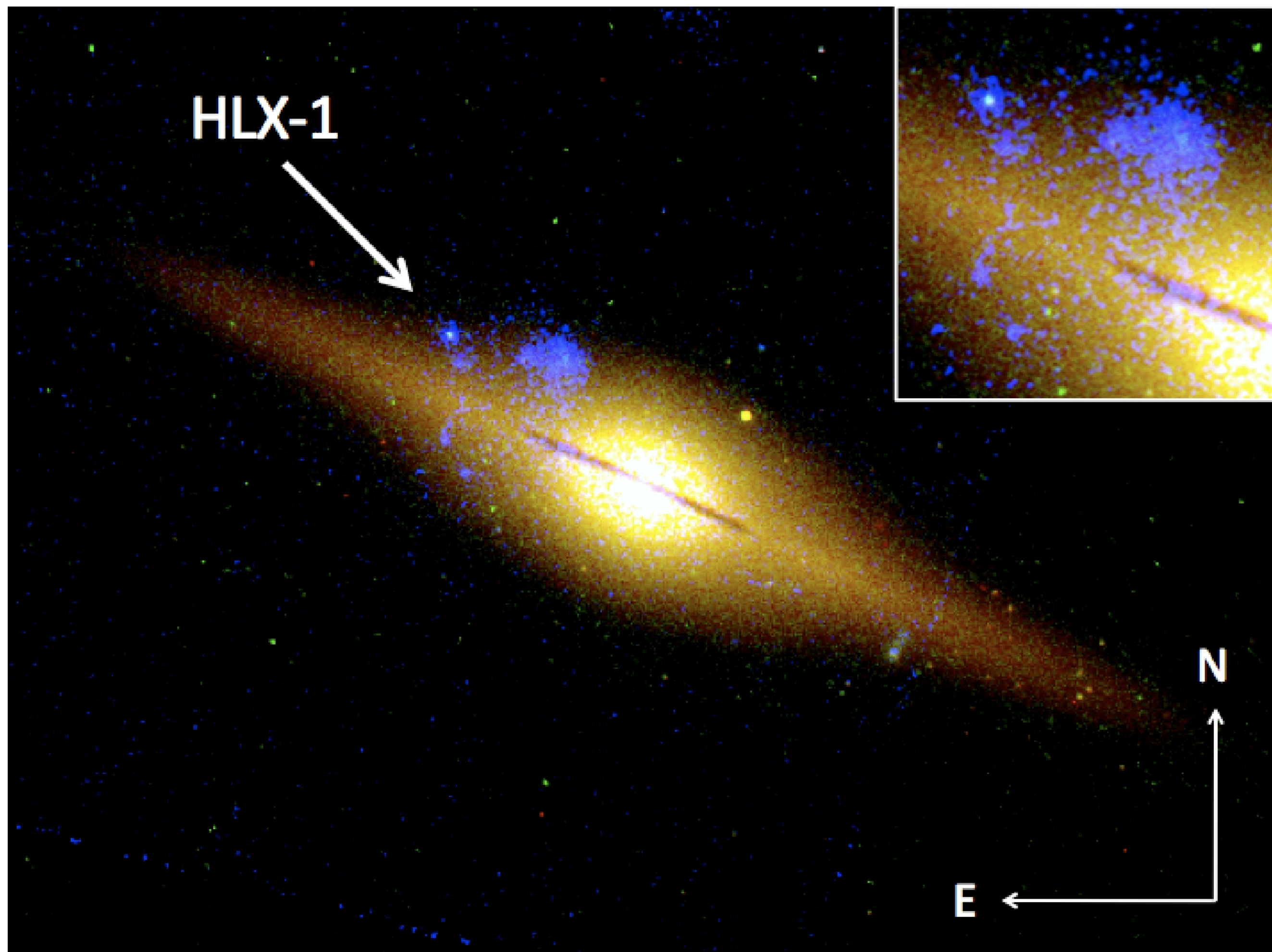
1) Intermediate-mass BHs (IMBHs)

definition: BHs with mass 10^{2-5} M☉

OBSERVATIONAL EVIDENCES: none, just hints

1* Hyperluminous X-ray source HLX-1 close to ESO 243-49

peak L_X~10^{42} ergs,
X-ray VARIABILITY,
redshift consistent
with ESO 243-49
(not a background object)
→ BH mass~10^4 M☉



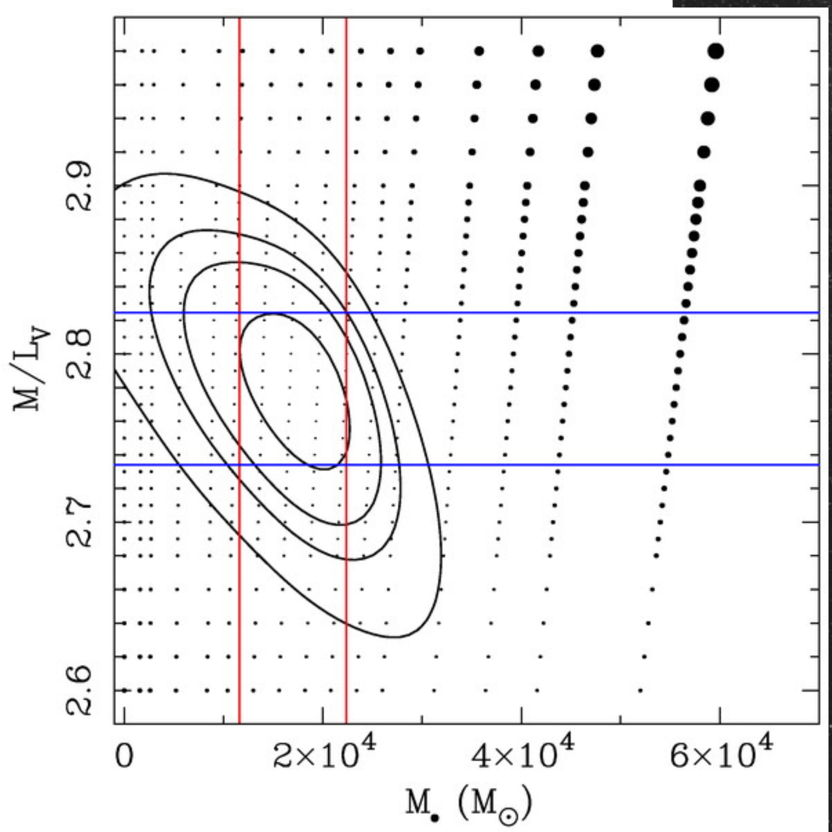
Farrell+ 2009, 2012;
Soria+ 2010, 2012;
Mapelli+ 2012, 2013

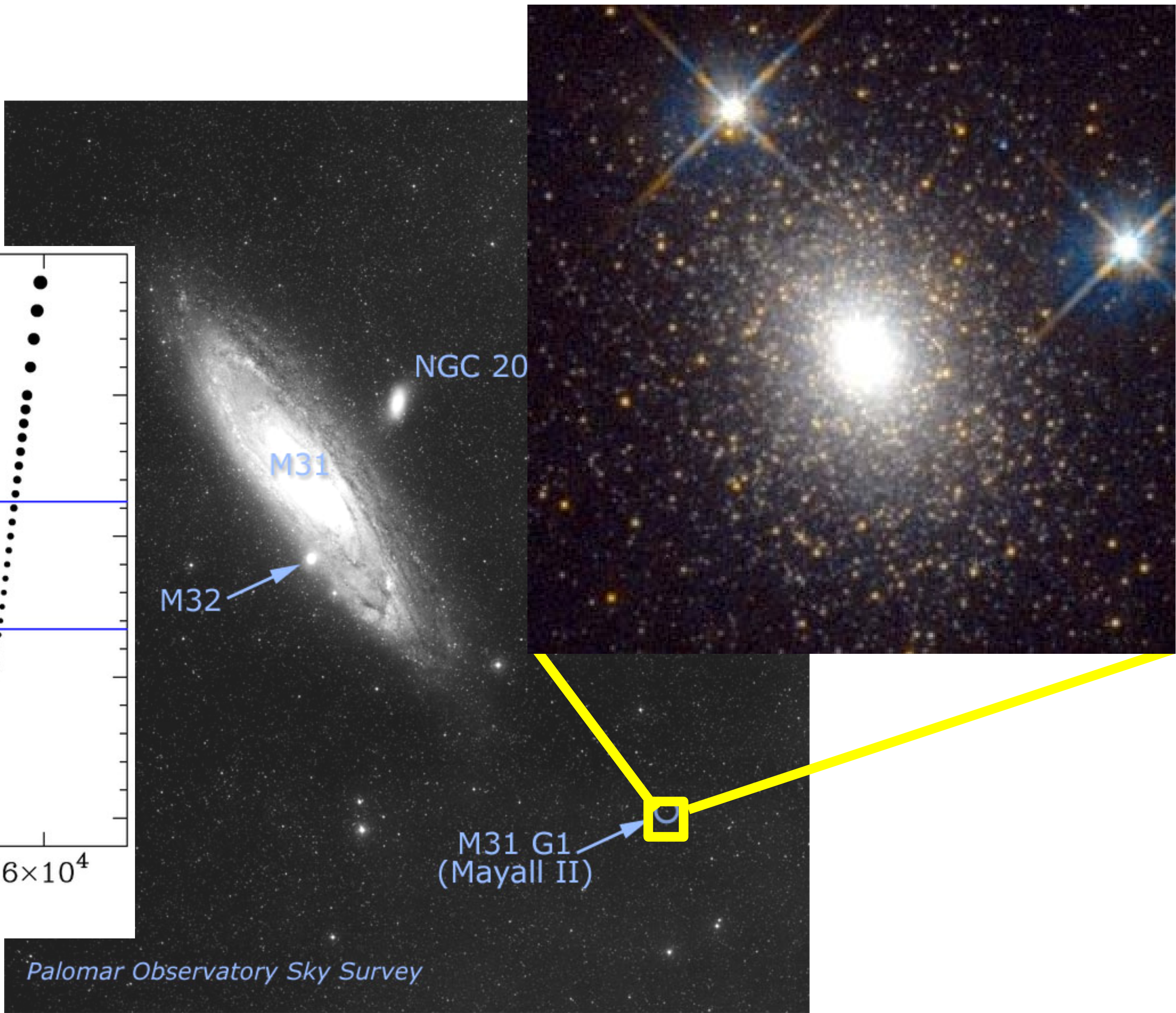
1) Intermediate-mass BHs (IMBHs)

definition: BHs with mass $10^{2\text{-}5}$ M⊙

2* centre of G1 globular cluster (dwarf nucleus?) in Andromeda

Central velocity distribution
+central M/L ratio
suggest BH mass~10^4 M⊙





Gebhardt+ 2005

1) Intermediate-mass BHs (IMBHs)

How do IMBHs form?

1- runaway collapse of stars at centre of star cluster (for systems with core collapse time < evolution of massive stars – e.g. young dense star clusters)

2- repeated mergers of BHs at centre of star cluster (for systems with core collapse time >> evolution of massive stars – e.g. globular clusters)

3- remnants of extremely metal poor stars (independent of environment)

4- low mass end of super-massive BHs (needs gas physics → different PhD course)

1) Intermediate-mass BHs (IMBHs)

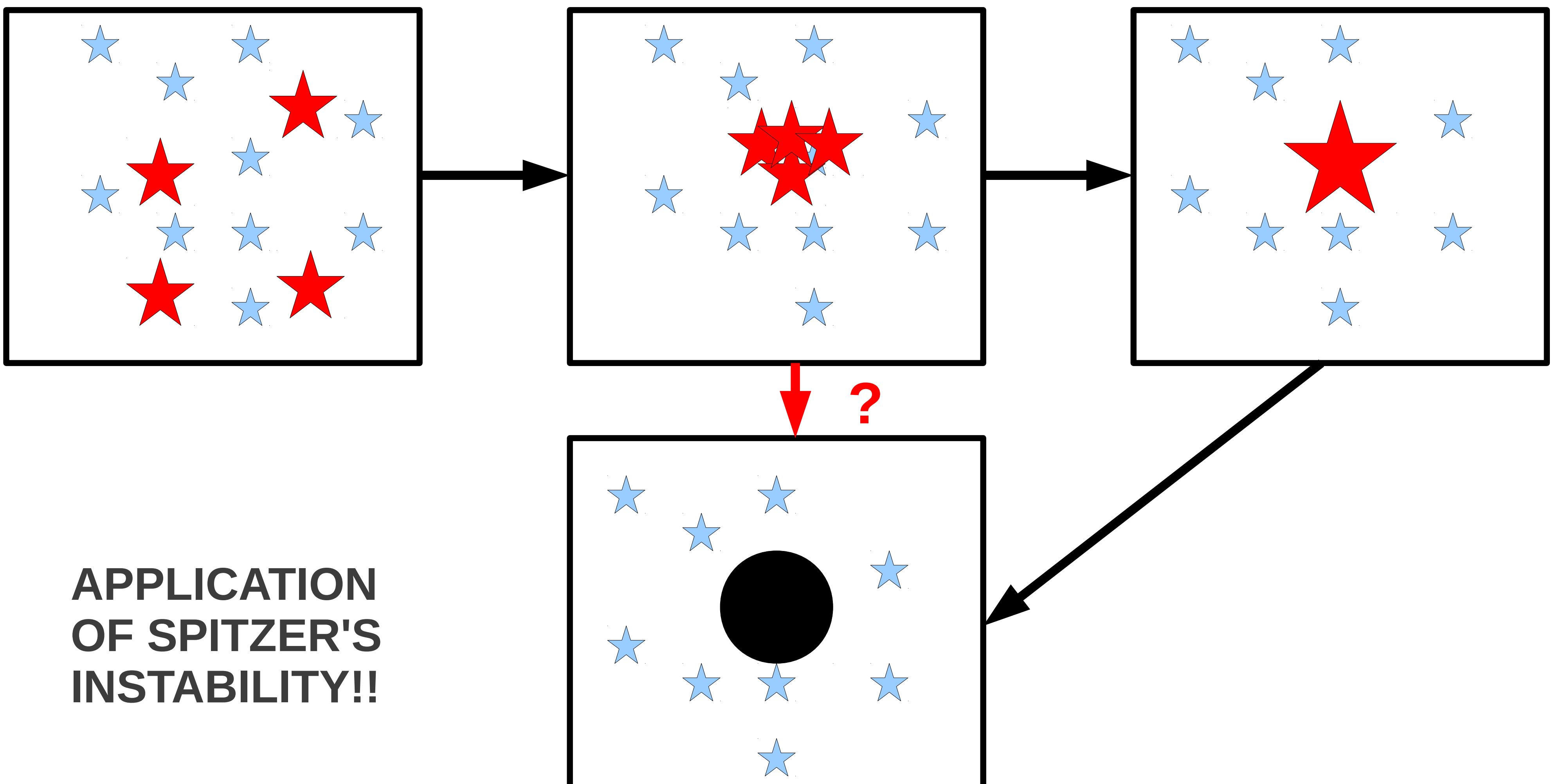
1- runaway collapse of stars at centre of star cluster

IDEA: mass segregation brings very massive stars to the centre

If timescale for mass segregation < timescale for stellar evolution

+ if encounter rate sufficiently high

Massive stars collide, merge and form a super-massive star, which collapses to a BH



APPLICATION OF SPITZER'S INSTABILITY!!

1) Intermediate-mass BHs (IMBHs)

1- runaway collapse of stars at centre of star cluster

Formalism by Portegies Zwart & McMillan 2002

IDEA: very hard binaries sink to the centre and likely collide with other stars/binaries unless they are ejected.

The product of the first collisions is SO MASSIVE that it triggers other collisions (=is the main collision target) Starting a RUNAWAY PROCESS

→ Maximum mass that can be grown in a dense star cluster If all collisions involve the same star

$$\frac{dM_{raway}}{dt} = R_{coll}\,\delta m_{coll}$$

Where R_{coll} = collision rate, δm_{coll} = mass transferred per collision on average

1) Intermediate-mass BHs (IMBHs)

1- runaway collapse of stars at centre of star cluster
Formalism by Portegies Zwart & McMillan 2002

IDEA: very hard binaries sink to the centre and likely collide with other stars/binaries unless they are ejected

ESTIMATE of R_{coll}

Maximum recoil velocity for a binary not to be ejected

$$v_{rec} = v_{esc} = 2\,\sigma$$

Definition of v_{rec}

$$v_{rec} = \frac{m_3}{m_T}\,v_{fin} = \frac{m_3}{m_T}\sqrt{\frac{m_3\,(m_1+m_2)}{m_e\,(m_a+m_b)}\,{v_\infty}^2 + \frac{2\,m_T}{m_e\,(m_a+m_b)}\,\Delta E_b}$$

$$= \frac{m}{3\,m}\sqrt{\frac{m\,(2\,m)}{m\,(2\,m)}\,\sigma^2 + \frac{2\,(3\,m)}{m\,(2\,m)}\,\Delta E_b} = \frac{1}{3}\sqrt{\sigma^2 + \frac{3}{m}\,\frac{m}{2\,m}\,E_b}$$

Nearly equal mass cluster

$$\Delta E_b = \frac{m}{2\,m}\,E_b$$

1) Intermediate-mass BHs (IMBHs)

1- runaway collapse of stars at centre of star cluster

Formalism by Portegies Zwart & McMillan 2002

$$v_{rec} = v_{esc} = 2\,\sigma \qquad (1)$$

$$v_{rec} = \frac{1}{3}\sqrt{\sigma^2 + \frac{3}{2\,m}\,E_b} \qquad (2)$$

Combining (1) and (2) $$36\,\sigma^2 = \sigma^2 + \frac{3}{2\,m}\,E_b$$

$$E_b = \frac{70\,m}{3}\,\sigma^2 = 70\,k_B\,T \sim 10^2\,k_B\,T$$

$$\frac{1}{2}m\,\sigma^2 = \frac{3}{2}\,k_B\,T$$

E_b is the binding energy exchanged by a hard binary during its life (i.e. before it is ejected).

1) Intermediate-mass BHs (IMBHs)

1- runaway collapse of stars at centre of star cluster

Formalism by Portegies Zwart & McMillan 2002

We calculate now the number of binaries necessary to reverse core collapse (estimated as 10% of the total potential energy of the cluster, Goodman 1987):

$$N_{bin}\, 10^2\, k_B\, T \sim 0.1\, |W| = 0.1\, (2\, \frac{3}{2}\, N\, k_B\, T)$$

$$\longrightarrow N_{bin} \sim 10^{-3}\, N$$

Hard binary formation rate:

$$R_{bin} \sim 10^{-3}\, \frac{N}{t_{rlx}}$$

Assuming that ~ each hard binary undergoes <=1 collision, we estimate the collision rate

$$R_{coll} \sim 10^{-3}\, f_{coll}\, \frac{N}{t_{rlx}}$$

1) Intermediate-mass BHs (IMBHs)

1- runaway collapse of stars at centre of star cluster
Formalism by Portegies Zwart & McMillan 2002

IDEA: very hard binaries sink to the centre and likely collide with other stars/binaries unless they are ejected

ESTIMATE of δm_{coll}

From dynamical friction timescale

$$t_{df} \sim \frac{\langle m \rangle}{100\, M_\odot} \frac{0.138\, N}{\ln\left(0.11\, M/100\, M_\odot\right) \left(\frac{R^3}{G\, M}\right)^{1/2}}$$

where <*m*>= average star mass, *M*=total cluster mass, *N*= number of stars

We estimate the minimum mass of star that can sink to the centre in a time *t*

$$m_{df} = 2\, M_\odot \left(\frac{1\,\mathrm{Myr}}{t}\right) \left(\frac{R}{1\,\mathrm{pc}}\right)^{3/2} \left(\frac{m}{1\, M_\odot}\right)^{1/2}$$

mass that can be acquired after a collision (!!!)

$$\delta m_{coll} \sim m_{df} \sim \frac{t_{rlx}}{t} \langle m \rangle \ln \Lambda$$

1) Intermediate-mass BHs (IMBHs)

1- runaway collapse of stars at centre of star cluster

Formalism by Portegies Zwart & McMillan 2002

$$\frac{dM_{raway}}{dt} = 10^{-3}\, f_{coll}\, \frac{N}{t}\, \langle m \rangle\, \ln \Lambda$$

$M_{raway} \sim 10^{2\text{-}3}\ M_\odot$ for a dense young cluster with $t_{coll} < 10$ Myr

1st CONDITIO SINE QUA NON:

core collapse time << massive star evolution time

→ $t_{coll} < 3\text{-}25$ Myr

2nd CONDITIO SINE QUA NON:

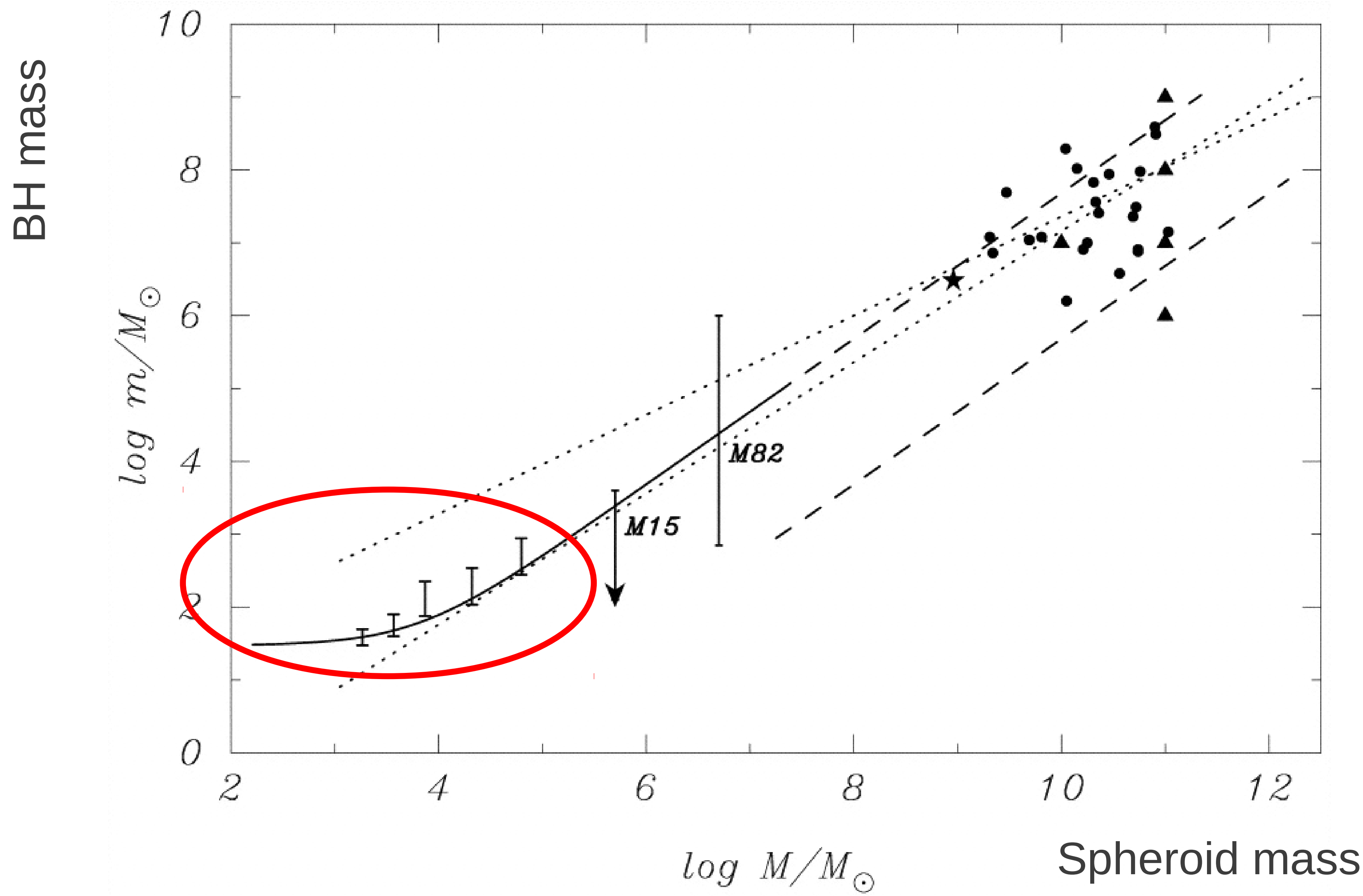
STAR CLUSTER SUFFICIENTLY MASSIVE AND CONCENTRATED

1) Intermediate-mass BHs (IMBHs)

1- runaway collapse of stars at centre of star cluster

Formalism by Portegies Zwart & McMillan 2002

Confirmed by simulations



1) Intermediate-mass BHs (IMBHs)

1- runaway collapse of stars at centre of star cluster

Formalism by Portegies Zwart & McMillan 2002

MAIN ISSUE: MASS LOSSES!!!

(1) during merger

Recent simulations show mass losses

up to 25% of total mass

(Gaburov, Lombardi & Portegies Zwart 2010, MNRAS, 402, 105)

(2) by stellar winds

After merger the super-massive star will

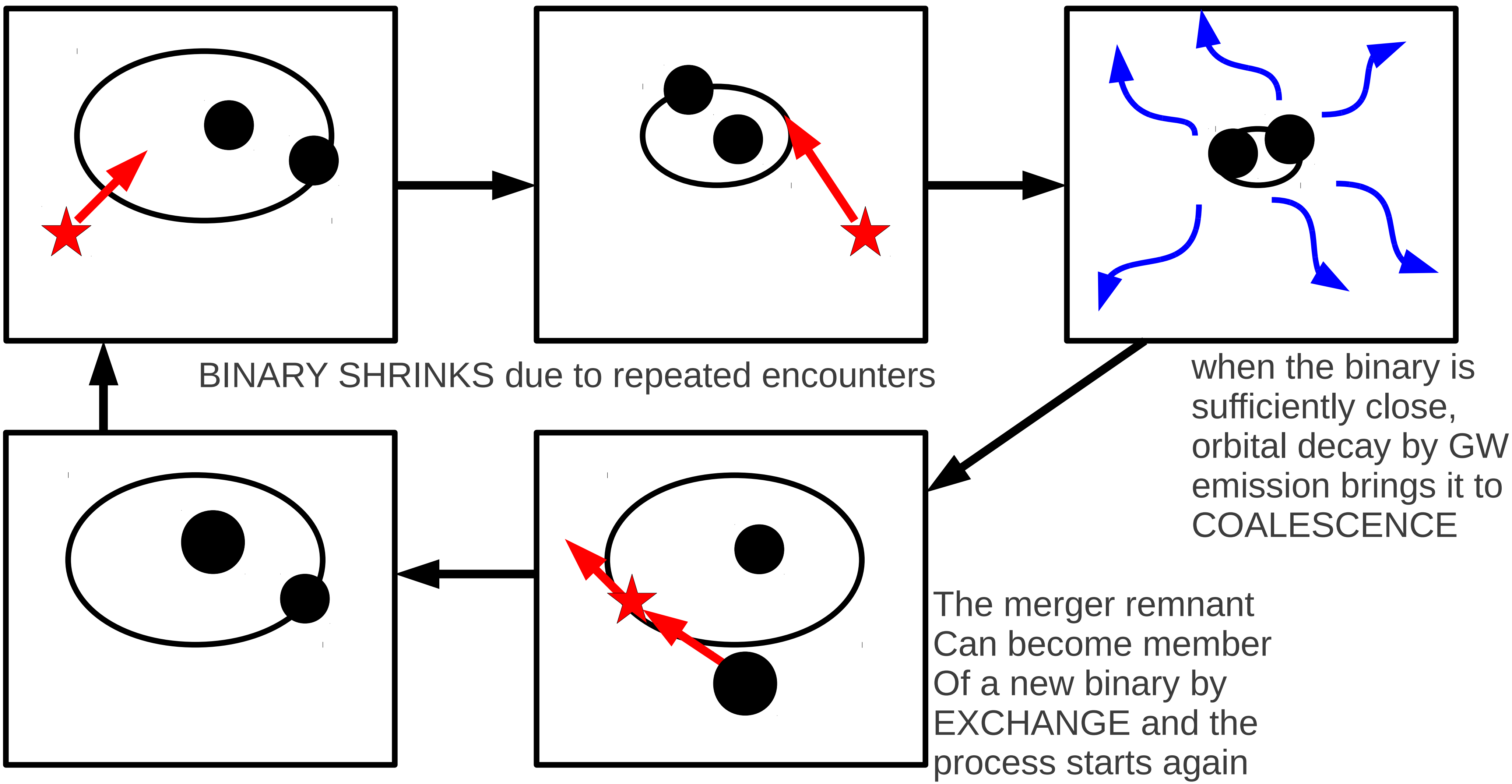
be very unstable (radiation pressure dominated)

1) Intermediate-mass BHs (IMBHs)

2- repeated mergers

Formalism by Miller & Hamilton (2002)

In a old cluster stellar BHs can grow in mass because of repeated mergers with the companion triggered by 3-body encounters



1) Intermediate-mass BHs (IMBHs)

2- repeated mergers

Formalism by Miller & Hamilton (2002)

MAIN PROBLEM: seed BH must avoid ejection before merger

$$v_{rec} = \frac{m_3}{m_T}\sqrt{\frac{m_3\,(m_1+m_2)}{m_e\,(m_a+m_b)}\,v_\infty^2 + \frac{2\,m_T}{m_e\,(m_a+m_b)}\,\Delta E_b} \sim \frac{m_3}{m_T}\sqrt{\frac{2\,m_T}{m_3\,(m_1+m_2)}\,\Delta E_b}$$

$$m_1 + m_2 \gg m_3$$

$$\sim \frac{m_3}{m_T}\sqrt{\frac{2\,m_T}{m_3\,(m_1+m_2)}\,\frac{\xi\,m_3}{(m_1+m_2)}\,E_b} \sim \frac{m_3}{m_1+m_2}\sqrt{\frac{2\,\xi}{m_T}\,E_b}$$

We find the minimum binding energy for EJECTION ($E_{b,min}$) by imposing $v_{rec}=v_{esc}$:

$$\Rightarrow E_{b,\,min} \sim \frac{(m_1+m_2)^3}{2\,\xi\,m_3^2}\,v_{esc}^2$$

where we assumed $m_1+m_2\sim m_T$

$$E_{b,\,min} \sim 2\times10^{50}\ \mathrm{erg}\left(\frac{m_1}{50\,M_\odot}\right)^3\left(\frac{m_3}{10\,M_\odot}\right)^{-2}\left(\frac{\xi}{0.2}\right)^{-1}\left(\frac{v_{esc}}{50\,\mathrm{km\ s^{-1}}}\right)^2$$

1) Intermediate-mass BHs (IMBHs)

2- repeated mergers

Formalism by Miller & Hamilton (2002)

MAIN PROBLEM: seed BH must avoid ejection before merger

Orbital separation in merger regime (see lecture 3):

$$a_{GW} \sim 3 \times 10^{11} \text{ cm} \left(\frac{t_{GW}}{10^6 \text{ Myr}}\right)^{1/4} \left(\frac{m_1}{50\, M_\odot}\right)^{1/2} \left(\frac{m_2}{10\, M_\odot}\right)^{1/4}$$

Binding energy in merger regime:

$$E_{b,\,merg} = \frac{G\, m_1\, m_2}{2\, a_{GW}} \sim 2 \times 10^{50} \text{ erg} \left(\frac{t_{GW}}{10^6 \text{ Myr}}\right)^{-1/4} \left(\frac{m_1}{50\, M_\odot}\right)^{1/2} \left(\frac{m_2}{10\, M_\odot}\right)^{3/4}$$

COMPARING $E_{b,min}$ with $E_{b,merg}$:

$$x = \frac{E_{b,\,min}}{E_{b\,merg}} \sim \left(\frac{m_1}{50\, M_\odot}\right)^{5/2} \left(\frac{m_2}{10\, M_\odot}\right)^{-11/4} \left(\frac{t_{GW}}{10^6 \text{ Myr}}\right)^{1/4}$$

If $x>1$ BINARY MERGES BEFORE EJECTION

If $x<1$ BINARY IS EJECTED BEFORE MERGER

1) Intermediate-mass BHs (IMBHs)

2- repeated mergers

Formalism by Miller & Hamilton (2002)

In a old cluster stellar BHs can grow in mass because of repeated mergers with the companion triggered by 3-body encounters

Number of 3-body encounters for a BH to merge with its companion (from lecture 3):

$$N_{merg} = \frac{1}{\xi} \frac{m_T}{\langle m \rangle} \ln\left(\frac{a_0}{a_{GW}}\right)$$

Time required for 1 merger:

$$dt = -\frac{\sigma}{2\,\pi\,G\,\xi\,\rho}\,\frac{da}{a^2} \longrightarrow \int_0^{t_{GW}} dt = -\frac{\sigma}{2\,\pi\,G\,\xi\,\rho}\int_{a_0}^{a_{GW}}\frac{da}{a^2}$$

$$t_{GW} = \frac{\sigma}{2\,\pi\,G\,\xi\,\rho}\left(\frac{1}{a_{GW}} - \frac{1}{a_0}\right)$$

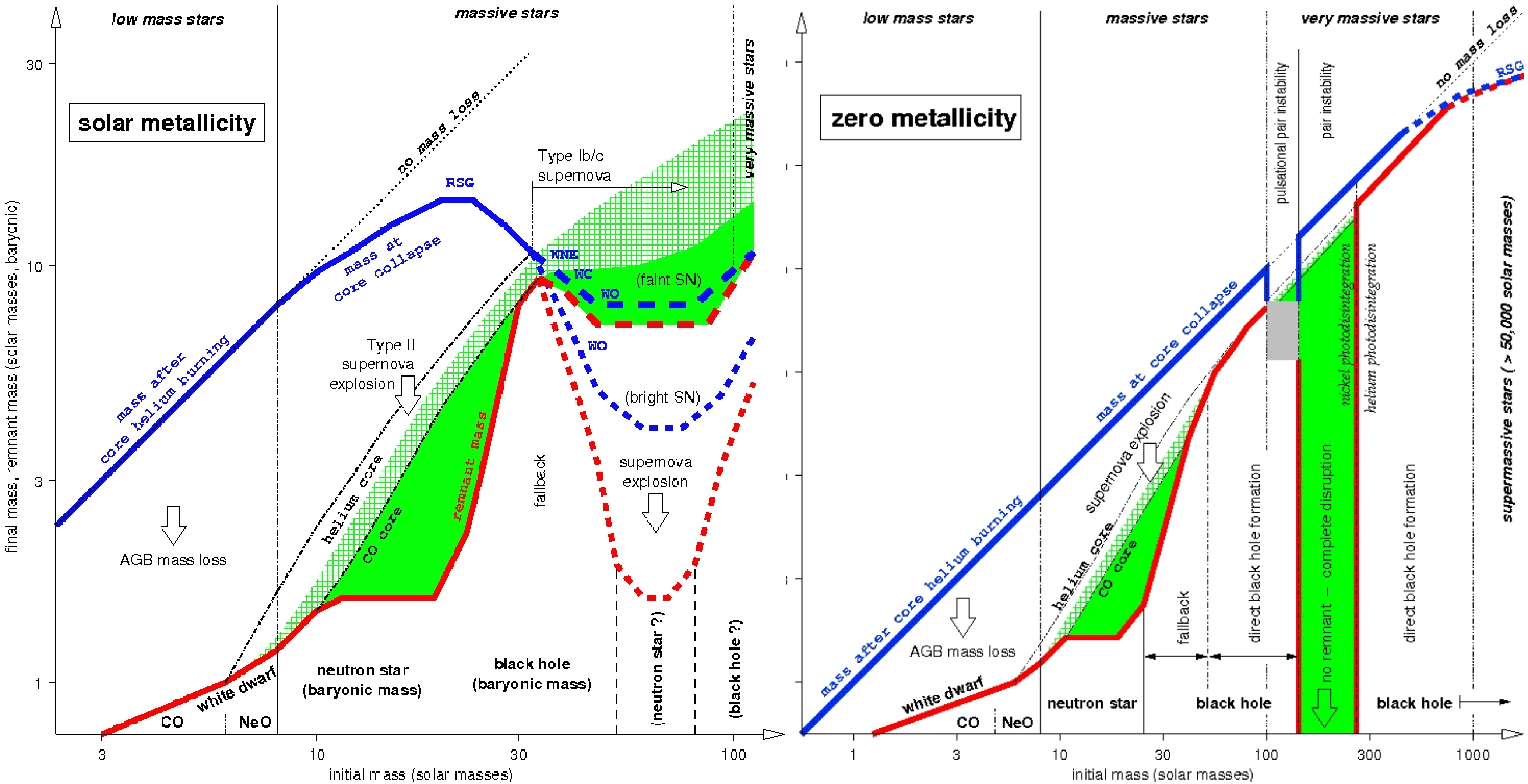
$$t_{GW} \sim 3\times 10^8\ \mathrm{yr}\left(\frac{\sigma}{10\ \mathrm{km\ s^{-1}}}\right)\left(\frac{\xi}{0.2}\right)^{-1}\left(\frac{\rho}{10^6\ M_\odot\ \mathrm{pc^{-3}}}\right)^{-1}\left(\frac{a_{GW}}{1\ \mathrm{AU}}\right)^{-1}$$

INEFFICIENT!!!!!

1) Intermediate-mass BHs (IMBHs)

3- remnants of extremely metal-poor stars

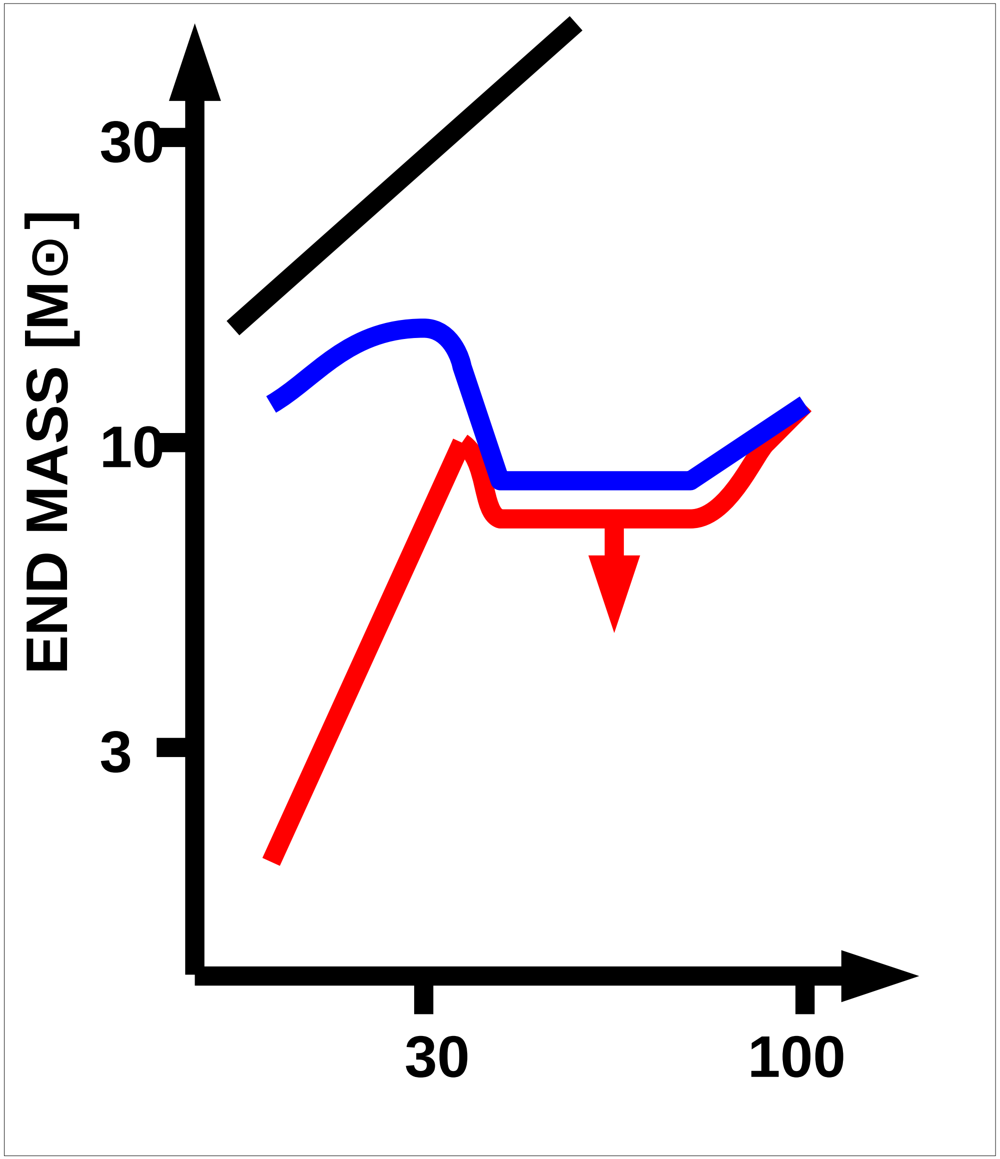
Formalism by Heger et al. (2003)

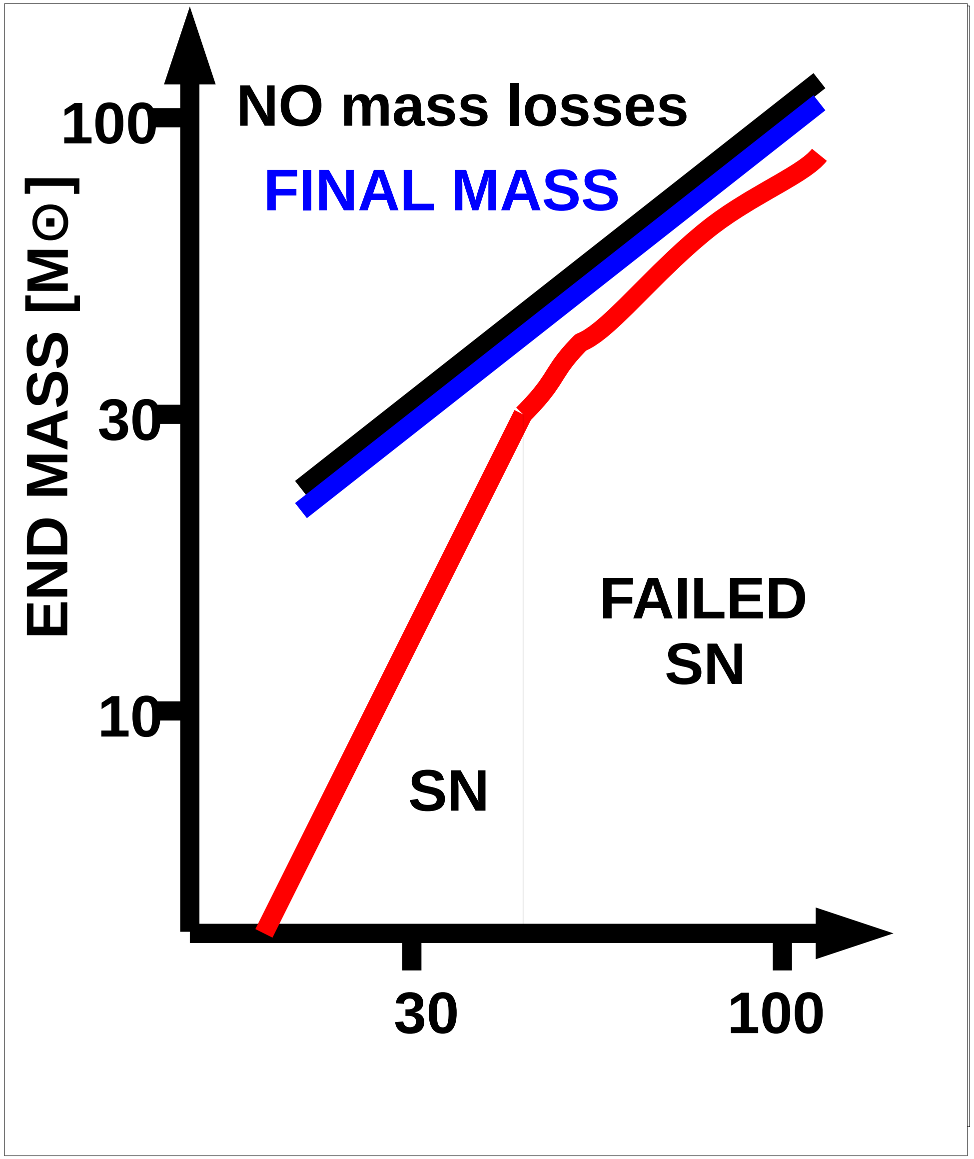


1) Intermediate-mass BHs (IMBHs)

3- remnants of extremely metal-poor stars

Formalism by Heger et al. (2003)





1) Intermediate-mass BHs (IMBHs)

3- remnants of extremely metal-poor stars

Formalism by Heger et al. (2002)

TWO INGREDIENTS:

1) STELLAR WINDS depend on METALLICITY

$$\dot{M}(Z) \propto \left(\frac{Z}{Z_\odot}\right)^\alpha$$

$$\alpha = 0.5 - 0.9$$

Vink+ (2001)

at low Z, stars lose less mass by stellar winds!

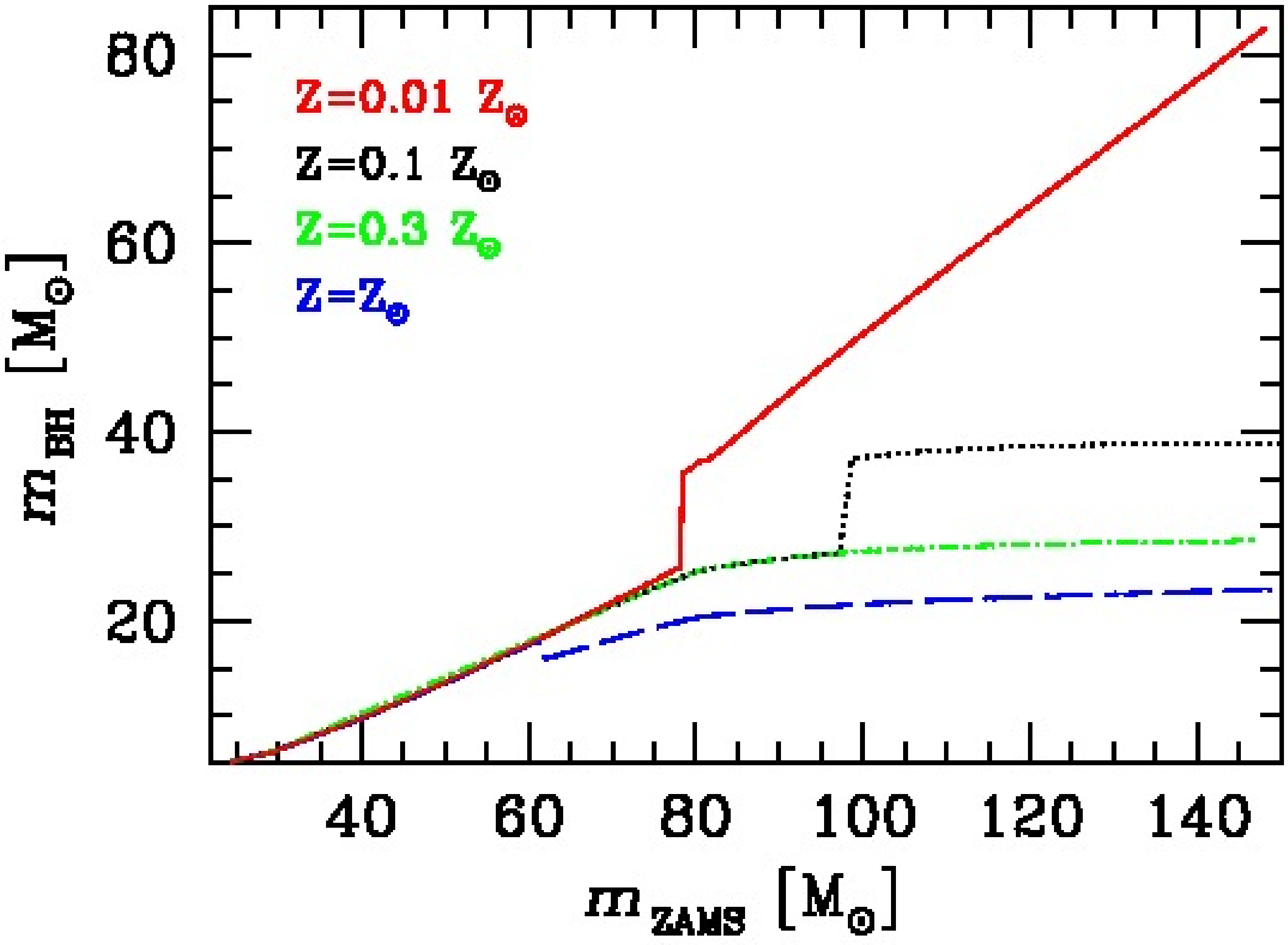
2) IF FINAL MASS SUFFICIENTLY HIGH (> 40 Msun),
SN EXPLOSION CANNOT SUCCEED:
almost NO EJECTA and direct collapse to BHs
(FAILED SUPERNOVAE, Fryer 1999)

1) Intermediate-mass BHs (IMBHs)

3- remnants of extremely metal-poor stars

Formalism by Heger et al. (2002)

NOT ONLY AT ZERO METALLICITY



MM+09; Zampieri & Roberts 2009; Belczynski+2010; Fryer+2012; MM+2013

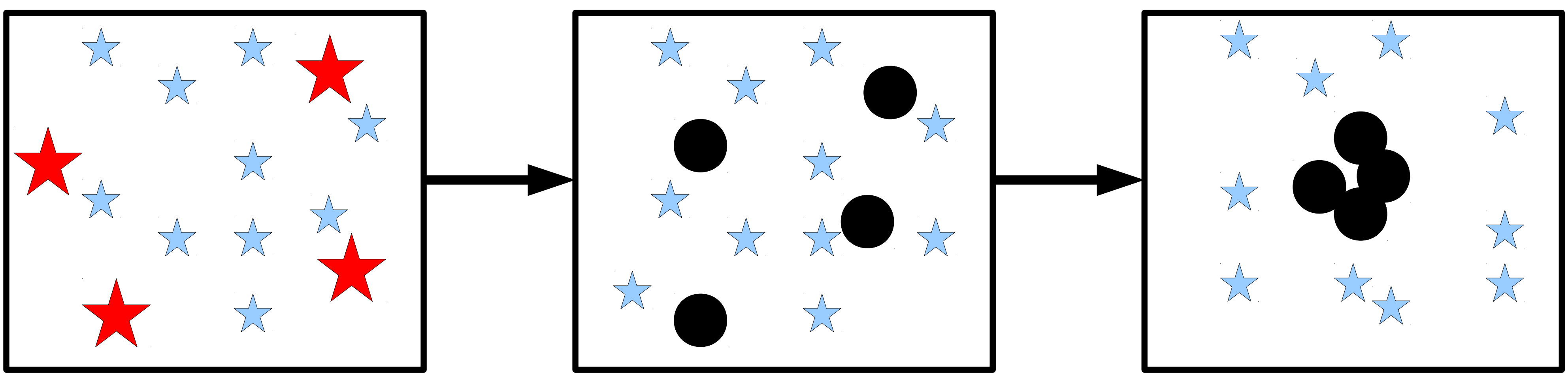
2) BHs eject each other?

Note: valid for globular clusters! Why? BHs form before progenitors segregate to the centre (no runaway collapse)

N_{bh}~10^2 expected to form in GCs

Segregate to the centre in

$$t_{df} \sim 3 \times 10^7 \text{ yr} \left(\frac{\sigma}{10\,\text{km s}^{-1}}\right) \left(\frac{r_c}{1\,\text{pc}}\right)^2 \left(\frac{m_{BH}}{10\,M_\odot}\right)^{-1}$$



2) BHs eject each other?

Note: valid for globular clusters! Why? BHs form before progenitors segregate to the centre (no runaway collapse)

What happens when BHs are in the core?

1) Total mass of BHs sufficiently large to have SPITZER'S INSTABILITY →
   * formation of a **dynamically decoupled core of BHs**!!!
   * **core collapse** for BHs on faster timescale than expected for stars
   * **efficient formation of BH-BH** binaries
   * **fast ejection of all lighter stars** from the BH dominated core
   * ejections of (nearly) all single BHs in halo or out of cluster
   * **ejections of binary BHs** with x>1

$$x = \frac{E_{b,\,min}}{E_{b\,merg}} \sim \left(\frac{m_1}{50\,M_\odot}\right)^{5/2} \left(\frac{m_2}{10\,M_\odot}\right)^{-11/4} \left(\frac{t_{GW}}{10^6\ \mathrm{Myr}}\right)^{1/4}$$

How many BHs left in the cluster?? 0, 1, 2, boh...

How massive? It depends whether mechanism by Miller & Hamilton (2002) is efficient or not – i.e. it depends on the mass of available seeds

From Kulkarni, Hut & McMillan 1993, Nature 364, 421
Sigurdsson & Hernquist 1993, Nature 364, 423

2) BHs eject each other?

Note: valid for globular clusters! Why? BHs form before progenitors segregate to the centre (no runaway collapse)

What happens when BHs are in the core?

2) Total mass of BHs relatively small with respect to stars →
- efficient formation of BH-BH binaries on standard core-collapse time
- ejection of lighter stars from the core (BH-BH binaries harden)
- ejections of (nearly) all single BHs in halo or out of cluster
- ejections of binary BHs with *x*>1

$$x = \frac{E_{b,\,min}}{E_{b\,merg}} \sim \left(\frac{m_1}{50\,M_\odot}\right)^{5/2} \left(\frac{m_2}{10\,M_\odot}\right)^{-11/4} \left(\frac{t_{GW}}{10^6\ \mathrm{Myr}}\right)^{1/4}$$

How many BHs left in the cluster?? 0, 1, 2, boh...

How massive? It depends whether mechanism by Miller & Hamilton (2002) is efficient or not – i.e. it depends on the mass of available seeds

From Kulkarni, Hut & McMillan 1993, Nature 364, 421
Sigurdsson & Hernquist 1993, Nature 364, 423

2) BHs eject each other?

Note: valid for globular clusters! Why? BHs form before progenitors segregate to the centre (no runaway collapse)

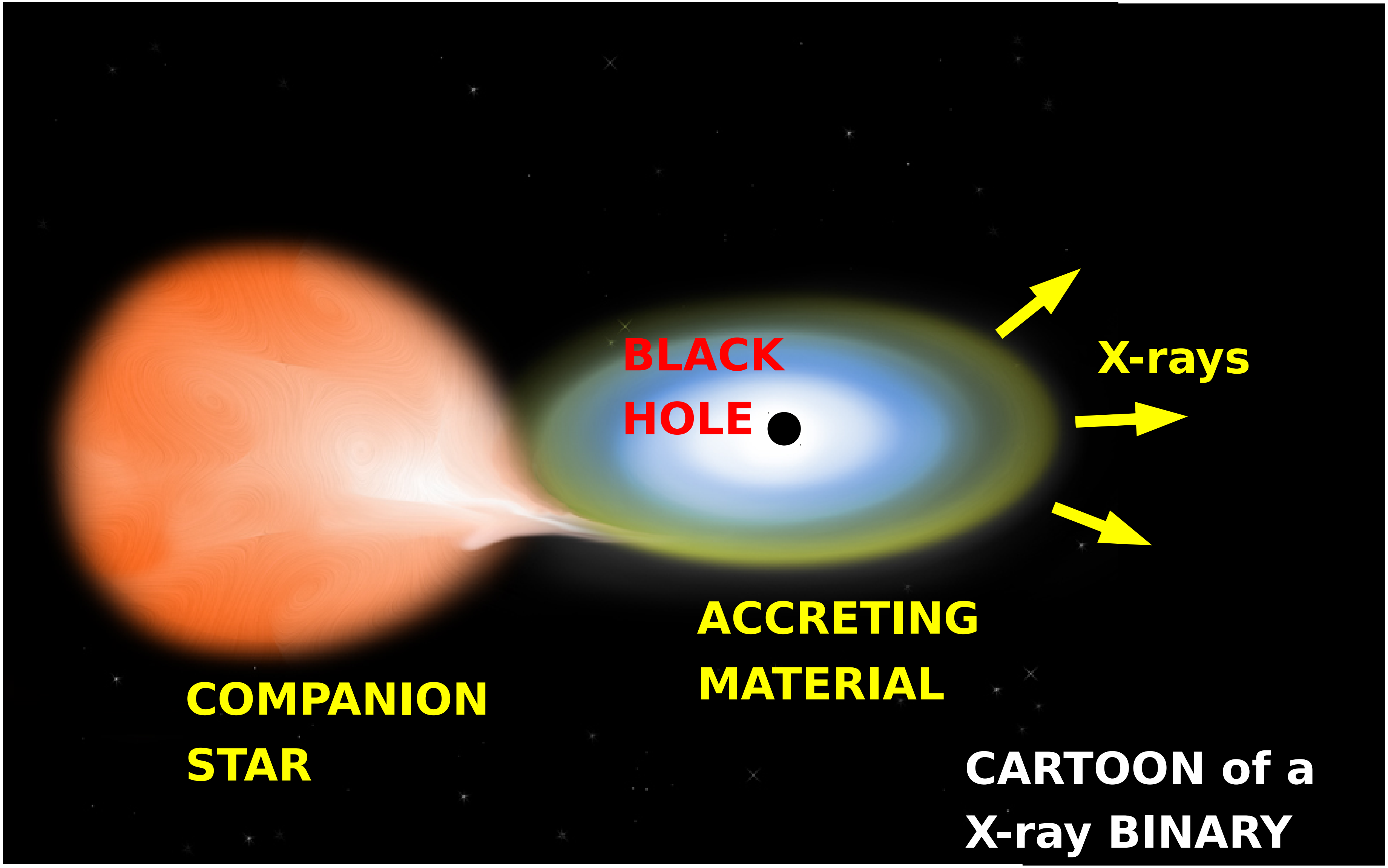
What do data tell us?

* The only 2 strong **BH candidates in MW GCs are 2 RADIO SOURCES** in globular cluster M22 (Strader et al. 2012, Nature, 490, 71). No X-ray detection (<10^{30} erg/s) → log L_R/L_X> -2.6 (too high for NSs)

* 5 sources in GCs of elliptical galaxies are strong BH candidates for X-ray variability:
  - NGC 4472 (Maccarone et al. 2007, Nature, 445, 183)
  - NGC 4472 (2nd source, Maccarone et al. 2011, MNRAS, 410, 1655)
  - NGC 3379 (Brassington et al. 2010, ApJ, 725, 1805)
  - NGC 1399 (Irwin et al. 2010, ApJ, 712, L1)
  - NGC 1399 (2nd source, Shih et al. 2010, ApJ, 721, 323)

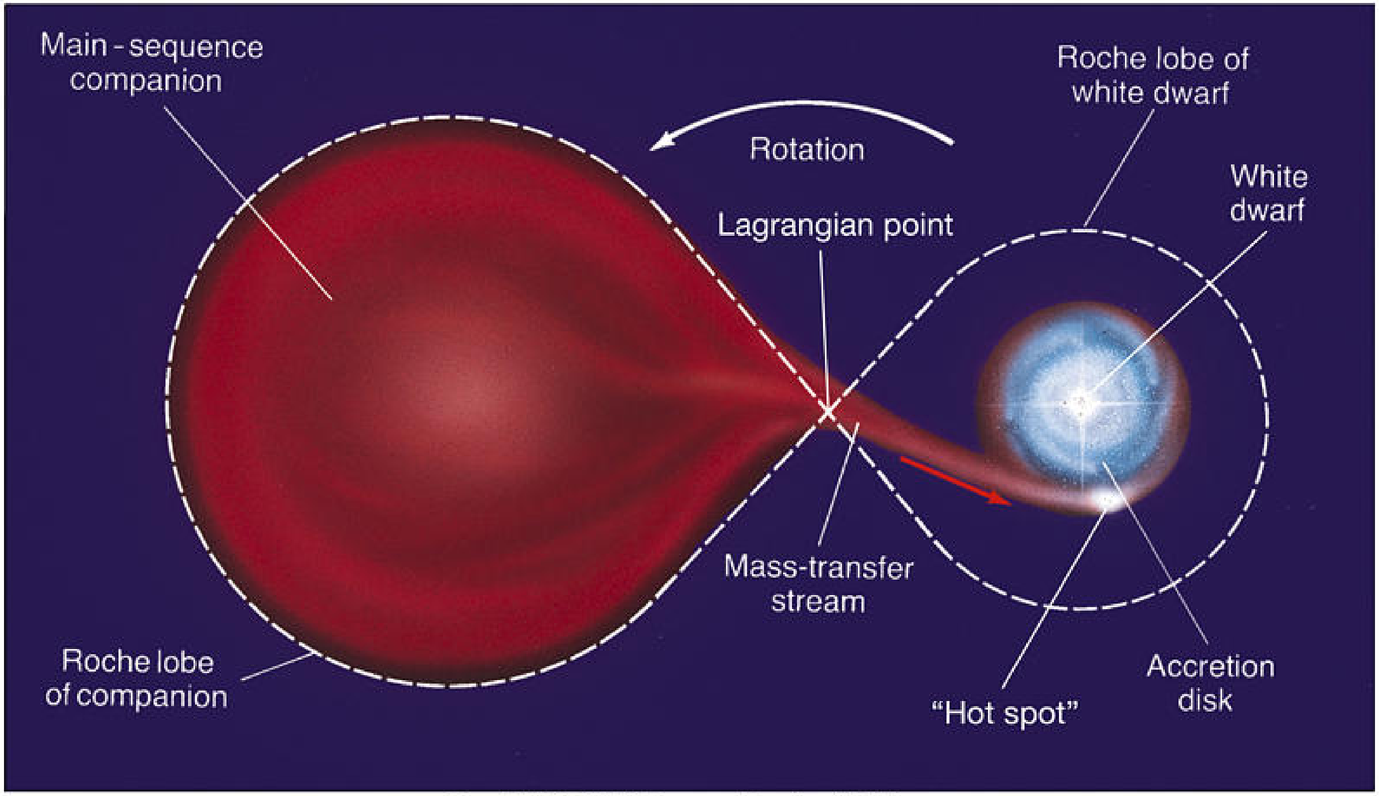
3) Effects of 3-body on X-ray binaries (formation and escape)

Compact object accreting matter from companion star via Roche lobe overflow or stellar winds



3) Effects of 3-body on X-ray binaries

Compact object accreting matter from companion star via Roche lobe overflow or stellar winds

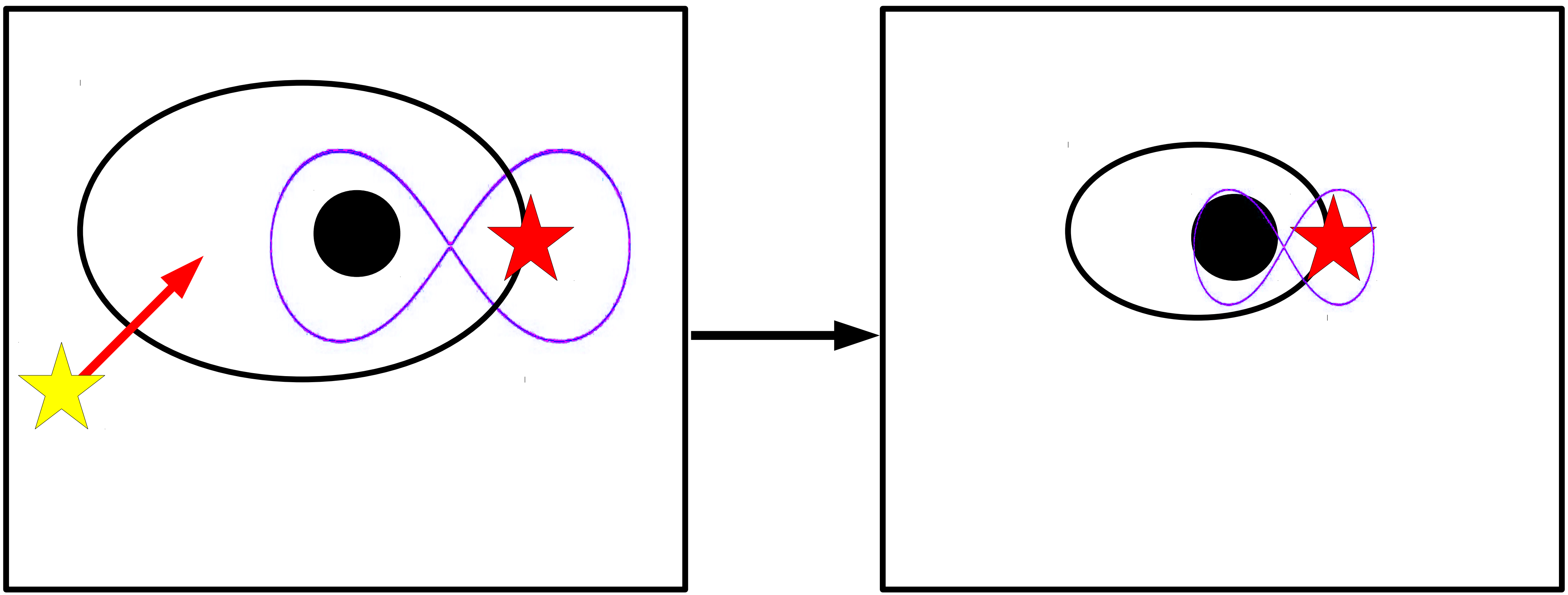


Luca Zampieri's course

$$r_{RL} = \frac{0.49\, a\, (M_1/M_2)^{2/3}}{0.6\, (M_1/M_2)^{2/3} + \ln\left[1 + (M_1/M_2)^{1/3}\right]}$$

3) Effects of 3-body on X-ray binaries

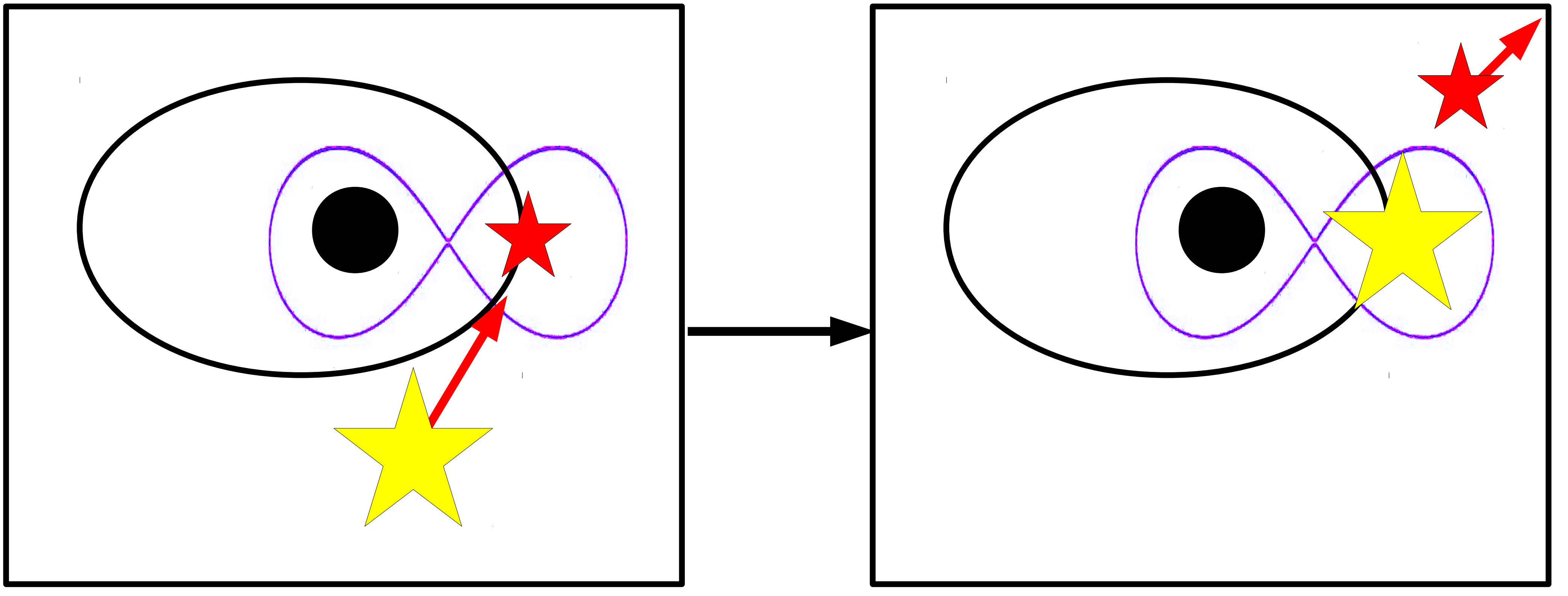
Which is the effect of 3-body encounters on X-ray binaries?



After 3-body encounters, the semi-major axis shrinks and the radius of the companion equals the Roche lobe

3) Effects of 3-body on X-ray binaries

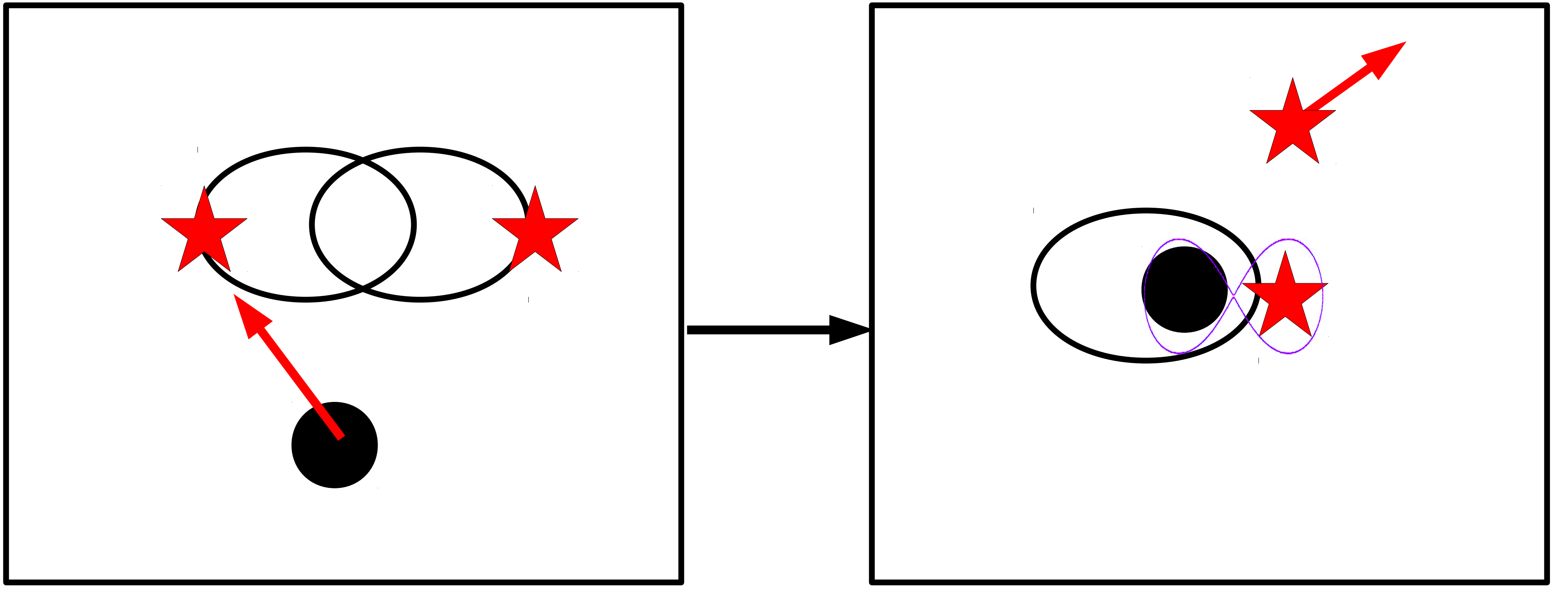
Which is the effect of 3-body encounters on X-ray binaries?



Exchanges are very important: (1) bring stars with larger radius in the binary

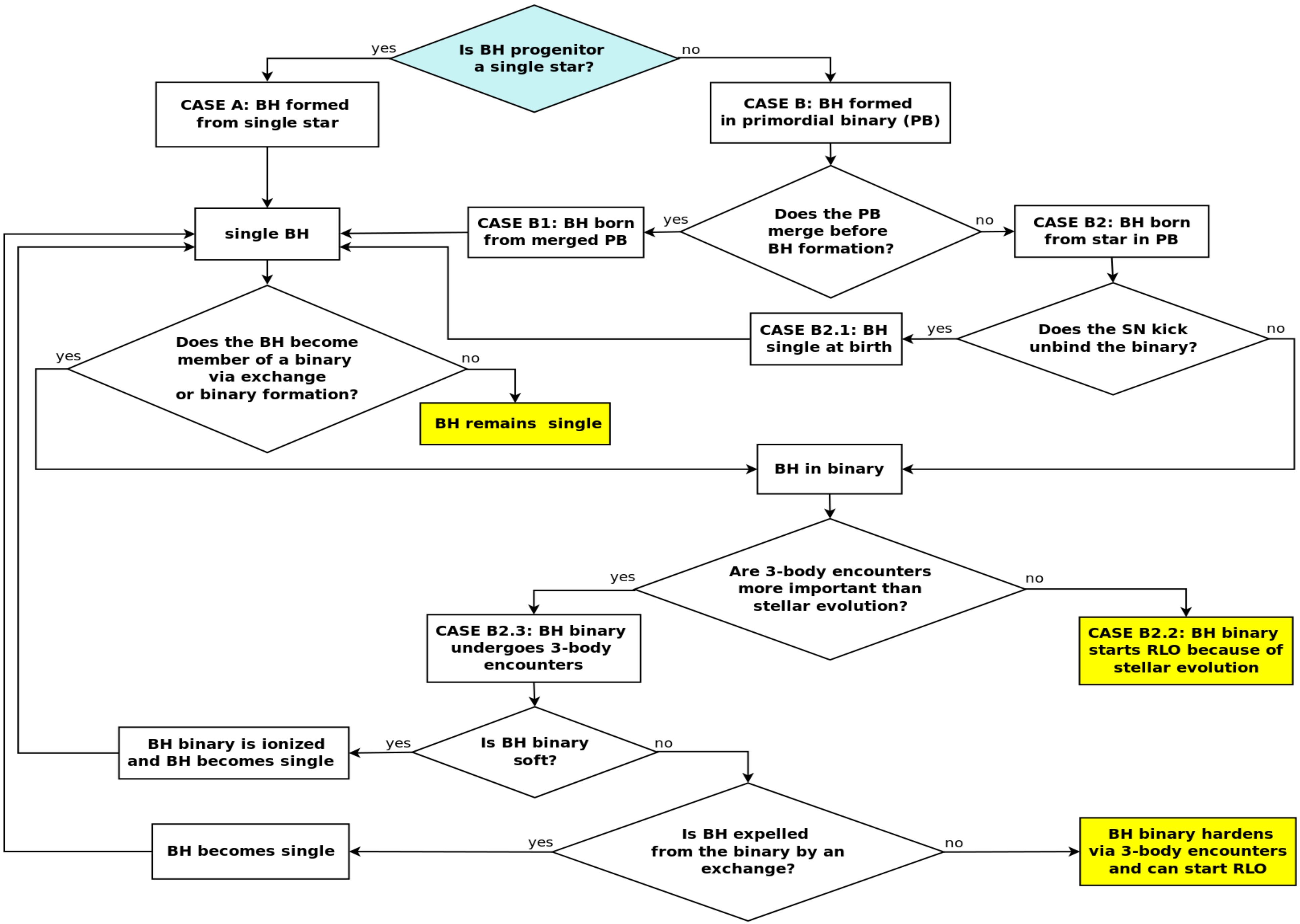
3) Effects of 3-body on X-ray binaries

Which is the effect of 3-body encounters on X-ray binaries?



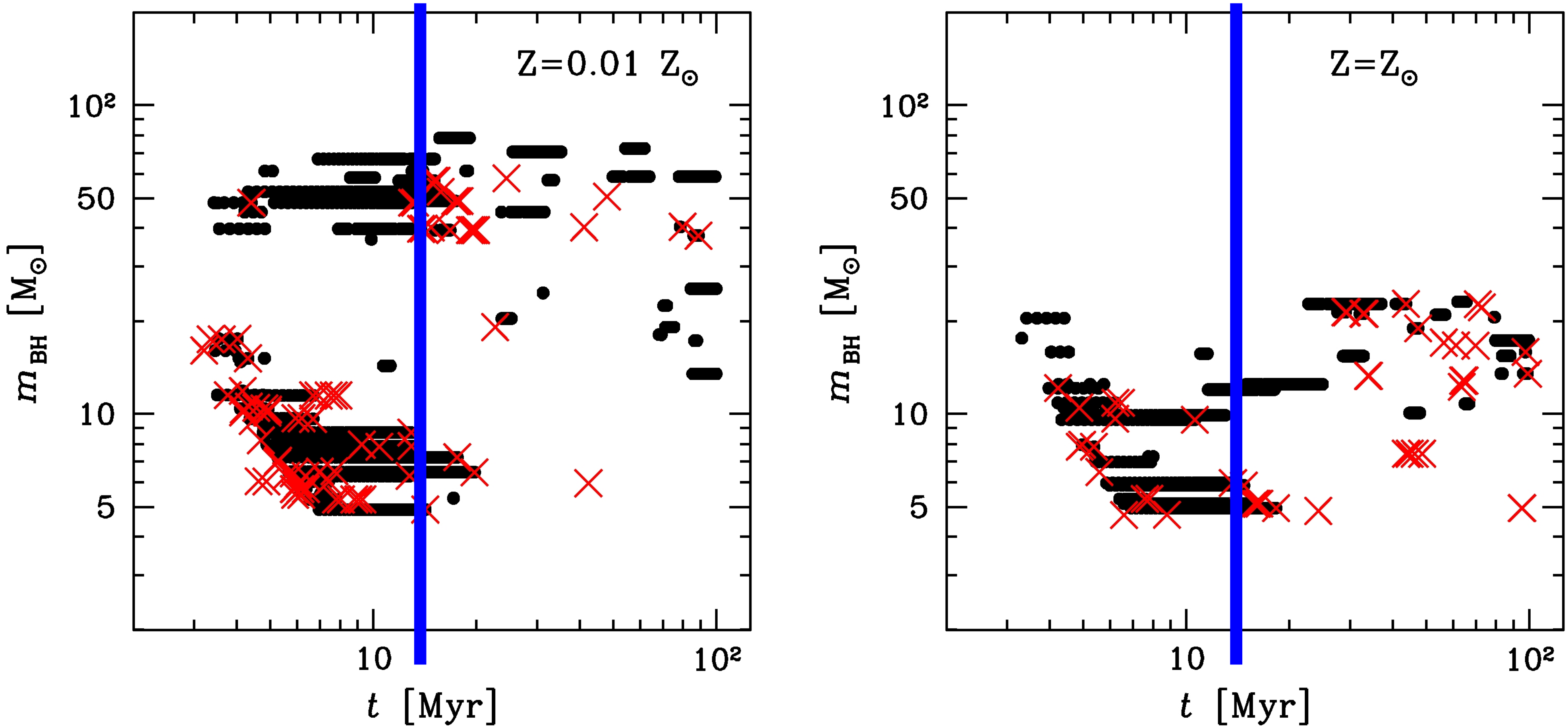
Exchanges are very important: (2) bring single BHs in binaries

3) Effects of 3-body on X-ray binaries



3) Effects of 3-body on X-ray binaries

X-ray binaries from stellar evolution switch on in the first stages
X-ray binaries from DYNAMICS switch on after 3-body encounters affect the binary

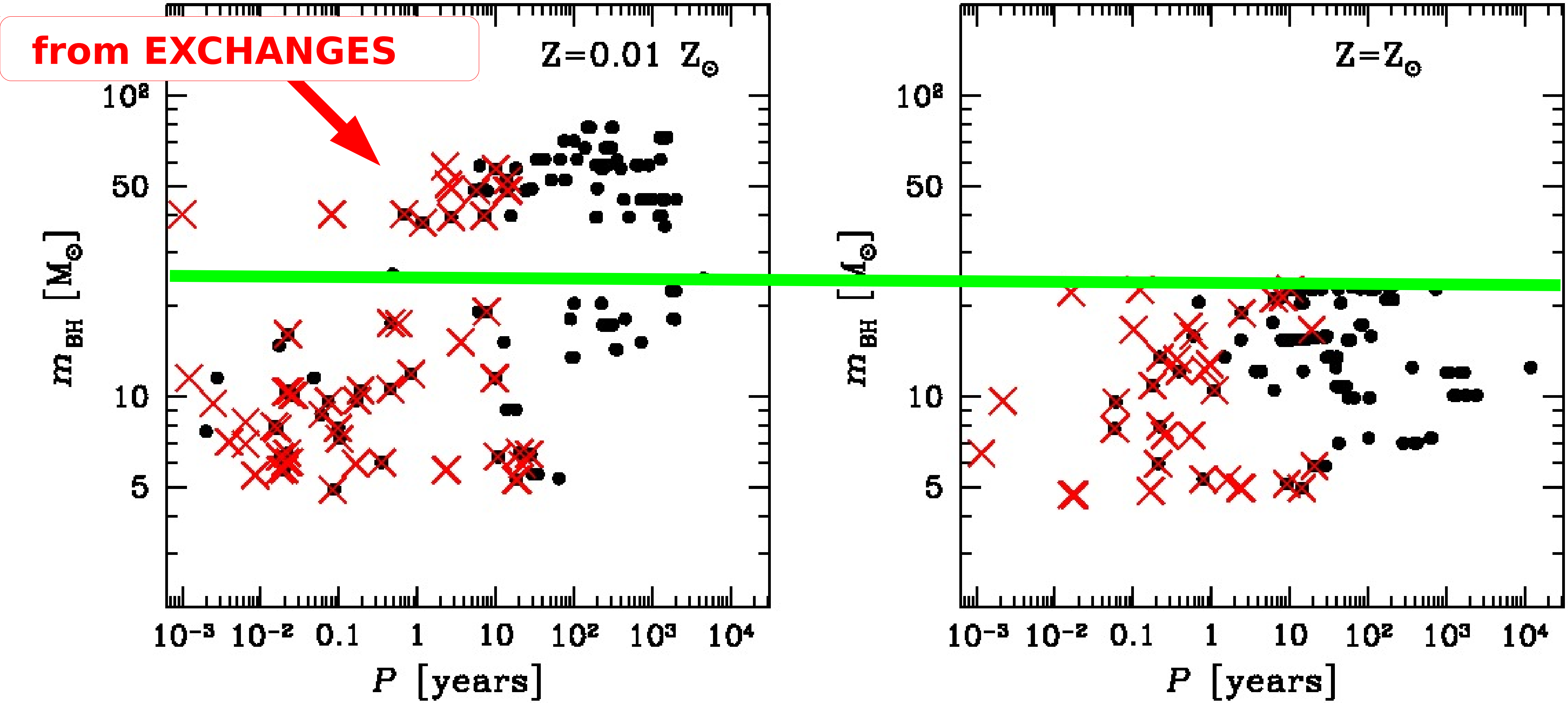


X = RLO systems
● = wind-accreting systems

MM+2013

3) Effects of 3-body on X-ray binaries

X-ray binaries from stellar evolution switch on in the first stages
X-ray binaries from DYNAMICS switch on after 3-body encounters affect the binary



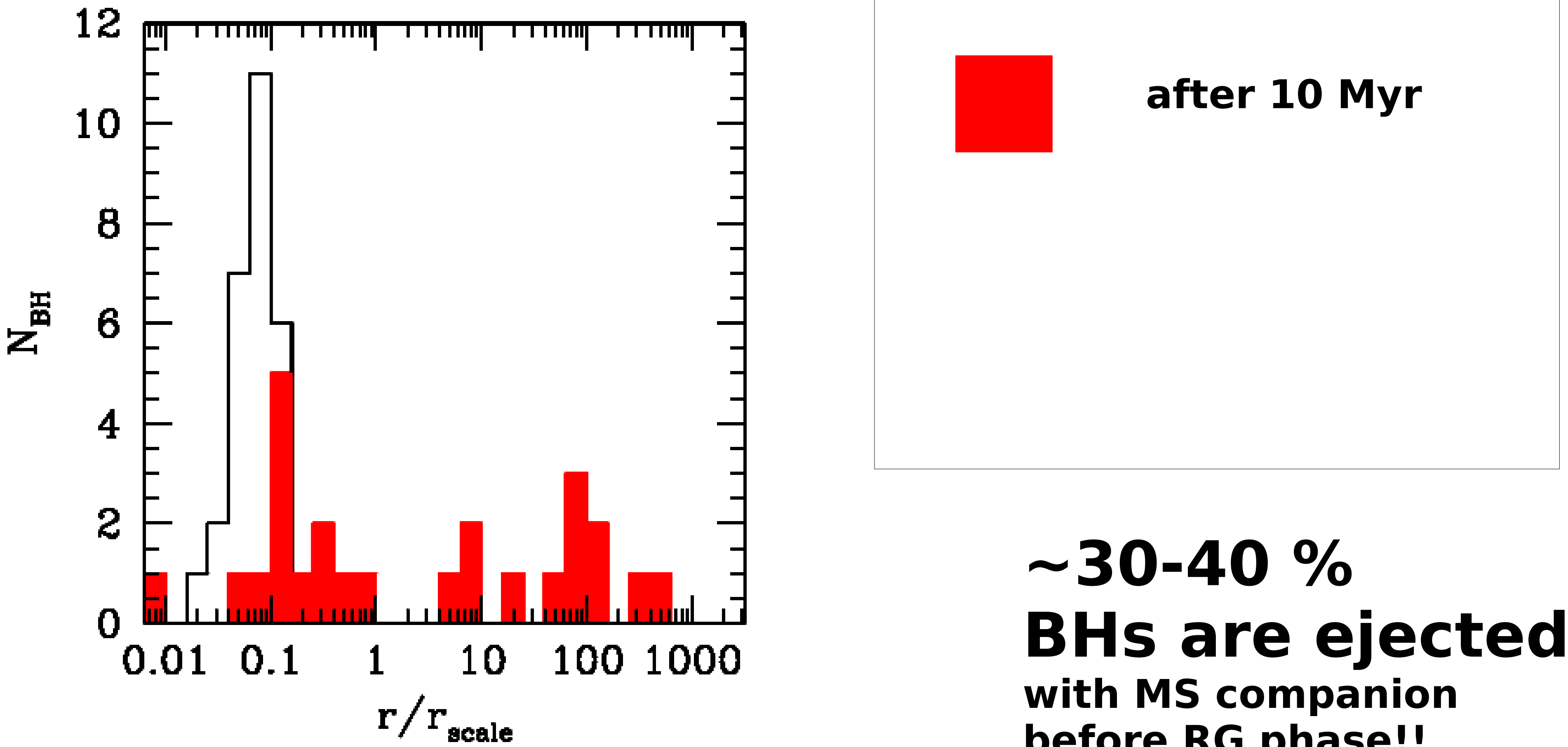
X = RLO systems

● = wind-accreting systems

MM+2013

3) Effects of 3-body on X-ray binaries

Simulations of young star clusters + MSBH binary with Starlab:



~30-40 %

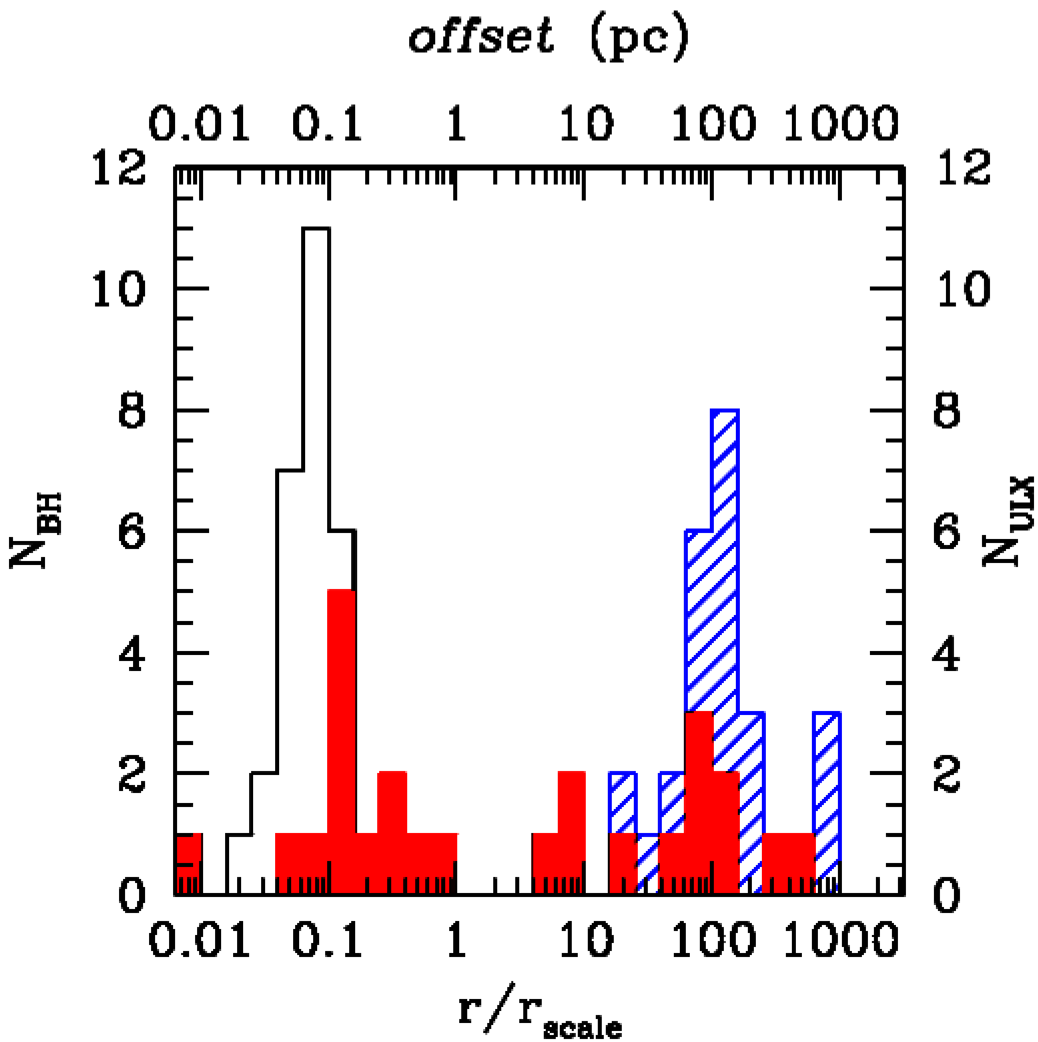
BHs are ejected

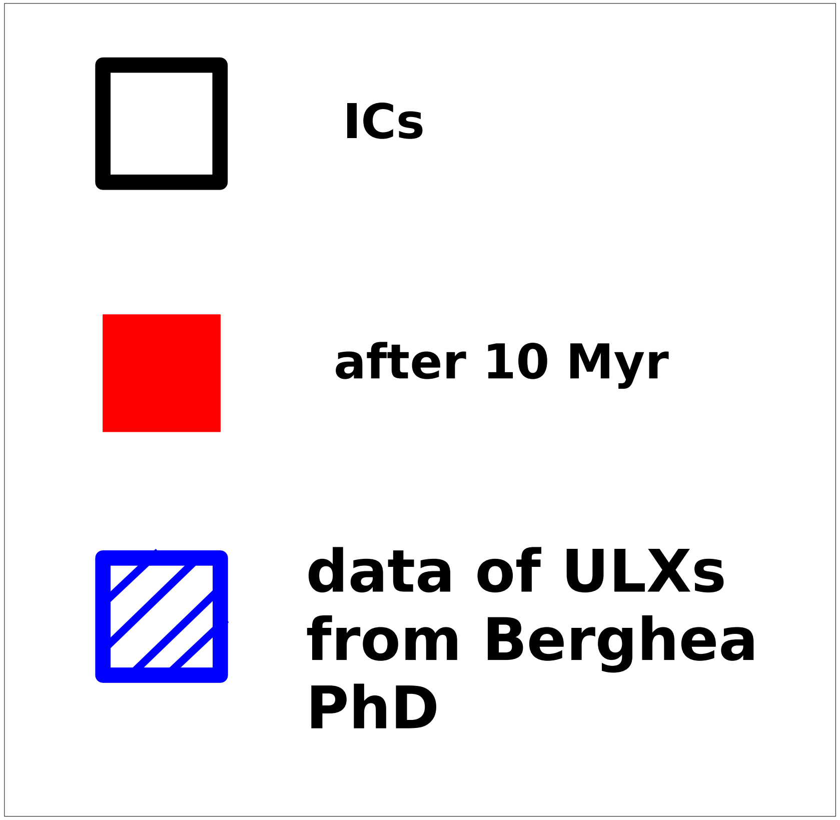
with MS companion before RG phase!!

MM et al. 2011

3) Effects of 3-body on X-ray binaries

Simulations of young star clusters + MSBH binary with Starlab:



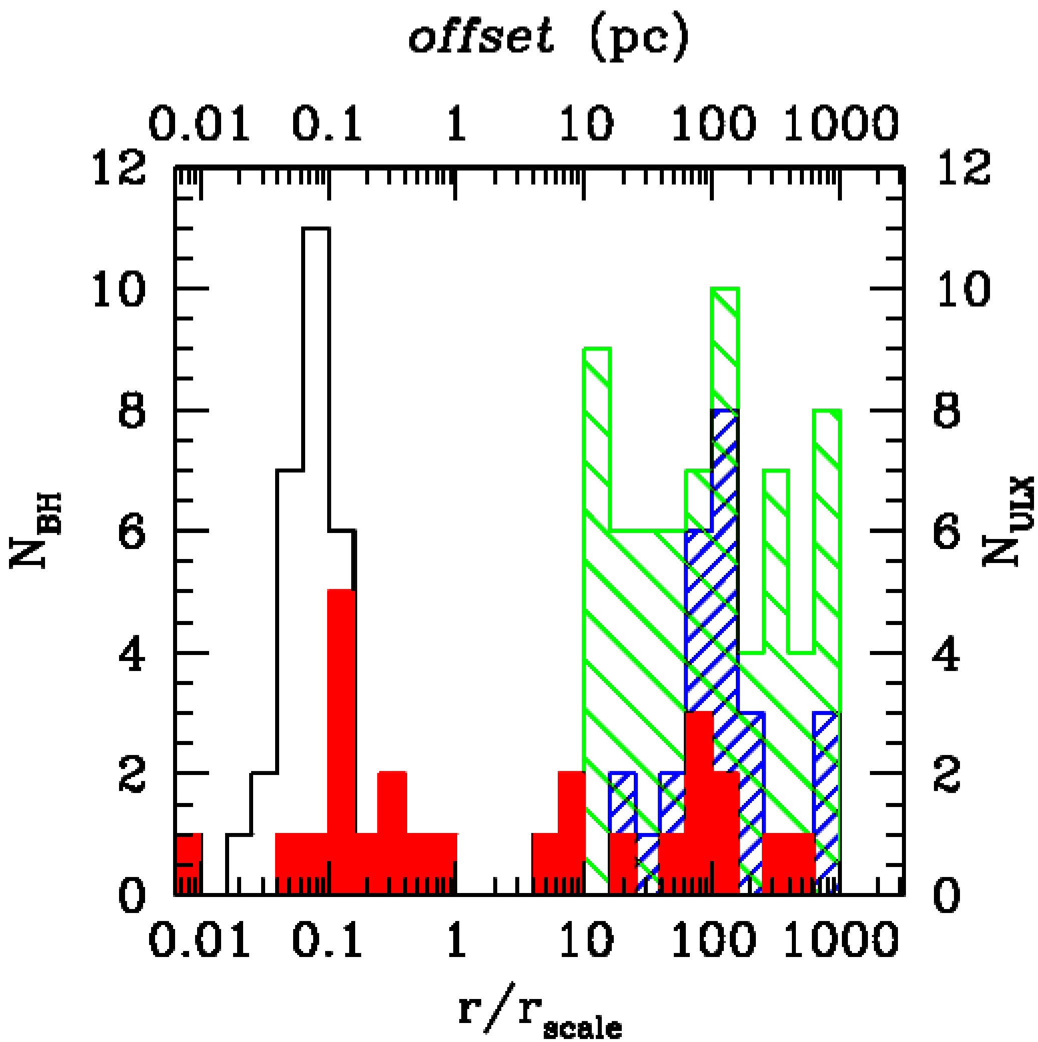


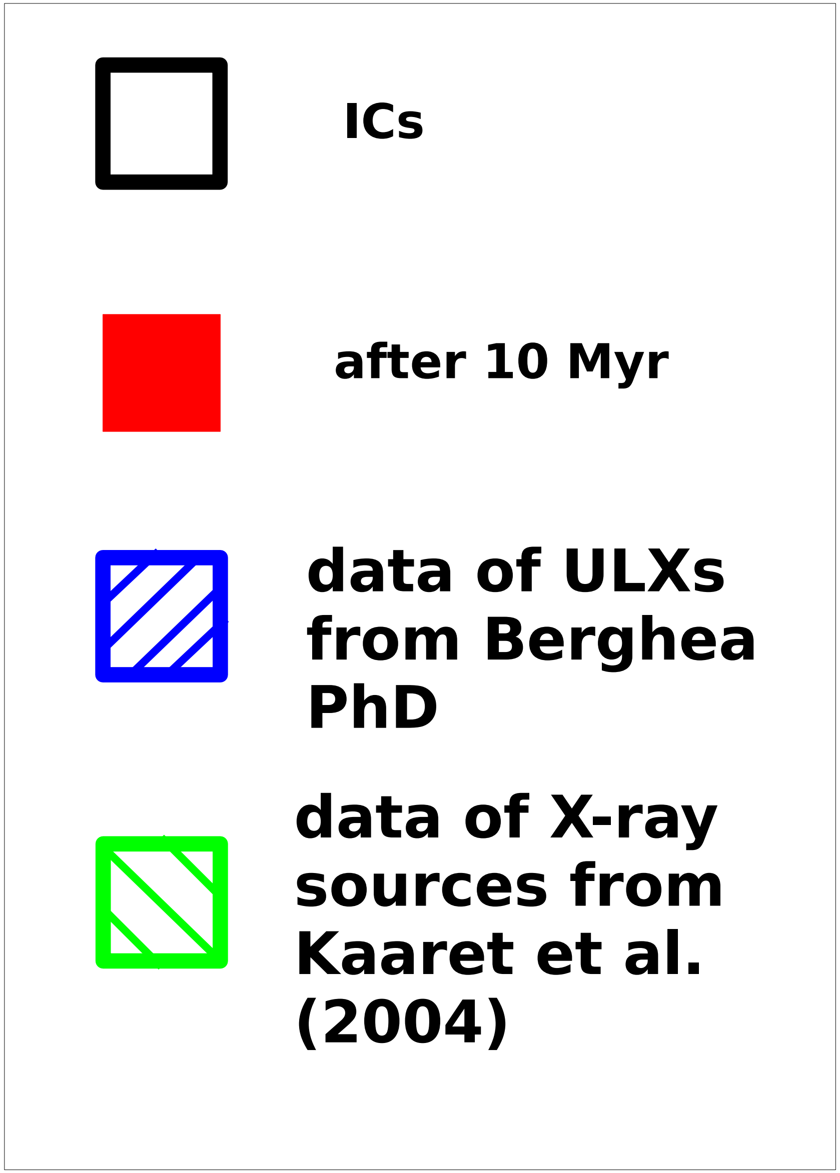
~30-40 %
BHs are ejected
with MS companion before RG phase!!

MM et al. 2011

3) Effects of 3-body on X-ray binaries

Simulations of young star clusters + MSBH binary with Starlab:





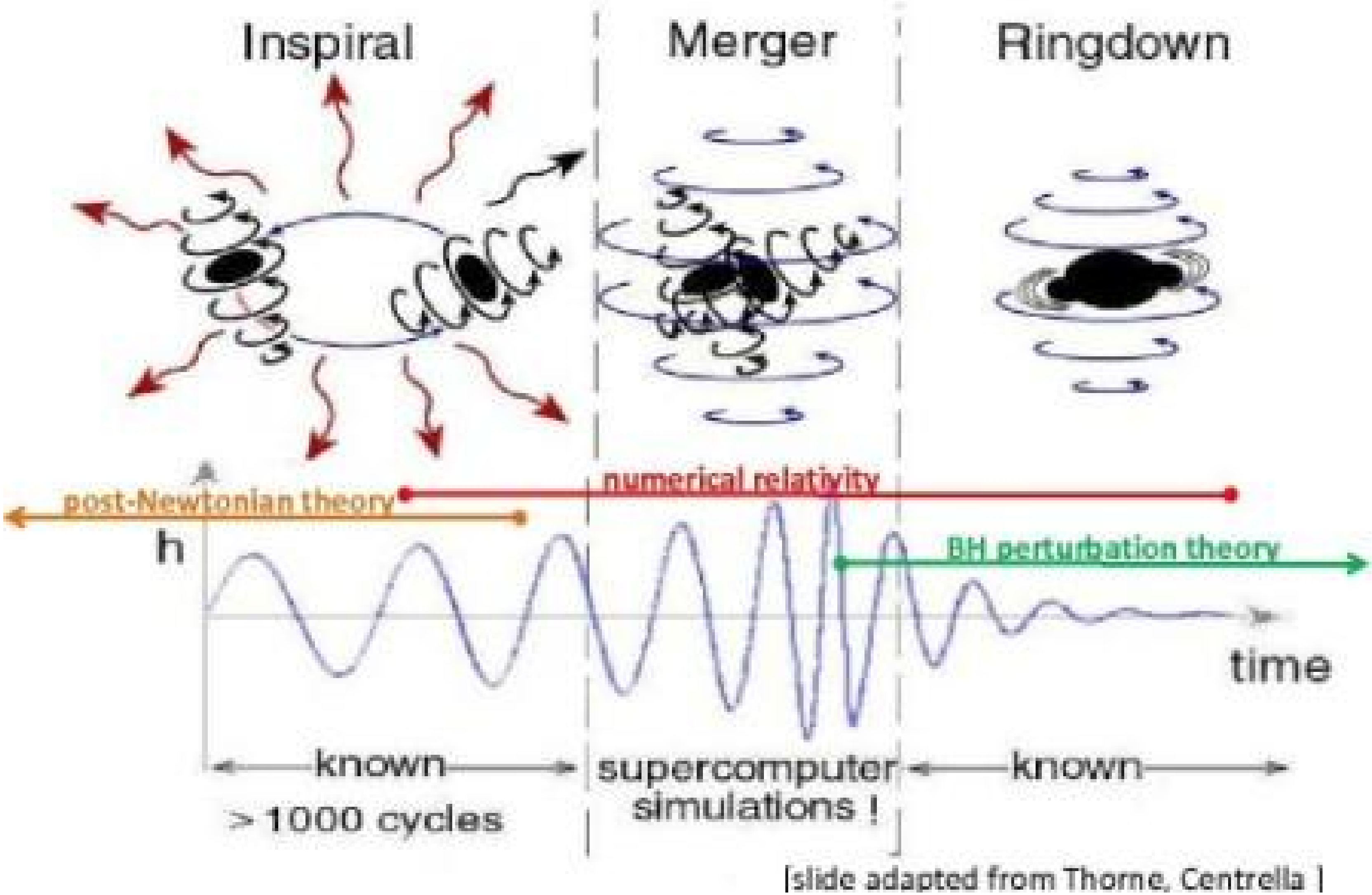
MM et al. 2011

3b) Effects of 3-body on GW sources

GWs:= perturbations of space-time that propagate as WAVES,
Predicted by Einstein's theory

It can be shown that merging compact-object binaries are SOURCEs of GWs

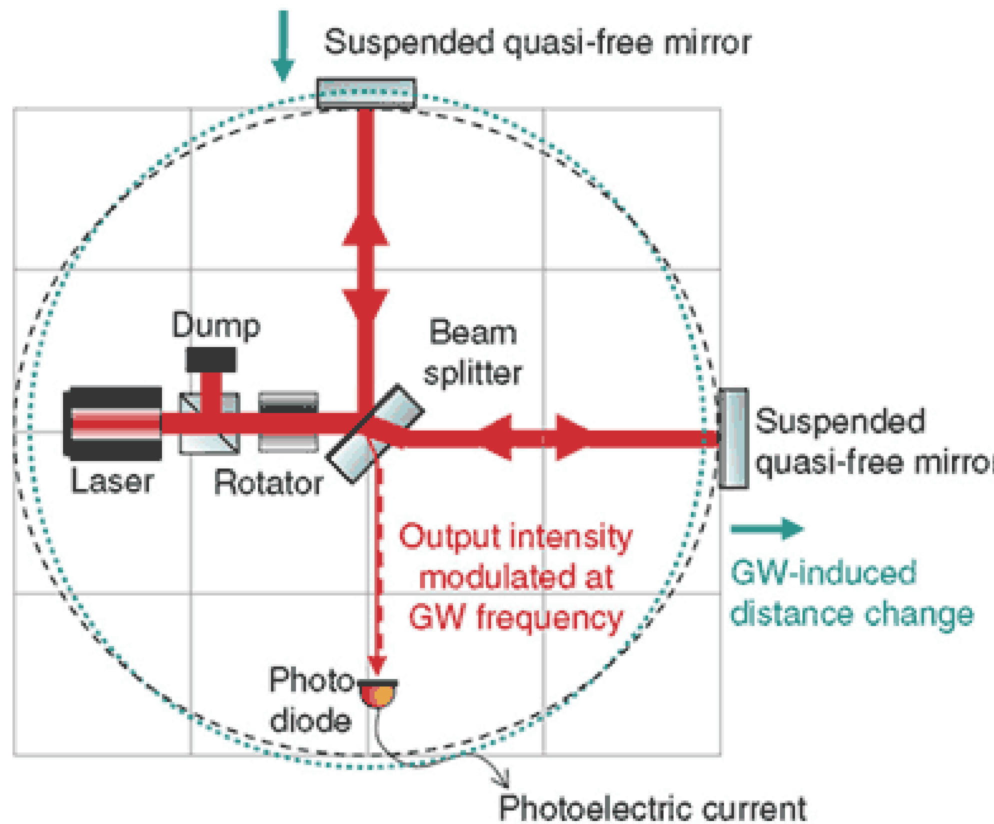
Cartoon of BH coalescence:



3b) Effects of 3-body on GW sources

* Only INDIRECT evidence by orbital decay of NS-NS binaries
(Hulse & Taylor)

* In 2015-2016 the second-generation ground based detectors
Advanced VIRGO and Advanced LIGO start operating !!!!!!!!!!!



3b) Effects of 3-body on GW sources

* Only INDIRECT evidence by orbital decay of NS-NS binaries
(Hulse & Taylor)

* In 2015-2016 the second-generation ground based detectors
Advanced VIRGO and Advanced LIGO start operating !!!!!!!!!!!

* BH-BH, BH-NS and NS-NS are sources of GWs

* In star clusters BH-BH and BH-NS are among the most massive
BINARIES: (i) form efficiently by **exchange**
(ii) are hard → **shrink by 3-body encounters**

and are LONG LIVED (because BH and NS do not evolve)

→ **ENHANCEMENT of GW sources by 3-body encounters?**

To be checked with SIMULATIONS for young clusters!!!

References:

*** Portegies Zwart & McMillan, 2002, ApJ, 576, 899**

* Miller & Hamilton, 2002, MNRAS, 330, 232

* Heger et al. 2003, arXiv:astro-ph/0211062

* Kulkarni, Hut & McMillan 1993, Nature 364, 421

* Sigurdsson & Hernquist 1993, Nature 364, 42

* Mapelli et al. 2013, MNRAS, submitted

* Mapelli et a. 2011, MNRAS, 416, 1756