A First Course in General Relativity

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(2nd Edition, Cambridge University Press, 2009)

Solutions to Selected Exercises

(Version 1.0, November 2009)

To the user of these solutions:

This document contains solutions to many of the Exercises in the second edition of *A First Course in General Relativity*. The textbook offers an extensive collection of exercises, some of which prove results omitted from the text, or form the basis for later exercises, or provide the foundation for results in later chapters. Doing exercises is integral to the process of learning a subject as complex and conceptually challenging as general relativity. These solutions, therefore, are meant to help users of this book master exercises that they might have had difficulty with. It is assumed that an attempt has already been made at solving the problem!

Solutions are provided to those problems which might present particular conceptual challenges; those which require just routine algebra, and those which require computer solutions, are generally not included.

I hope you find the solutions helpful!

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Chapter 1

1 $3.7\times10^{-24}\,\mathrm{kg\,m^{-1}}$; $3.5\times10^{-43}\,\mathrm{kg\,m}$; 10^{-7}; $10^{-4}\,\mathrm{kg}$; $1.1\times10^{-12}\,\mathrm{kg\,m^{-3}}$; $10^{3}\,\mathrm{kg\,m^{-3}}$; $3.7\times10^{-16}\,\mathrm{kg\,m^{-3}}$.

2 $3\times10^{6}\,\mathrm{m\,s^{-1}}$; $9\times10^{35}\,\mathrm{N\,m^{-2}}$; $3.3\times10^{9}\,\mathrm{s}$; $9\times10^{16}\,\mathrm{J\,m^{-3}}$; $9\times10^{17}\,\mathrm{m\,s^{-2}}$.

3 (a)–(k) See the top panel in Figure 1. Note that most of the items continue into other quadrants (not shown). (l) See the bottom panel in the same figure. Note that light always travels at speed 1 in this frame even after reflecting off of a moving mirror.

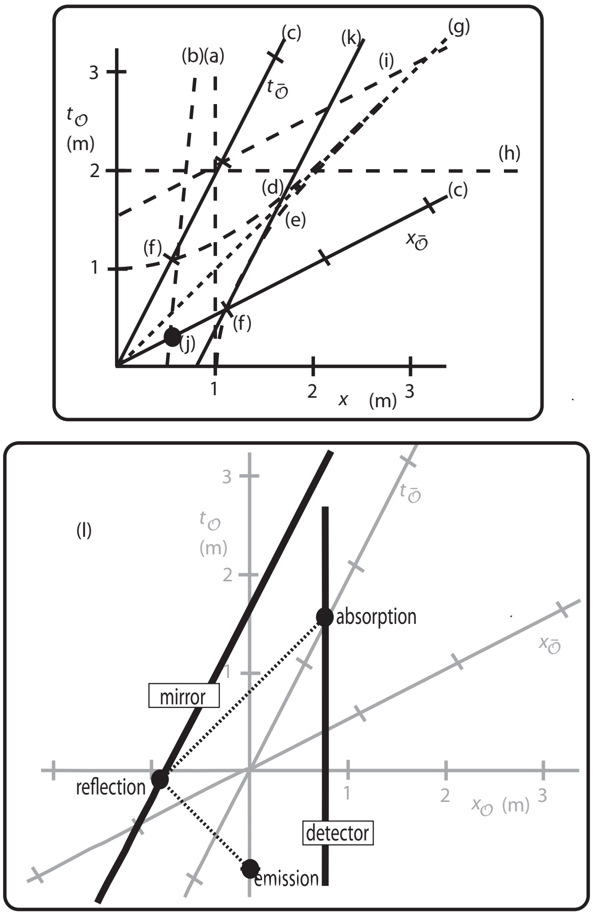

Figure 1: Solution to Ex. 3 of Chapter 1. See the solution text for explanation.

5 (a) See the top panel in Figure 2. The particles (short dashed world lines) are emitted at event $\mathcal{A}$ and reach the detectors at events $\mathcal{B}$ and $\mathcal{C}$. The detectors send out their signals (more short dashed world lines) at events $\mathcal{D}$ and $\mathcal{E}$, which arrive back at the spatial origin at event $\mathcal{F}$. The line $\mathcal{BC}$ joining the two reception events (long dashed) at the detectors is parallel to the x=axis in our diagram, which means that they occur at the same time in this frame. Note that the lines $\mathcal{DF}$ and $\mathcal{EF}$ are tilted over more than the lines $\mathcal{AB}$ and $\mathcal{AC}$ because the returning signals go at speed 0.75 while the particles travel only at speed 0.5.

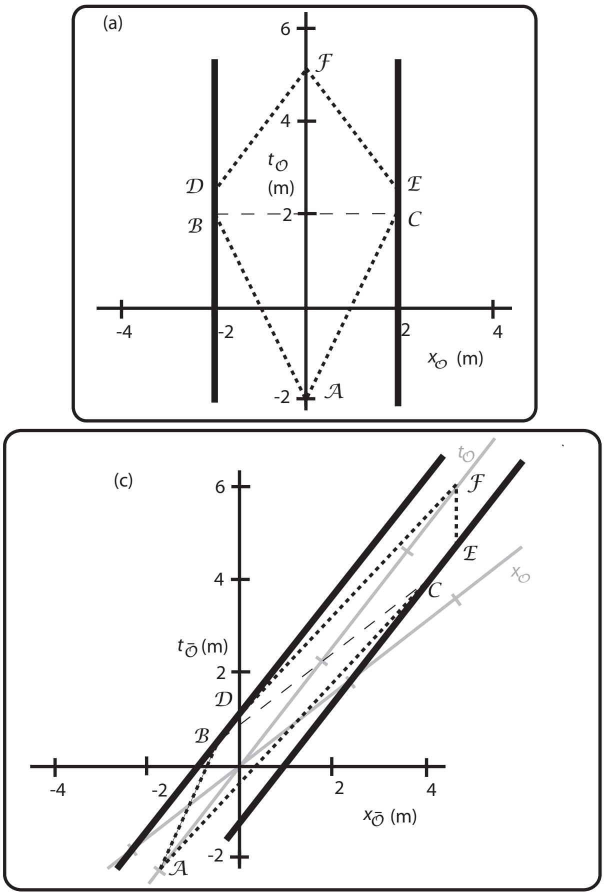

Figure 2: Solution to Ex. 5 of Chapter 1. See the solution text for explanation.

(b) The experimenter knows that the detectors are equidistant from $x = 0$, and that the signals they send out travel at equal speeds. Therefore they have equal travel times in this frame. Since they arrive at the same time, they must have been sent out at the same time. This conclusion depends on observer-dependent things, such as the fact that the signals travel at equal speeds. Therefore the conclusion, while valid in this particular frame, might not be valid in others.

(c) See the bottom panel in Figure 2. This is drawn using the axes of frame $\bar{\mathcal{O}}$, which we draw in the usual horizontal and vertical directions, since any observer would normally draw his/her axes this way. In this frame, the frame $\mathcal{O}$ moves forwards at speed 0.75. These axes are drawn in gray, and calibrated using invariant hyperbolae (not shown), just as in the solution to Exercise 1.3. Then the events are located according to their coordinate locations on the axes of $\mathcal{O}$. Note that to do this, lines of constant $t_{\mathcal{O}}$ must be drawn parallel to the $x_{\mathcal{O}}$ axis, and lines of constant $x_{\mathcal{O}}$ must be drawn parallel to the $t_{\mathcal{O}}$ axis. One such line is the long-dashed line $\mathcal{BC}$. The detectors (heavy lines) must pass through the points $x_{\mathcal{O}} = \pm 2$ m on the $x_{\mathcal{O}}$-axis. The two signal-emission events $\mathcal{D}$ and $\mathcal{E}$ are clearly not simultaneous in this frame, although they are simultaneous in $\mathcal{O}$: $\mathcal{D}$ occurs much earlier than $\mathcal{E}$. Note also that the signal sent at the event $\mathcal{E}$ and received at $\mathcal{F}$ remains at rest in $\bar{\mathcal{O}}$, since it was sent backwards at speed 0.75 in frame $\mathcal{O}$, exactly the same speed as the frame $\bar{\mathcal{O}}$.

(d) The interval is easily computed in frame $\mathcal{O}$ because the events $\mathcal{D}$ and $\mathcal{E}$ have zero separation in time and are separated by 4 m in x. So the squared interval is 16 m^2. To compute it in frame $\bar{\mathcal{O}}$, measure as carefully as you can in the diagram the coordinates $t_{\bar{\mathcal{O}}}$ and $x_{\bar{\mathcal{O}}}$ for both emission events. Given the thickness of the lines representing the detectors, you will not get exactly 16, but you should come close.

6 Write out all the terms.

7 $M_{00} = \mu^2 - \alpha^2, M_{01} = \mu\nu - \alpha\beta, M_{11} = \nu^2 - \beta^2, M_{22} = a^2, M_{33} = b^2, M_{02} = M_{03} = M_{12} = M_{13} = M_{23} = 0$.

8 (c) Use various specific choices of Δx^i; e.g. $\Delta x = 1, \Delta y = 0, \Delta z = 0 \Rightarrow M_{11} = -M_{00}$.

10 Null; spacelike; timelike; null.

11 "Asymptotic" refers to the behavior of the curves for large values of x and t. But when these variables are sufficiently large, one can neglect a and b, and then one has approximately $-t^2 + x^2 = 0$, leading to $t = \pm x$. This approximation is better and better as t and x get larger and larger.

13 The principle of relativity implies that if time dilation applies to one clock (like one based on light travel times over known distances, which is effectively the sort we use to calibrate our time coordinate), then it applies to all (like the pion half-life). Algebra gives the result.

14 (d) For (a), 3.7×10^{-5}.

16 (a) In Fig. 1.14, we want the ratio $\bar{t}/t$ for event $\mathcal{B}$ (these are its time-coordinate differences from $\mathcal{A}$). The coordinates of $\mathcal{B}$ in $\mathcal{O}$ are $(t, \boldsymbol{v}t)$. The first line of Eq. (1.12) implies $\bar{t} = t(1 - \boldsymbol{v}^2)^{1/2}$.

17 (a) $12\,\mathrm{m}$.

(b) $1.25 \times 10^{-8}\,\mathrm{s} = 3.75\,\mathrm{m}$; $211\,\mathrm{m}^2$: spacelike.

(c) $9\,\mathrm{m}$; $20\,\mathrm{m}$. (d) No: spacelike separated events have no unique time ordering. (e) If one observer saw the door close, all observers must have seen it close. The finite speed of transmission ($< c$) of the shock wave along the pole prevents it behaving rigidly. The front of the pole may stop when it hits the wall, but the back keeps moving and can't be stopped until after the door has shut behind it. Many apparent paradoxes in special relativity are resolved by allowing for the finite speed of transmission of pressure waves: for a body to be perfectly rigid would violate special relativity, since the communication among its different parts would have to happen instantaneously.

18 (b) $\tanh[N\ \tanh^{-1}(0.9)] \approx 1 - 2(19)^{-N}$.

19 (c) The analog of the interval is the Euclidean distance $x^2 + y^2$. The analog of the invariant hyperbola is the circle $x^2 + y^2 = a^2$. The circle allows one to transfer the measure of length along, say, the x-axis up to the y-axis.

21 The easiest way to demonstrate these theorems is graphically: put two arbitrary events onto a spacetime diagram, and join them. If they are time-like separated, then the line joining them will be more vertical than a light-line, and so there is a frame in which it is parallel to the time-axis, in which frame it will represent a line of fixed spatial position: the two events will be at the same spatial point in this new frame. This proves (a). The proof of (b) is similar.

Chapter 2

1 (a) -4; (b) 7, 1, 26, 17; (c) same as (b); (d) -15, 27, 30, -2 (different from (b) and (c) because the sum is on the second index, not the first); (e) $A^0 B_0 = 0$, $A^2 B_3 = 0, A^3 B_1 = 12$, etc.; (f) -4; (g) the subset of (e) with indices drawn from (1, 2, 3) only.

2 (c) γ, α free; μ, λ dummy; 16 equations.

5 (b) No.

6 To get the basis vectors of the second frame, compose velocities to get $v = 0.882$.

8 (b) The difference of the two vectors is the zero vector.

10 Choose each of the basis vectors for $\vec{A}$, getting the four equations in Eq. (2.13) in turn.

12 (b) (35/6, $-37/6$, 3, 5).

14 (a) $-(0.75/1.25) = -0.6$ in the z direction.

15 (a) $(\gamma, \gamma v, 0, 0)$; (b) $(\gamma, \gamma v^x, \gamma v^y, \gamma v^z)$ with $\gamma = (1 - \boldsymbol{v} \cdot \boldsymbol{v})^{-1/2}$. (d) $v^x = U^x/U^0 = 0.5$, etc.

18 (a) Given $\vec{a} \cdot \vec{a} > 0$, $\vec{b} \cdot \vec{b} > 0$, $\vec{a} \cdot \vec{b} = 0$; then $(\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = \vec{a} \cdot \vec{a} + 2\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b} > 0$.

(b) If $\vec{a}$ is timelike, use the frame in which $\vec{a} \rightarrow (a, 0, 0, 0)$.

19 (b) $v = \alpha t(1 + \alpha^2 t^2)^{-1/2}$, $\alpha = 1.1 \times 10^{-16}\,\mathrm{m}^{-1}$; $t = 2.0 \times 10^{17}\,\mathrm{m} = 6.7 \times 10^8\,\mathrm{s} = 22\,\mathrm{yr}$.

(c) $v = \tanh\,(\alpha\tau), x = \alpha^{-1}[(1 + \alpha^2 t^2)^{1/2} - 1]$, 10 yr.

22 (a) 4 kg, 3.7 kg, 0.25 $(\boldsymbol{e}_x + \boldsymbol{e}_y)$; (b) $(3, -\frac{1}{2}, 1, 0)$ kg, 3 kg, 2.8 kg, $-\frac{1}{6}\boldsymbol{e}_x + \frac{1}{3}\boldsymbol{e}_y$, 0.2 $\boldsymbol{e}_y$.

23 $E = m + \frac{1}{2}m|\boldsymbol{v}|^2 + \frac{3}{8}m|\boldsymbol{v}|^4 + \ldots$; $|\boldsymbol{v}|^2 = 2/3$.

24 Work in the CM frame.

25 (a) Lorentz transformation of $\vec{p}$. (b) $v = 2\cos\theta/(1 + \cos^2\theta)$.

27 The cooler, because ratio of the rest masses is $1 + 1.1 \times 10^{-16}$.

31 $2h\nu\cos\theta$; $h\nu\cos\theta$.

33 $E_{\max} = 8 \times 10^5\, m_{\rm p}$, above the γ-ray band.

Chapter 3

3 (b) (i) -1; (ii) 2; (iii) -7; (iv) same.

4 (b) $\tilde{p} \to (-1/4, -3/8, 15/8, -23/8)$; (b) $-5/8$; (d) yes.

5 The order of the multiplication of numbers does not matter, nor does the order in which the sums are done; summation over $\bar{\alpha}$ produces the unit matrix because the two Λ matrices are inverse to one another; summation over μ is just multiplication by the unit identity matrix.

6 (a) Consider $\tilde{p} = \tilde{\omega}^0$, an element of the basis dual to $\{\vec{e}_\alpha\}$; (b) (1, 0, 0, 1).

8 See Figure 3 for the diagram. Note that in the representation of a one-form as a set of surfaces, the extent of the surfaces and the number of them is not important. The critical property is their orientation and spacing. The one-form $\tilde{d}t$ has surfaces separated by one unit in the t-direction so that, when the basis vector $\vec{e}_0$ is drawn on top of it, it crosses exactly one surface, and when the other basis vectors are placed on it they cross no surfaces (they lie in the surfaces).

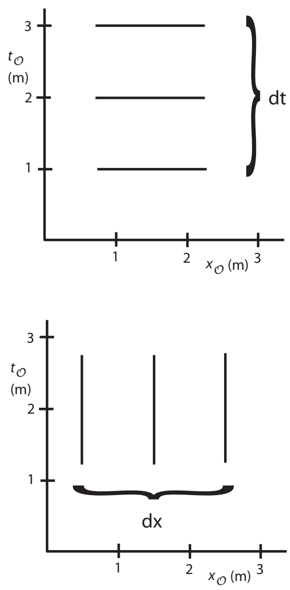

Figure 3: Solution to Ex. 8 of Chapter 3. See the solution text for explanation.

9 $\tilde{\mathrm{d}}T(\mathcal{P}) \to (-15, -15)$; $\tilde{\mathrm{d}}T(\mathcal{L}) \to (0, 0)$.

10 (a) A partial derivative *wrt* x^β holds all x^α fixed for $\alpha \neq \beta$, so $\partial x^\alpha / \partial x^\beta = 0$ if $\alpha \neq \beta$. Of course, if $\alpha = \beta$ then $\partial x^\alpha / \partial x^\beta = 1$.

12 (a) By definition, $\tilde{n}(\vec{V}) = 0$ if $\vec{V}$ is tangent to S, so if $\vec{V}$ is not tangent, then $\tilde{n}(\vec{V}) \neq 0$.

(b) If $\vec{V}$ and $\vec{W}$ point to the same side of S then there exists a *positive number* α such that $\vec{W} = \alpha\vec{V} + \vec{T}$, where $\vec{T}$ is tangent to S. Then $\tilde{n}(\vec{W}) = \alpha\tilde{n}(\vec{V})$ and both have the same sign.

(c) On the Cartesian basis, the components of $\tilde{n}$ are $(\beta, 0, 0)$ for some β. Thus any $\tilde{n}$ is a multiple of any other.

16 (e) 10 and 6 in four dimensions.

18 (b) $\vec{q} \to (-1, -1, 1, 1)$.

20 (a) In matrix language, $\Lambda^{\bar{\alpha}}{}_{\beta} A^\beta$ is the product of the matrix $\Lambda^{\bar{\alpha}}{}_{\beta}$ with the column vector A^β, while $\Lambda^{\alpha}{}_{\bar{\beta}} p_\alpha$ is the product of the *transpose* of $\Lambda^{\alpha}{}_{\bar{\beta}}$ with the column vector p_α. Since $\Lambda^{\alpha}{}_{\bar{\beta}}$ is inverse to $\Lambda^{\bar{\alpha}}{}_{\beta}$, these are the same transformation if $\Lambda^{\alpha}{}_{\bar{\beta}}$ equals the transpose of its inverse.

21 (a) The associated vectors for $t = 0$ and $t = 1$ point *inwards*.

24 (b) No. The arguments are different geometrical objects (one vector, one one-form) so they cannot be exchanged.

25 Use the inverse property of $\Lambda^{\alpha}{}_{\bar{\beta}}$ and $\Lambda^{\bar{\alpha}}{}_{\beta}$.

26 (a) $A^{\alpha\beta} B_{\alpha\beta} = -A^{\beta\alpha} B_{\alpha\beta} = -A^{\beta\alpha} B_{\beta\alpha} = -A^{\mu\nu} B_{\mu\nu} = -A^{\alpha\beta} B_{\alpha\beta}$. Therefore $2A^{\alpha\beta} B_{\alpha\beta} = 0$. The justification for each of the above steps: antisymmetry, symmetry, relabel dummy indices, relabel dummy indices.

28 Arbitrariness of $\vec{U}$.

30 (a) Since $\vec{D} \cdot \vec{D} = -x^2 + 25t^2x^2 + 2t^2 \neq -1, \vec{D}$ is not a four-velocity field.

(f) $5t$.

(h) $-5t$ because of (e).

(i) the vector gradient of ρ has components $\{\rho^{,\alpha}\} = (-2t, 2x, -2y, 0)$.

(j) $\nabla_{\vec{U}} \vec{D} \to [t^2, 5t^3 + 5x(1 + t^2), \surd 2(1 + t^2), 0]$.

33 (d) Given any matrix (A) in $0(3)$, let (Λ) be the 4×4 matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & (A) & \\ 0 & & & \end{pmatrix}.$$

Show that this is in $L(4)$ and that the product of any two such matrices is one of the same type, so that they form a subgroup. These are pure rotations of the spatial axes (relative velocities of the two frames are zero). Transformations like Eq. (1.12) are pure boosts (spatial axes aligned, relative velocity nonzero). The most general Lorentz transformation involves both boost and rotation.

34 (c) $g_{uu} = g_{vv} = 0, g_{uv} = -1/2, g_{uy} = g_{uz} = g_{yz} = 0, g_{yy} = g_{zz} = 1.$

(e) $\tilde{\mathrm{d}}u = \tilde{\mathrm{d}}t - \tilde{\mathrm{d}}x,\ \tilde{\mathrm{d}}v = \tilde{\mathrm{d}}t + \tilde{\mathrm{d}}x, \mathbf{g}(\vec{e}_{u,}\ ) = -\tilde{\mathrm{d}}v/2, \mathbf{g}(\vec{e}_{v,}) = -\tilde{\mathrm{d}}u/2.$

Notice that the basis dual to $\{\vec{e}_u, \vec{e}_v\}$, which is $\{\tilde{\mathrm{d}}u, \tilde{\mathrm{d}}v\}$, is not the same as the set of one-forms mapped by the metric from the basis vectors.

Chapter 4

1 (a) No. (b) Yes. (c) Dense traffic can indeed resemble a continuum, and traffic congestion on highways often shows wave-like behavior, with cars moving through a compression zone (slow speed) into a rarefaction zone (fast speed) and back into a compression zone. See, for references, the Wikipedia article on "Traffic Flow". (d) This is one situation where cars are not a continuum: they move through the intersection one by one. (e) If the plasma is dense enough then it is a continuum, but very rarified plasmas, particularly in astronomy, can be difficult to describe fully this way.

2 Particles contributing to this flux need not be moving exclusively in the x direction. Moreover, consider a change to the non-orthogonal coordinates $(t, x, y' =: x + y, z)$. A surface of constant x is unchanged, so the flux across it is unchanged, but the 'x direction', which is the direction in which now t, y', and z are constant, is in the old $\vec{e}_x + \vec{e}_y$ direction. Is the unchanged flux now to be regarded as a flux in this new direction as well? The loose language carries an implicit assumption of orthogonality of the coordinates in it.

3 (a) In Galilean physics $\boldsymbol{p}$ changes when we change frames, but in relativity $\vec{p}$ does not: only its components change.

(b) This is because the usual Galilean momentum is only a three-vector. However, if in Galilean spacetime we define a four-vector $(m, \boldsymbol{p})$ then the Galilean transformation changes this to $(m, \boldsymbol{p} - mv)$, where v is the relative velocity of the two frames. This is an approximation to the relativistic one (see Eq. (2.21)) in which terms of order v^2 are neglected.

4 The required density is, by definition, N^0 in the frame in which $\vec{U} \to (1, 0, 0, 0)$. In this frame $\vec{N} \cdot \vec{U} = -N^0$.

8 (a) Consider a two-dimensional space whose coordinates are, say, p and T, each point of which represents an equilibrium state of the fluid for that p and T. In such a space the $\mathrm{d}\rho$ of Eq. (4.25) is just $\langle \tilde{\mathrm{d}}\rho, \vec{\Delta} \rangle$ where $\vec{\Delta}$ is whatever vector points from the old state to the new one, the change in state contemplated in Eq. (4.25). Since we want Eq. (4.25) to hold for arbitrary $\vec{\Delta}$, it must hold in the one-form version. See B.F. Schutz, *Geometrical Methods of Mathematical Physics* (Cambridge University Press, Cambridge, 1980).

(b) If $\tilde{\Delta} q = \tilde{\mathrm{d}} q$, then $T\partial S/\partial x^i = \partial q/\partial x^i$, where x^i is either p or T. The identity $\partial^2 q/\partial T \partial p = \partial^2 q/\partial p \partial T$ implies $(\partial T/\partial p)(\partial S/\partial T) = \partial S/\partial p$, which will almost never be true.

9 Follow the steps leading to Eq. 4.33 but changing the first index '0' to, say, 'x'. Use the form of Newton's law that says that the force is the rate of change of the momentum. Interpret the terms analogous to those on the left-hand-side of Eq. 4.30 as fluxes of momentum.

11 (a) By definition of 'rotation': see Exer. 20b, § 3.10.

(b) Suppose M has the property $O^{\mathrm{T}}MO = M$ for any orthogonal matrix O. Consider the special case of a rotation about x^3, where $O_{11} = \cos\theta, O_{12} = \sin\theta, O_{21} = -\sin\theta, O_{22} = \cos\theta, O_{33} = 1$, all other elements zero. Then $O^T MO = M$ for arbitrary θ implies $M_{13} = M_{23} = M_{31} = M_{32} = 0, M_{11} = M_{22}$, and $M_{21} = -M_{12}$. By relabeling, a rotation about x implies $M_{12} = M_{21} = 0, M_{33} = M_{22}$. Therefore M is proportional to I.

13 $U_\alpha = \eta_{\alpha\gamma}U^\gamma; \eta_{\alpha\gamma}$ is constant, so $\eta_{\alpha\gamma,\beta} = 0; U^\alpha{}_{,\beta}U^\gamma\eta_{\alpha\gamma} = U^\gamma{}_{,\beta}u^\alpha\eta_{\gamma\alpha}$ (relabeling) $= U^\gamma{}_{,\beta}U^\alpha\eta_{\alpha\gamma}$ (symmetry of η).

14 Since $\vec{U} \to_{\mathrm{MCRF}} (1,\ 0,\ 0,\ 0)$, multiplying by U_α and summing on α picks out the zero component in the MCRF.

16 There is no guarantee that the MCRF of one element is the same as that of its neighbor.

20 (a) In Eq. 4.58, let $\vec{V}$ be $\vec{N}$ but now show that the expression does not equal zero, but instead equals the integral of ϵ over the four-dimensional volume. Interpret the result as the difference between the change in the number of particles and the number that have entered over the boundaries. Show that this means that ϵ is the rate of creation of particles per unit volume per unit time.

(b) F^0 is the rate of generation of energy per unit volume, and F^i is the ith component of the force. F^α is the only self-consistent generalization of the concept of force to relativity.

21 (a) $T^{\alpha\beta} = \rho_0 U^\alpha U^\beta, \vec{U} \to \gamma(1, \beta, 0, 0), \gamma = (1-\beta^2)^{-1/2}$.

(b) At any point on the ring, the particles have speed ωa. At position (x, y) we have $U^\alpha \to \gamma(1, -\omega y, \omega x, 0), \gamma = (1-\omega^2 a^2)^{-1/2}$. In the inertial frame their number density is $N[2\pi^2 a(\delta a)^2]^{-1}$, which equals

nU^0, where n is the number density in their rest frame. Therefore $n = N[2\pi^2\gamma a(\delta a)^2]^{-1}$ and $T^{\alpha\beta} = mnU^\alpha U^\beta$.

(c) Add (b) to itself with $\omega \to -\omega$. For example, $T^{0x} = 0$ and $T^{xx} = 2mn\omega^2 y^2\gamma^2$.

22 No bias means that T^{ij} is invariant under rotations. By Exer. 11 $T^{ij} = p\delta^{ij}$ for some p. Since $T^{0i} = 0$ in the MCRF, Eq. (4.36) holds. Clearly $\rho = \gamma nm$, where $\gamma = (1 - v^2)^{-1/2}$. The contributions of each particle to T^{zz}, say, will be the momentum flux it represents. For a particle with speed v in the direction (θ, ϕ), this is a z component of momentum $m\gamma v \cos\theta$ carried across a $z =$ const. surface at a speed $v\cos\theta$. If there are n particles per unit volume, with random velocities, then $T^{zz} = n(m\gamma v)(v)$ times the average value of $\cos^2\theta$ over the unit sphere. This is $\frac{1}{3}$, so $T^{zz} = p = \gamma nmv^2/3$. Thus, $p/\rho = v^2/3 \to \frac{1}{3}$ as $v \to 1$. (In this limit $m\gamma$ remains finite, the energy of each photon.)

23 This exercise prepares us for computing the quadrupole radiation of gravitational waves in Ch. 9. Note that, although the stress-energy tensor's components vanish outside a bounded region of space, the domain of integration in the spatial integrals in this exercise can still be taken to be all of space: since the divergence vanishes outside the system, integrating over the exterior does not add anything. This seemingly small detail will be used in all parts of this exercise.

(a) The integral is over spatial variables, so the partial derivative with respect to time may be brought inside, where it operates only on $T^{0\alpha}$. Then use the identity $T^{\mu\nu}{}_{,\nu} = 0$ to replace $T^{0\alpha}{}_{,0}$ by $-T^{j\alpha}{}_{,j}$. This is a spatial divergence, so its integral over d^3x converts to a surface integral, by Gauss' law in three dimensions. Since the region of integration in space is unbounded, this surface can be taken anywhere outside the domain of the body, where the stress-energy components all vanish. That means that the surface integral vanishes, and this proves the result.

(b) The computation follows that in (a) closely. First bring one time-derivative inside the integral, replace $T^{00}{}_{,0}$ by $-T^{0k}{}_{,k}$, and this time integrate by parts on x^k. The integrated term is a full divergence, and vanishes as above. But the integration by parts leaves another term involving $\partial(x^i x^j)/\partial x^k$. Since partial derivatives of one coordinate with respect to another are zero, the only terms that survive this differentiation are where the indices i or j equal k. We can write the result using Kronecker deltas: $\partial(x^i x^j)/\partial x^k = \delta^i{}_k x^j + \delta^j{}_k x^i$. The integral involves this expression multiplied by T^{0k} and summed on k. This summation leaves a relatively simple integrand: $-T^{0i}x^j - T^{0j}x^i$. But this integral still has one further time-derivative outside it. Bring this one in now, let it operate on the components of $T^{\mu\nu}$ (it does not affect the coordinates x^i and x^j), and again replace these time-derivatives by the spatial divergences. Again integrate by parts and evaluate the full divergence on a surface outside the body. There will be more Kronecker deltas, but when summed they will give the simple result.

(c) This is a variation on the procedure in (b). Follow the same steps, but there are now more factors of x^k so the result will be the one given in the problem.

24 (c) $R/x = [(1-v)/(1+v)]^{1/2}$.

25 (h) $E^{\bar{x}} = E^x$; let $E^y\boldsymbol{e}_y + E^z\boldsymbol{e}_z$ be called $E_\perp$, the part of $\boldsymbol{E}$ perpendicular to $\boldsymbol{v}$. Similarly, let $\boldsymbol{E}'_\perp = E^{\bar{y}}\boldsymbol{e}_{\bar{\nu}} + E^{\bar{z}}\boldsymbol{e}_{\bar{z}}$. Then $\boldsymbol{E}'_\perp = \gamma(\boldsymbol{E}_\perp + \boldsymbol{v}\times\boldsymbol{B})$.

Chapter 5

3 (b) (i) Good except at origin $x = y = 0$: usual polar coordinates. (ii) Undefined for $x < 0$, fails at $x = 0$, good for $x > 0$: maps the right-hand plane of (x, y) onto the whole plane of (ξ, η). (iii) Good except at origin and infinity: an inversion of the plane through the unit circle.

4 A vector has a slope which is the ratio of its y- and x-components. For the given vector this is $(\mathrm{d}y/\mathrm{d}\lambda)/(\mathrm{d}x/\mathrm{d}\lambda) = \mathrm{d}y/\mathrm{d}x$, which is the slope of the curve.

5 (a) and (b) have same path, the unit circle $x^2 + y^2 = 1$. But their tangent vectors are different even at the same point, because the parametrization is different. Additionally, for this problem the points $\lambda = 0$ in (a) and $t = 0$ in (b) are different.

6 Recall the solution to Exer. 3.8.

7

$$\begin{aligned}
\Lambda^{1'}{}_1 &= x/r &&= \cos\theta; \\
\Lambda^{2'}{}_1 &= -y/r^2 &&= -\tfrac{1}{r}\sin\theta; \\
\Lambda^{1'}{}_2 &= y/r &&= \sin\theta; \\
\Lambda^{2'}{}_2 &= x/r^2 &&= \tfrac{1}{r}\cos\theta; \\
\Lambda^{1}{}_{1'} &= x/r &&= \cos\theta; \\
\Lambda^{2}{}_{1'} &= y/r &&= \sin\theta; \\
\Lambda^{1}{}_{2'} &= -y &&= -r\sin\theta; \\
\Lambda^{2}{}_{2'} &= x &&= r\cos\theta.
\end{aligned}$$

8 (a)

$$\begin{aligned}
f &= r^2(1 + \sin 2\theta) \\
V^r &= r^2(\cos^3\theta + \sin^3\theta) + 6r(\sin\theta\cos\theta) \\
V^\theta &= r\sin\theta\,\cos\theta(\sin\theta - \cos\theta) + 3(\cos^2\theta - \sin^2\theta) \\
W^r &= \cos\theta + \sin\theta \\
W^\theta &= (\cos\theta - \sin\theta)/r
\end{aligned}$$

(b)

$$(\tilde{\mathrm{d}}f)_x = 2x + 2y; (\tilde{\mathrm{d}}f)_y = 2x + 2y$$
$$(\tilde{\mathrm{d}}f)_r = 2r(1 + \sin 2\theta) = \partial f/\partial r$$
$$(\tilde{\mathrm{d}}f)_\theta = 2r^2 \cos 2\theta = \Lambda^1{}_{2'}(\tilde{\mathrm{d}}f)_x + \Lambda^2{}_{2'}(\tilde{\mathrm{d}}f)_y$$

(c)(i)

$$\begin{aligned}
(\tilde{V})_r &= V^r = r^2(\cos^3\theta + \sin^3\theta) + 6r(\sin\theta\cos\theta) \\
(\tilde{V})_\theta &= r^2 V^\theta = r^3 \sin\theta\,\cos\theta(\sin\theta - \cos\theta) + 3r^2(\cos^2\theta - \sin^2\theta) \\
(\tilde{W})_r &= W^r = \cos\theta + \sin\theta \\
(\tilde{W})_\theta &= r^2 W^\theta = r(\cos\theta - \sin\theta)
\end{aligned}$$

(c)(ii) Same result by a different method, e.g.:

$$(\tilde{W})_r = \Lambda^1{}_{1'}(\tilde{W})_x + \Lambda^2{}_{1'}(\tilde{W})_y = \cos\theta + \sin\theta$$

10 The key thing is to prove linearity. The lower index, associated with the derivative, is linear in whatever vector we give it, just as is the derivative of a scalar function. So if we double the vector argument, we get a derivative twice as large, since it has to approximate the change in the function when we go twice as far. Similarly, the one-form argument is linear (associated with the upper index) because of Eq. 5.52.

11 (a) The Christoffel symbols are zero in Cartesian coordinates so the result is:

$$V^x{}_{,x} = V^x{}_{;x} = \partial V^x/\partial x = 2x;\ V^x{}_{,y} = 3;\ V^y{}_{,x} = 3;\ V^y{}_{,y} = 2y.$$

(b) Although it is possible to do this using matrices, the straightforward expansion of the summations in the transformation equation is less error prone. Thus, the $r - r$ component of $\nabla\vec{v}$ is

$$\begin{aligned}
V^r{}_{;r} &= \Lambda^{1'}{}_\alpha \Lambda^\beta{}_{1'} V^\alpha{}_{,\beta} \\
&= \Lambda^{1'}{}_1 \Lambda^1{}_{1'} V^1{}_{,1} + \Lambda^{1'}{}_2 \Lambda^1{}_{1'} V^2{}_{,1} + \Lambda^{1'}{}_1 \Lambda^2{}_{1'} V^1{}_{,2} + \Lambda^{1'}{}_2 \Lambda^2{}_{1'} V^2{}_{,2} \\
&= 2r(\cos^3\theta + \sin^3\theta) + 6\sin\theta\cos\theta.
\end{aligned}$$

Other components are:

$$V^r{}_{;\theta} = 2r^2 \sin\theta\cos\theta(\sin\theta - \cos\theta) + 3r(\cos^2\theta - \sin^2\theta)$$
$$V^\theta{}_{;r} = 2\sin\theta\cos\theta(\sin\theta - \cos\theta) + 3(\cos^2\theta - \sin^2\theta)/r$$
$$V^\theta{}_{;\theta} = 2r\sin\theta\cos\theta(\sin\theta + \cos\theta) - 6\sin\theta\cos\theta$$

(c) This gives the same as (b).

(d) $2(x+y)$.

(e) $2r(\sin\theta + \cos\theta)$, same as (d).

(f) Same as (d).

12 (a) Same components as in Exer. 11a.

(b),(c) These components are the same for both (b) and (c) and are related to the answers given for Exer. 11c as follows: $p_{r;r} = V^r{}_{;r}, p_{r;\theta} = V^r{}_{;\theta}, p_{\theta;r} = r^2 V^\theta{}_{;r}, p_{\theta;\theta} = r^2 V^\theta{}_{;\theta}$. It happens that $p_{\theta;r} = p_{r;\theta}$ for this one-form field. This is not generally true, but happens in this case because $\tilde{p}$ is the gradient of a function.

14 Two selected results: $A^{r\theta}{}_{;\theta} = r(1 + \cos\theta - \tan\theta)$; $A^{rr}{}_{;r} = 2r$.

15 Of the first-derivative components, only $V^\theta{}_{;\theta} = 1/r$ is nonzero. Of the second-derivative components, the only nonzero ones are $V^\theta{}_{;r;\theta} = -1/r^2, V^r{}_{;\theta;\theta} = -1$, and $V^\theta{}_{;\theta;r} = -1/r^2$. Note that this vector field is just the unit radial vector of polar coordinates.

17

$$\frac{\partial \vec{e}_{\mu'}}{\partial x^{\nu'}} = \left[\frac{\partial}{\partial x^\beta}(\Lambda^\alpha{}_{\mu'}\vec{e}_\alpha)\right]\Lambda^\beta{}_{\nu'} = \Lambda^\alpha{}_{\mu'}\Lambda^\beta{}_{\nu'}\frac{\partial \vec{e}_\alpha}{\partial x^\beta} + \Lambda^\alpha{}_{\mu',\beta}\Lambda^\beta{}_{\nu'}\vec{e}_\alpha$$
$$\Rightarrow \Gamma^{\lambda'}{}_{\mu'\nu'} = \Lambda^\alpha{}_{\mu'}\Lambda^\beta{}_{\nu'}\Lambda^{\lambda'}{}_\gamma\Gamma^\gamma{}_{\alpha\beta} + \Lambda^\alpha{}_{\mu',\beta}\Lambda^\beta{}_{\nu'}\Lambda^{\lambda'}{}_\alpha$$

20 $\Gamma^\nu{}_{\alpha\beta} = \frac{1}{2}g^{\nu\mu}(g_{\mu\alpha,\beta} + g_{\mu\beta,\alpha} - g_{\alpha\beta,\mu} + c_{\alpha\mu\beta} + c_{\beta\mu\alpha} - c_{\mu\alpha\beta})$.

21 (a) Compute the vectors $(\mathrm{d}t/\mathrm{d}\lambda,\ \mathrm{d}x/\mathrm{d}\lambda)$ and $(\mathrm{d}t/\mathrm{d}a,\ \mathrm{d}x/\mathrm{d}a)$ and show they are orthogonal.

(b) For arbitrary a and λ, x and t obey the restriction $|x| > |t|$. The lines $\{x > 0, t = \pm x\}$ are the limit of the $\lambda =$ const. hyperbolae as $a \to 0^+$, but for any finite λ the limit $a \to 0^+$ takes an event to the origin $x = t = 0$. To reach $x = t = 1$, for example, one can set $\lambda = \ln(2/a)$, which sends $\lambda \to \infty$ as $a \to 0$.

(c) $g_{\lambda\lambda} = -a^2$, $g_{aa} = 1$, $g_{a\lambda} = 0$, $\Gamma^\lambda_{a\lambda} = 1/a$, $\Gamma^a{}_{\lambda\lambda} = a$, all other Christoffel symbols zero. Note the close analogy to Eqs. (5.3), (5.31), and (5.45). Note also that $g_{\lambda\lambda}$ (the time-time component of the metric) vanishes on the null lines $|x| = |t|$, another property we will see again when we study black holes.

22 Use Eq. (5.68).

Chapter 6

1 (a) Yes, singular points depend on the system.

(b) Yes: if we define $r = (x^2 + y^2)^{1/2}$, the map $\{X = (x/r)\tan(\pi r/2), Y = (y/r)\tan(\pi r/2)\}$ shows that, as a manifold, the interior of the unit circle ($r < 1$) is indistinguishable from the whole plane (X, Y arbitrary). This map distorts distances, but the metric is not part of the definition of the manifold.

(c) No, discrete.

(d) This consists of the unit circle and the coordinate axes. It has the structure of a one-dimensional manifold everywhere except at the five intersection points, such as (1, 0).

2 (a) Normally no metric is used on this manifold: since the axes represent physically different quantities, with different units, no combination like $p^2 + q^2$ is normally useful or meaningful.

(b) The usual Euclidean metric is normally used here.

(d) On the different segments the one-dimensional Euclidean metric (length) is used.

4 (a) Symmetry on (γ', μ') means there are $\frac{1}{2}n(n+1)$ = ten independent pairs (γ', μ'). Since α can assume four values independently, there are 40 coefficients.

(b) The number of symmetric combinations $(\lambda', \gamma', \mu')$ is $\frac{1}{6}n(n+1) \times (n+2) = 20$, times four for α, gives 80.

(c) Two symmetric pairs of ten independent combinations each give 100.

5 Carefully repeat the flat-space argument in a local inertial frame.

6 $g^{\alpha\beta}g_{\beta\mu,\alpha} = g^{\beta\alpha}g_{\beta\mu,\alpha}$ (symmetry of metric) $= g^{\alpha\beta}g_{\alpha\mu,\beta}$ (relabeling dummy indices) $= g^{\alpha\beta}g_{\mu\alpha,\beta}$ (symmetry of metric). This cancels the second term in brackets.

9 For polar coordinates in two-dimensional Euclidean space, $g = r^2$. In three dimensions it is $g = r^4 \sin^2\theta$. Notice in these cases $g > 0$, so formulae like Eqs. (6.40) and (6.42), which are derived from Eq. (6.39), should have $-g$ replaced by g.

10 The vector maintains the angle it makes with the side of the triangle as it moves along. On going around a corner the angle with the new side exceeds that with the old by an amount which depends only on the interior angle at that corner. Summing these changes gives the result.

15 Consider a spacelike curve $\{x^\alpha(\lambda)\}$ parametrized by λ, and going from $x^\alpha(a)$ to $x^\alpha(b)$. Its length is $\int_a^b [g_{\alpha\beta}(\mathrm{d}x^\alpha/\mathrm{d}\lambda)(\mathrm{d}x^\beta/\mathrm{d}\lambda)]^{1/2}/\mathrm{d}\lambda$. Assume that λ is chosen so that the integrand is constant along the given curve. Now change the curve to $\{x^\alpha(\lambda)+\delta x^\alpha(\lambda)\}$ with $\delta x^\alpha(a)=\delta x^\alpha(b)=0$. The first-order change in the length is, after an integration by parts, $\int_a^b [\frac{1}{2}g_{\alpha\beta,\gamma}U^\alpha U^\beta - \mathrm{d}(g_{\gamma\alpha}U^\alpha)/\mathrm{d}\lambda]\delta x^\gamma\, \mathrm{d}\lambda$, where $U^\alpha \equiv \mathrm{d}x^\alpha/\mathrm{d}\lambda$. For a geodesic, the term in square brackets vanishes.

18 (b) Each pair $\alpha\beta$ or $\mu\nu$ is antisymmetric, so has six independent combinations for which the component need not vanish. Each pair can be chosen independently from among these six, but the component is symmetric under the exchange of one pair with the other. (c) Eq. (6.69) allows us to write Eq. (6.70) as $R_{\alpha[\beta\mu\nu]}=0$. There are only $n(n-1)\times(n-2)/6=$ four independent choices for the combination $\beta\mu\nu$, by antisymmetry. In principle α is independent, but if α equals any one of β,μ, and ν then Eq. (6.70) reduces to one of Eq. (6.69). So Eq. (6.70) is at most four equations determined only by, say, the value α. However, Eq. (6.69) allows us also to change Eq. (6.70) to $R_{\beta[\alpha\mu\nu]}=0$. Thus, if we had earlier taken, say, $\alpha=1$ and $\beta=2$, then that equation would have been equivalent to the one with $\alpha=2$ and $\beta=1$: all values of α give the same equation.

25 Use Eqs. (6.69) and (6.70).

28 (a) We desire the components $g_{\alpha\beta}$ in the new (spherical) coordinates, so we need to compute derivatives of the Cartesian with respect to the spherical, e.g.

$$\partial x/\partial r = \sin\theta\cos\phi,$$

and so on. Then we compute the new components from the transformation law, e.g.

$$g_{r\theta} = \frac{\partial x}{\partial r}\frac{\partial x}{\partial\theta}g_{xx} + \frac{\partial x}{\partial r}\frac{\partial y}{\partial\theta}g_{xy} + \dots,$$

summing over all components of the metric in Cartesian coordinates. Since most of the components are zero the algebra is not very long.

(b) If one moves on the surface of the sphere then one is at constant r, so that $\mathrm{d}r = 0$. With this in the line element, only the second and third rows and columns of the matrix are relevant.

(c) $g^{\theta\theta} = r^{-2}$, $g^{\phi\phi} = (r\sin\theta)^{-2}$, $g^{\theta\phi} = 0$.

29 $R_{\theta\phi\theta\phi} = \sin^2\theta$.

30 The most sensible coordinates to use are Euclidean: unwrap the cylinder so that it lies flat and put a flat coordinate system on it. To make a cylinder you just have to remember that when you come to the edge of the paper you jump to the other side, where the join was. But locally, this makes no difference to the geometry. Since the metric is Euclidean, its derivatives all vanish, and the Riemann tensor vanishes.

32 Compare with Exer. 34, § 3.10.

35 The nonvanishing algebraically independent Christoffel symbols are: ${\Gamma^t}_{tr} = \Phi'$, ${\Gamma^r}_{tt} = -\Phi' \exp(2\Phi - 2\Lambda), {\Gamma^r}_{rr} = \Lambda', {\Gamma^r}_{\theta\theta} = -r\exp(-2\Lambda)$, ${\Gamma^r}_{\phi\phi} = -r\sin^2\theta\exp(-2\Lambda)$, ${\Gamma^\theta}_{r\theta} = {\Gamma^\phi}_{\bar{r}\phi} = r^{-1}, {\Gamma^\theta}_{\phi\phi} = -\sin\theta\cos\theta$, ${\Gamma^\phi}_{\theta\phi} = \cot\theta$. Here primes denote r derivatives. (Compare these with the Christoffel symbols you calculated in Exer. 29.) As explained in the solution to Exer. 18, in calculating $R_{\alpha\beta\mu\nu}$ we should concentrate on the pairs $(\alpha\beta)$ and $(\mu\nu)$, choosing each from the six possibilities $(tr, t\theta, t\phi, r\theta, r\phi, \theta\phi)$. Because $R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta}$, we do not need to calculate, say, $R_{\theta\phi r\phi}$ after having calculated $R_{r\phi\theta\phi}$. This gives 21 independent components. Again following Exer. 18, one of the components with four distinct indices, say $R_{tr\theta\phi}$, can be calculated from others. We catalog, therefore, the following 20 algebraically independent components:

$$\begin{aligned}
\boldsymbol{R}_{trtr} &= [\Phi'' - (\Phi')^2 - \Phi'\Lambda'] \exp(2\Phi), \\
\boldsymbol{R}_{trt\theta} &= \boldsymbol{R}_{trt\phi} = \boldsymbol{R}_{trr\theta} = \boldsymbol{R}_{trr\phi} = 0, \\
\boldsymbol{R}_{t\theta t\theta} &= -\boldsymbol{r}\Phi' \exp(2\Phi - 2\Lambda), \\
\boldsymbol{R}_{t\theta t\phi} &= \boldsymbol{R}_{t\theta r\theta} = \boldsymbol{R}_{t\theta r\phi} = \boldsymbol{R}_{t\theta\theta\phi} = 0, \\
\boldsymbol{R}_{t\phi t\phi} &= -\boldsymbol{r}\Phi' \sin^2\theta \exp(2\Phi - 2\Lambda), \\
\boldsymbol{R}_{t\phi r\theta} &= \boldsymbol{R}_{t\phi r\phi} = \boldsymbol{R}_{t\phi\theta\phi} = 0, \\
\boldsymbol{R}_{r\theta r\theta} &= r\Lambda', \\
\boldsymbol{R}_{r\theta r\phi} &= \boldsymbol{R}_{r\theta\theta\phi} = \boldsymbol{R}_{r\phi\theta\phi} = 0, \\
\boldsymbol{R}_{r\phi r\phi} &= r \sin^2\theta \Lambda', \\
\boldsymbol{R}_{\theta\phi\theta\phi} &= r^{-2} \sin^2\theta [1 - \exp(-2\Lambda)].
\end{aligned}$$

See if you can use the spherical symmetry and time independence of the metric to explain why certain of these components vanish. Also compare $R_{\theta\phi\theta\phi}$ with the answer to Exer.29 and see if you can explain why they are different.

36 Since $\phi = 0$ is already inertial, we can look for a coordinate transformation of the form $x^{\alpha'} = (\delta^\alpha{}_\beta + L^\alpha{}_\beta)x^\beta$, where $L^\alpha{}_\beta$ is of order ϕ. The solution to Exer. 17, § 5.9, gives $\Gamma^{\lambda'}{}_{\mu'\nu'}$, which must vanish at P. Since $\Lambda^{\alpha'}{}_\beta = \delta^\alpha{}_\beta + L^\alpha{}_\beta + L^\alpha{}_{\mu,\beta}x^\mu$, we find $L^\alpha{}_{(\beta,\nu)} = \frac{1}{2}\Gamma^\alpha{}_{\beta\nu}$ at P. The antisymmetric part, $L^\alpha{}_{[\beta,\nu]}$, is undetermined, and represents a Lorentz transformation of order ϕ. Since we are only looking for *an* inertial system, we can set $L^\alpha{}_{[\beta,\nu]} = 0$. Calculating $\Gamma^\alpha{}_{\beta\nu}$ at P (as in Exer. 3, § 7.6, below) gives the new coordinates. In particular, the equation $x^{i'} = 0$ gives the motion of the origin of the new frame, whose acceleration is $\mathrm{d}^2x^i/\mathrm{d}t^2 = -\Gamma^i{}_{tt} = -\phi_{,i}$. We shall interpret this in the next chapter, where we identify ϕ as the Newtonian gravitational potential and see that this acceleration expresses the equivalence principle.

37 (b) The ranges of the coordinates must be deduced: $0 < \chi < \pi, 0 < \theta < \pi, 0 < \phi < 2\pi$. Then the volume is $2\pi^2 r^3$.

38 $2\pi r \sin\theta$.

Chapter 7

1 Consider a fluid at rest, where $U^i = 0$ and $U^0 = 1$ in a local inertial frame. Then $\partial n/\partial t = R$, so Eq. (7.3) implies creation (or destruction) of particles by the curvature.

2 $g^{00} = -(1-2\phi), g^{ij} = \delta^{ij}(1+2\phi)$.

3 $\Gamma^0{}_{00} = \dot{\phi}$, $\Gamma^0{}_{0i} = \phi_{,i}$, $\Gamma^0{}_{ij} = -\dot{\phi}\delta_{ij}$, $\Gamma^i{}_{00} = \phi_{,i}$, $\Gamma^i{}_{0j} = -\dot{\phi}\delta_{ij}$, $\Gamma^i{}_{jk} = \delta_{jk}\phi_{,i} - \delta_{ij}\phi_{,k} - \delta_{ik}\phi_{,j}$.

7 (a)(i) In the Minkowski metric, all metric components are independent of the given coordinates (t, x, y, z), so the momentum components (p_t, p_x, p_y, p_z) are conserved.

(a)(ii) The Schwarzschild metric components are independent of time t, so the component p_t is conserved. Also, the metric does not depend on the coordinate ϕ, so that the associated momentum p_ϕ is conserved. In part (b)(ii) below we will see that this is an angular momentum.

(a)(iii) The Kerr metric is a somewhat more complicated generalization of Schwarzschild. Like Schwarzschild, however, the components are all independent of the two coordinates t and ϕ. Therefore there are two conserved momentum components, p_t and p_ϕ.

(a)(iv) The Robertson-Walker metric does depend on time t, but like the previous two it is independent of the coordinate ϕ. Therefore p_ϕ is conserved.

(b)(i) This coordinate transformation is to spherical coordinates, and it produces a metric whose components are independent of the angle ϕ. This means that p_ϕ is conserved, the angular momentum about the axis of our coordinates. But this axis could have been chosen to point in any direction, so there are three independent conserved components of angular momentum, usually referred to as (J_x, J_y, J_z). This exhausts the set of conserved quantities.

(b)(ii) Now we can see that there are more conserved quantities for Schwarzschild. The metric is written in coordinates that look like the spherical coordinates of Minkowski spacetime that we have just derived, and indeed the way it depends on the angles θ and ϕ is identical to Minkowski spacetime in spherical coordinates. This means that the metric is in fact spherically symmetric. Now, since no metric component depends on ϕ, then certainly p_ϕ is conserved. But we can also imagine performing exactly the kinds of coordinate transformations (rotations) on these coordinates as one would perform in Minkowski spacetime if one wanted to re-orient the symmetry axis of the coordinates in a different direction, and that would again produce a metric independent of the new angle ϕ. So, just as in flat spacetime, there are actually three conserved quantities associated with angles, three angular momentum components. There are no other conservation laws for Schwarzschild.

(b)(iii) There is no spherical symmetry for the Kerr metric, so we do not find any further conserved quantities.

(b)(iv) The Robertson-Walker metric is spherically symmetric in the same way as Schwarzschild and Minkowski are. Therefore there are three conserved angular momentum components. But linear momentum is not conserved.

9 (a)

$$\begin{aligned}
\boldsymbol{R}_{0i0j} &= \phi_{,ij} + \delta_{ij}\phi_{,00}, \\
\boldsymbol{R}_{0ijk} &= \phi_{,0j}\delta_{ik} - \phi_{,0k}\delta_{ij}, \\
\boldsymbol{R}_{ijkl} &= \delta_{ik}\phi_{,jl} + \delta_{jl}\phi_{,ik} - \delta_{jl}\phi_{,ik} - \delta_{jk}\phi_{,il}.
\end{aligned}$$

(c) The 'acceleration', in Newtonian language, is $-\phi_{,i}$. The difference between the accelerations of nearby particles separated by ξ^j is therefore $-\xi^j\phi_{.ij}$.

10 (b) In terms of a Lorentz basis, the following vector fields are Killing fields: $\vec{e}_t, \vec{e}_x, \vec{e}_y, \vec{e}_z, x\vec{e}_y - y\vec{e}_x, y\vec{e}_z - z\vec{e}_y, z\vec{e}_x - x\vec{e}_z, t\vec{e}_x + x\vec{e}_t, t\vec{e}_y + y\vec{e}_t, t\vec{e}_z + z\vec{e}_t$. Any linear combination with constant coefficients of solutions to Eq. (7.45) is a solution to Eq. (7.45), which means that the analogous fields to these in another Lorentz frame are derivable from these.

Chapter 8

2 (c) (i) 7.425×10^{-11} m^{-2}; (ii) 8.261×10^{-12} m^{-2}; (iii) 1.090×10^{-16} m^{-1}; (iv) 2.756×10^{-12}.

(d) $\boldsymbol{m}_{\rm PL} = 2.176\times10^{-8}$ kg; $t_{\rm PL} = 5.390\times10^{-44}$ s. Typical elementary-particle lifetimes are 10^{-24} s or greater. The heaviest known particles are less than 10^{-23} kg.

3 (a)(i) -2.122×10^{-6}; (ii) -9.873×10^{-9}; (iii) -6.961×10^{-10}; (iv) 9.936×10^{-5}.

(b) We and everything on Earth are in free-fall around the Sun, so the value of the Sun's potential does not matter, nor even the value of its gradient (the gravitational acceleration the Sun produces on us). What we do experience is the tidal force of the Sun, which is comparable in size to that of the Moon, and which raises ocean tides and distorts Earth's shape. The tidal forces arise in the second derivatives of the Sun's potential, *i.e.* the differences in the Sun's acceleration across Earth. We experience Earth's gravity, even though its potential is much smaller than the Sun's, because we are not in free-fall on Earth. The forces we feel from our weight are actually the forces of the ground or floor pushing up on us, stopping us from falling.

5 (a) This is an exercise in "index gymnastics". Make sure that you use all possible symmetries, in this case the freedom to exchange the order of partial derivatives. (b) Since the expression is linear, a gauge transformation is just the same as adding a pure-gauge value, and we have just shown that this vanishes. Therefore the Riemann tensor at linear order is gauge-invariant.

9 (b) No contradiction because this expression was derived by applying the gauge condition Eq. 8.33, as is computed explicitly in the next exercise, Ex. 10. One could use this condition to remove the time-derivatives from the time-components of Eq. 8.42, at the expense of making them more complicated to write down. The important thing is that the gauge condition Eq. 8.33 creates a relationship among the differential equations in Eq. 8.42, so that there are still only 6 independent equations with second-time derivatives.

11 Gauge transformation: $A_\mu \to A_\mu + f_{,\mu}$. Lorentz gauge condition: $A^\mu{}_{,\mu} = 0$. These are very similar to Eq. 8.24 and Eq. 8.33. The differences are all to do with the extra index needed in general relativity because our fundamental field is a tensor (the metric) rather than a vector.

13 The ratio of momentum to mass is velocity, so the ratio of the corresponding densities, T^{0i}/T^{00}, will be of order v, where v is a typical velocity. In the low-velocity limit this will be small. The argument for the stresses is similar but more subtle. In a fluid the stress components are just pressures, and when one uses statistical mechanics to derive pressure from the random motions of particles of the gas, one finds $p \sim \rho v^2$. Therefore in this case $T^{ij}/T^{0i} \ll 1$. In more general materials, the stresses can be made of direct forces between particles, but even here they must be of the same size: too great a stress would create forces that would accelerate a body to relativistic speeds, violating the low-velocity assumption.

17 (a) $\boldsymbol{M} = \boldsymbol{R}v^2 = (C^3/2\pi P^2) \sim 1.3\,\mathrm{km} \sim 1\,\mathrm{M}_\odot$. (b) $100\,\mathrm{km} \sim 68\,\mathbf{M}_\odot$.

18 (a)$|\Lambda| \lesssim 4 \times 10^{-35}\,\mathrm{m}^{-2}$. (b) This value of Λ represents a mass density that one would get by spreading the mass of the Sun over the whole Solar System. This is very much bigger than the average density of the universe, since the nearest star is much further away than Pluto, and on top of that there are vast empty spaces between galaxies. So a cosmological constant of this size would have enormous and easily observable consequences for the evolution of the universe. It follows that a Λ of the same size as the cosmological mass density would not be observable in the Solar System.

19 (a) $T^{00} = \rho$, $T^{01} = -\rho\Omega x^2$, $T^{02} = \rho\Omega x^1$, $T^{03} = 0$. The components T^{ij} are not fully determined by the given information, but they must be of order $\rho v^i v^j$, i.e. of order $\rho\Omega^2 R^2$.

(b) Since $\nabla^2 \bar{h}^{00} = -16\pi\rho, \bar{h}^{00}$ is just minus four times the Newtonian potential, $\bar{h}^{00} = 4M/r$ exactly. For $\bar{h}^{0i}$ we have

$$\bar{h}^{0i} = -4\rho\Omega \int y^2 |x - y|^{-1} \mathrm{d}^3 y.$$

Use the binomial expansion

$$|x - y|^{-1} = r^{-1}[1 + \boldsymbol{x} \cdot \boldsymbol{y}/r^2 + 0(R/r)^2].$$

By symmetry,

$$\int y^i \mathrm{d}^3 y = 0, \quad \int y^i y^j \mathrm{d}^3 y = 0 \text{ if } i \neq j, \text{and}$$

$$\int y^1 y^1 \mathrm{d}^3 y = \int y^2 y^2 \mathrm{d}^3 y = \int y^3 y^3 \mathrm{d}^3 y = (4\pi/15) R^5.$$

This implies $\bar{h}^{01} = -(16\pi/15)\rho R^5 \Omega x^2/r^3$. In terms of the angular momentum J, we find

$$\bar{h}^{01} = -2Jx^2/r^3, \quad \bar{h}^{02} = 2Jx^1/r^3, \quad \bar{h}^{03} = 0.$$

These fall off as r^{-2} and are correct to order r^{-3}. A more careful study of the properties of the solutions of $\nabla^2 f = g$ would show that these are in fact exact: the higher-order terms all vanish in this simple situation. The components $\bar{h}^{ij}$ are small compared to $\bar{h}^{00}$ and $\bar{h}^{0i}$ because T^{ij} is small. Therefore Eq. (8.31) gives $h^{\mu\nu}$ and the metric

$$\begin{aligned} g_{00} &= -1 + 2M/r + 0(\Omega^2 R^2), \\ g_{01} &= 2Jx^2/r^3 + 0(\Omega^3 R^3), \quad g_{02} = -2Jx^1/r^3 + 0(\Omega^3 R^3), \quad g_{03} = 0, \\ g_{ij} &= \delta_{ij}(1 + 2M/r) + 0(\Omega^2 R^2). \end{aligned}$$

Compare this with Eq. (7.8). In standard spherical coordinates, g_{00} is the same, $g_{0r} = g_{0\theta} = 0, g_{0\phi} = -2J\sin^2\theta/r$, and the spatial line element is

$$\mathrm{d}l^2 = [1 + 2M/r - 0(\Omega^2 R^2)](\mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \; \mathrm{d}\phi^2).$$

(c) Such a particle obeys the geodesic equation with $p_0 := -E$ and $p_\phi := L$ constant, p^r and p^θ zero. To this order, the normalization $\vec{p} \cdot \vec{p} = -m^2$ implies $E = m(1 - M/r + L^2/2m^2r^2)$, just as in Newtonian theory (except for the rest mass). The r component of the geodesic equation implies, again to lowest order, $L = m(Mr)^{1/2}$, again as in Newtonian theory. One orbit, $\Delta\phi = 2\pi$, will take a time $\Delta t = (\mathrm{d}t/\mathrm{d}\phi)\Delta\phi$. Now $\mathrm{d}t/\mathrm{d}\phi = (\mathrm{d}t/\mathrm{d}\tau)/(\mathrm{d}\phi/\mathrm{d}\tau) = U^0/U^\phi = p^0/p^\phi$, and this can be expressed in terms of E, L, and the metric; a straightforward calculation gives $(\Delta t)_{\text{prograde}} - (\Delta t)_{\text{retrograde}} = -8\pi J/M$, independently of r. In principle, this allows measurement of a body's angular momentum by the study of particle orbits far from it.

(d) 0.16 ms.

Chapter 9

7 A solution of Eq. (9.22) is *uniquely* determined by the initial position and U^α. The function $U^\alpha = \delta^\alpha{}_0$ satisfies Eq. (9.22) for all time (by virtue of Eq. (9.23)), and so must be the unique solution for initial data in which $U^\alpha = \delta^\alpha{}_0$.

8 No: the equivalence principle.

9 For the light beam, $\mathrm{d}s^2 = 0 \Rightarrow \mathrm{d}t/\mathrm{d}x = (g_{xx}/|g_{tt}|)^{1/2} \approx 1 + \frac{1}{2}h^{TT}_{xx}(t)$. Therefore, if the remote particle is at coordinate location $x = \epsilon$, the time elapsed for a round trip of light is $\Delta t \approx (2 + \langle h^{\mathrm{TT}}_{xx}\rangle)\varepsilon$, where $\langle h^{\mathrm{TT}}_{xx}\rangle$ is some mean value of h^{TT}_{xx} during the time of flight of the photon. Since $\langle h^{\mathrm{TT}}_{xx}\rangle$ changes with time while ε does not, free particles do see accelerations relative to their neighbors.

10 (a) For example, $R_{tyxz} = -\frac{1}{2}\omega^2 h_{yz}$; (d) $\xi_t = -\frac{1}{2}B(x-t)^2, \xi_i = 0$.

13 One way to do this is with rotation matrices. But consider a more direct way. Write down the new basis vectors in terms of the old:

$$\vec{e}_{x'} = (\vec{e}_x + \vec{e}_y)/\sqrt{2}, \quad \vec{e}_{y'} = (\vec{e}_y - \vec{e}_x)/\sqrt{2}.$$

The components of the tensor $\boldsymbol{h}^{TT}$ are its values on the basis vectors, so its components in the new frame are, for example, $h^{TT}_{x'x'} = \boldsymbol{h}^{TT}(\vec{e}_{x'}, \vec{e}_{x'})$. Putting the expressions for the new basis vectors into this gives $h^{TT}_{x'x'} = h^{TT}_{xy} + (h^{TT}_{xx} + h^{TT}_{yy})/2$. The vanishing trace gives the desired result. (Note the sign errors in the statement of the problem!)

20 In the TT coordinates of the wave, choose the wave's direction to be z, the ellipse's principal axes to be x and y, and the masses' separation to be along the unit vector $s \to (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ in the usual spherical coordinates of the frame. Then in Eq. (9.45) we replace h^{TT}_{xx} by $h^{\mathrm{TT}}_{ss} \equiv s^\alpha s^\beta h^{\mathrm{TT}}_{\alpha\beta} = \sin^2\theta(\cos 2\phi + \mathrm{i}a\sin 2\phi)h^{\mathrm{TT}}_{xx}$. There will be no driving term if $\theta = 0$, i.e. if the masses lie along the direction of the wave's propagation (compare with Exer. **??**).

26 (a) Does not violate relativity: this is just a coordinate speed, dependent on the coordinate system. Relativity insists that the proper distance divided by proper time along a light ray should always be 1, but there is no constraint on the coordinate speed. (c) The statement is correct as long as the distance is small enough that there is a local inertial frame covering both ends of the light path. If the distance is larger, and the geometry is time-dependent, then there is not a unique meaning to the distance to a remote object: it depends on time and the path being measured. (e) This is a coordinate-independent result, since it is a proper time as measured on a single clock. So it would be the same in any coordinate system.

30 $I^{lm} = \sum_A m_{(A)} x^i_{(A)} x^i_{(A)}$.

31 (a) $I^{ij} = (4\pi/3)\delta^{ij} \int \rho r^4 \mathrm{d}r$; $\rlap{-}I^{ij} = 0$.

(b) $I^{ij} =$: Result of (a) $+ M a^i a^j$; $\rlap{-}I^{ij} = M(a^i a^j - \frac{1}{3}\,\delta^{ij} a^2)$.

(c)

$$\begin{aligned} I^{xx} &= a^2 M/5, \quad I^{yy} = b^2 M/5, \quad I^{zz} = c^2 M/5, \\ \rlap{-}I^{xx} &= (2a^2 - b^2 - c^2)M/15, \\ \rlap{-}I^{yy} &= (2b^2 - a^2 - c^2)M/15, \\ \rlap{-}I^{zz} &= (2c^2 - a^2 - b^2)M/15, \end{aligned}$$

all other components zero.

(d)

$$\begin{aligned} I^{xx} &= (a^2 \cos^2 \omega t + b^2 \sin^2 \omega t)M/5, \\ I^{yy} &= (b^2 \cos^2 \omega t + a^2 \sin^2 \omega t)M/5, \\ I^{zz} &= c^2 M/5, \\ I^{xy} &= \cos\,\omega t\,\sin\,\omega t(a^2 - b^2)M/5, \\ \rlap{-}I^{xx} &= [a^2(3\cos^2 \omega t - 1) + b^2(3\sin^2 \omega t - 1) - c^2]M/15, \\ \rlap{-}I^{yy} &= [b^2(3\cos^2 \omega t - 1) + a^2(3\sin^2 \omega t - 1) - c^2]M/15, \\ \rlap{-}I^{zz} &= (2c^2 - a^2 - b^2)M/15, \\ \rlap{-}I^{xy} &= I^{xy}, \end{aligned}$$

others zero.

(e) $I^{xx} = 2ma^2 = I^{yy}$, others zero;
$\rlap{-}I^{xx} = 2ma^2/3 = \rlap{-}I^{yy}, \rlap{-}I^{zz} = -4ma^2/3$, others zero.

(f) Same as (e).

(g) $I^{xx} = 2m(A^2 \cos^2 \omega t + Al_0 \cos \omega t + l_0^2/4)$, others zero;
$\rlap{-}I^{xx} = 2I^{xx}/3, \rlap{-}I^{yy} = -I^{xx}/3 = \rlap{-}I^{zz}$, others zero.

(h) $I^{xx} = (m + M)(A^2 \cos^2 \omega t + Al_0 \cos \omega t + l_0^2/4)$, other zero;
$\rlap{-}I^{xx} = 2I^{xx}/3, \rlap{-}I^{yy} = -I^{xx}/3 = \rlap{-}I^{zz}$, and $\omega^2 = k/\mu$, where $\mu = mM/(M+m)$ is the reduced mass.

35 There is no radiation in (a)–(c) and (e), and no *quadrupole* radiation in (f).

For (d), first put the time-dependence of $\mathcal{I}_{ij}$ from Exer. 31 into complex form, e.g., $\mathcal{I}_{xx} = \frac{1}{10}M(a^2 - b^2)\exp(-2\mathrm{i}\omega t)$. Then use Eqs. (9.84)–(9.86) with $\Omega = 2\omega$ and the correct permutations of the various indices. Results:

$$\begin{aligned}
\text{Along } x\text{-axis: } \bar{h}^{\mathrm{TT}}_{zz} &= -\bar{h}^{\mathrm{TT}}_{yy} = -\tfrac{2}{5}M\omega^2(a^2 - b^2)\exp[-2\mathrm{i}\omega(t-r)]/r, \\
\bar{h}^{\mathrm{TT}}_{zy} &= 0. \\
\text{Along } y\text{-axis: } \bar{h}^{\mathrm{TT}}_{zz} &= -\bar{h}^{\mathrm{TT}}_{xx} = \tfrac{2}{5}M\omega^2(a^2 - b^2)\exp[-2\mathrm{i}\omega(t-r)]/r, \\
\bar{h}^{\mathrm{TT}}_{xz} &= 0. \\
\text{Along } z\text{-axis: } \bar{h}^{\mathrm{TT}}_{xx} &= -\bar{h}^{\mathrm{TT}}_{yy} = -\tfrac{4}{5}M\omega^2(a^2 - b^2)\exp[-2\mathrm{i}\omega(t-r)]/r, \\
\bar{h}^{\mathrm{TT}}_{xy} &= \mathrm{i}\bar{h}^{\mathrm{TT}}_{xx}.
\end{aligned}$$

This means that the radiation is linearly polarized in the equatorial plane, circularly polarized along the z-axis. Notice there is no radiation if the body is axially symmetric, i.e. if $a = b$.

For (h), there is *no* radiation on the x-axis. On the y-axis:

$$\begin{aligned}
\bar{h}^{\mathrm{TT}}_{zz} &= -(m + M)\omega^2\{2A^2\exp[-2\mathrm{i}\omega(t-r)] + Al_0\exp[-\mathrm{i}\omega(t-r)]\}/r, \\
\bar{h}^{\mathrm{TT}}_{yy} &= -\bar{h}^{\mathrm{TT}}_{zz}, \\
\bar{h}^{\mathrm{TT}}_{xy} &= 0.
\end{aligned}$$

36 Without loss of generality choose the wave to be moving in the $x - y$ plane. The unit vector along its direction has components $n_x = \cos\theta$, $n_y = \sin\theta$.Then $P_{xx} = \sin^2\theta, P_{yy} = \cos^2\theta, P_{zz} = 1, P_{xy} = -\sin\theta\cos\theta$, others zero. By matrix multiplication or by just writing out all the summation terms, we find $\bar{h}^{\mathrm{TT}}_{xx} = f\sin^2\theta, \bar{h}^{\mathrm{TT}}_{yy} = f\cos^2\theta, \bar{h}^{\mathrm{TT}}_{xy} = -f\sin\theta\cos\theta, \bar{h}^{\mathrm{TT}}_{zz} = -f$, others zero, where $f = m\omega^2\sin^2\theta\{2A^2\exp[-2\mathrm{i}\omega(t-r)] + Al_0\exp[-\mathrm{i}\omega(t-r)]\}/r$. If we let l^i be $(-\sin\theta, \cos\theta, 0)$, then $\vec{l}$ and $\vec{e}_z$ are two orthogonal vectors perpendicular to the motion of the wave. We find $\bar{h}^{\mathrm{TT}}_{ll} := l^i l^j \bar{h}^{\mathrm{TT}}_{ij} = f$. Recalling $\bar{h}^{\mathrm{TT}}_{zz} = -f$, we see that the wave is always 100% linearly polarized with the ellipse's axes in the $x - y$ plane and parallel to $\vec{e}_z$: the component rotated by 45° is absent.

38 Let the wave be traveling in the $x - z$ plane at an angle θ with the z axis. Let $\vec{e}_y$ and $\vec{l} \to (\cos\theta, 0, -\sin\theta)$ be orthogonal vectors in the plane of polarization. Then $\bar{h}^{\rm TT}_{yy} = (1 - \frac{1}{2}\sin^2\theta)f, \bar{h}^{\rm TT}_{ly} := l^i\bar{h}^{\rm TT}_{iy} = -{\rm i}\cos\theta f, \bar{h}^{\rm TT}_{ll} := l^i l^j \bar{h}^{\rm TT}_{ij} = -\bar{h}^{\rm TT}_{yy}$, where $f = 2ml_0^2\omega^2\exp[-2{\rm i}\omega(t-r)]/r$. From Exer. **??**b we see that the wave is elliptically polarized with principal axes $\vec{l}$ and $\vec{e}_y$. The percentage of circular polarization is $|\bar{h}^{\rm TT}_{ly}/\bar{h}^{\rm TT}_{yy}|^2$, which ranges from zero at the equator to 100% at the poles.

39 (a) Let the stars orbit in the x-y plane, with the center of mass at the origin. If the polar coordinates of m are (r_1, θ), both functions of time, and if those of M are $(r_2, \theta + \pi)$, then their total separation is $r = r_1 + r_2$, with $r_1/r = M/(m + M)$, $r_2/r = m/(m + M)$. In terms of these variables the quadrupole tensor follows straightforwardly:

$$I^{xx} = \mu r^2 \cos^2\theta,$$
$$I^{yy} = \mu r^2 \sin^2\theta,$$
$$I^{xy} = I^{yx} = \mu r^2 \sin\theta\,\cos\theta,$$

with $\mu = mM/(m+M)$, which is called the *reduced mass* of the binary system.

Let the reduced orbit have eccentricity e and semi-major axis a. (Note that the statement of the problem incorrectly called a the distance of closest approach). The Newtonian equations imply

$$r(t) = \frac{a(1-e^2)}{1 + e\cos[\theta(t) - \theta_0]},$$

where θ_0 is the orientation of the major axis of the ellipse, the direction of their separation when they are closest. Similarly, the angular velocity of the orbit is

$$\Omega(t) = \frac{{\rm d}\theta}{{\rm d}t} = r(t)^{-2}[(m + M)a(1 - e^2)]^{1/2}.$$

The orbital period is

$$P = 2\pi\left[\frac{a^3}{m+M}\right]^{1/2}.$$

The total energy E and angular momentum L, being conserved, can be computed at any point; the point of closest approach $\theta = \theta_0$ is convenient. The results are $E = -mM/2a$, $L^2 = \mu^2(m+M)a(1-e^2)$. From these follow the desired relations:

$$\begin{aligned}
P^2 &= -\pi^2(M+m)^2\mu^3/2E^3,\\
e^2 &= 1 + 2EL^2/(M+m)^2\mu^3,\\
a &= -mM/2E.
\end{aligned}$$

(Recall that $E < 0$.)

(b) The trace of I^{jk} is μr^2, which is not a constant (unlike the case of circular orbits). The components of $\rlap{-}I^{jk}$ are:

$$\begin{aligned}
\rlap{-}I^{xx} &= \mu r^2(\cos^2\theta - \tfrac{1}{3}),\\
\rlap{-}I^{yy} &= \mu r^2(\sin^2\theta - \tfrac{1}{3}),\\
\rlap{-}I^{xy} &= \rlap{-}I^{yx} = \mu r^2 \sin\theta\,\cos\theta,\\
\rlap{-}I^{zz} &= -\tfrac{1}{3}\mu r^2.
\end{aligned}$$

(c) The radiation requires two time-derivatives of $\rlap{-}I^{jk}$.

40 Bring out the R^{-1} as r^{-1}, as in Eq. (9.103), but expand $t - R$ about $t - r$. Eq. (9.104) and its consequences still follow. The first term is Eq. (9.105) but higher terms depend on $(\mathrm{d}/\mathrm{d}t)^n \int \rho y_i y_k \ldots y_l y_m \,\mathrm{d}^3y$, where there are n factors of y_j. If ρ is spherical, then the integrations give indices of the form $\delta_{jk} \cdots \delta_{lm}$ (see Exer. 42) and permutations. The TT projection eliminates traces and so completely eliminates these terms.

Chapter 10

1 This is essentially a repeat of Exer. 28, except with the time-dimension present. However, the transformation affects only the spatial coordinates, so the solution follows in simple way.

3 The factor $e^{-\Phi}$ in Eq. 10.11 expands in the weak-field approximation to $1-\Phi$, and in this limit the relativistic function Φ just becomes the Newtonian potential ϕ. When multiplied by the expression for $E = -p_0$ given in Eq. ch07:eqn7.34, this factor cancels the gravitational potential energy, leaving just the rest-mass and kinetic energy.

4 First calculate the components of the Ricci tensor $R_{\alpha\beta} = g^{\mu\nu}R_{\alpha\mu\beta\nu}$. This would give, for example,

$$\begin{aligned} R_{tt} &= e^{-2\Lambda}R_{trtr} + r^{-2}R_{t\theta t\theta} + (r\sin\theta)^{-2}Rt\phi t\phi \\ &= [\Phi'' - (\phi')^2 - \Phi'\Lambda' - 2\Phi'/r]\exp(2\Phi - 2\Lambda). \end{aligned}$$

Then, from all the components of the Ricci tensor one forms the Ricci scalar, $R = g^{\alpha\beta}R_{\alpha\beta}$. Finally one forms the Einstein tensor, e.g.

$$G_{tt} = R_{tt} - \tfrac{1}{2}g_{tt}R.$$

5 The definition of a static metric is that this transformation must leave all quantities invariant, but it sends $t \to -t$ and hence $\mathrm{d}r/\mathrm{d}t \to -\mathrm{d}r/\mathrm{d}t$. Only if this vanishes can it be invariant.

7 The construction proceeds as described in Sec. 10.5, except that to integrate equations with p one has to specify the function S in the equation of state. It would be simple to give a function $S(r)$, but not very physical (unless S is constant through the star), because one can't know ahead of time how the star's structure will come out of the equations. It is more physical to give $S(m(r))$, so that the entropy is a function of the mass interior to a radius r. This might approximately describe a star formed by spherical collapse that conserves entropy, so that the rest-mass inside a given shell would be constant during the collapse and so would be the entropy, leading to a fixed relationship between entropy and interior rest-mass. Now, $m(r)$ is not exactly proportional to the rest mass, so giving $S(m(r))$ only approximates this situation. It would be possible, but more complicated, to compute the rest-mass interior to r and take the entropy to be a function of that. Since there is no differential equation for S one does not change the manner of integration; one just adds an auxiliary function or look-up table to determine the entropy at any radius and hence allow the pressure there to be computed.

8 (a) Use reasoning similar to that in Exer. 5. (b) Make sure to include the Christoffel symbols in the computation of the covariant derivative! These can be found in Exer. 35 of Sec. 6.9.

10 (b) $\exp(\Phi) = 0.999\,997$, 0.760; $z = 3 \times 10^{-6}$, 0.315.

15 Putting the power series into the differential equations and matching up powers of r shows that $\rho_1 = 0$ and

$$\rho_2 = -2\pi\rho_c(\rho_c + p_c)(\rho_c + 3p_c)/(3\Gamma_c p_c).$$

The power series solution for the three functions then becomes

$$\begin{aligned} \rho &= \rho_c + \rho_2 r^2, \\ p &= p_c(1 + \rho_2\Gamma_c/\rho_c r^2), \\ m &= 4\pi r^3\rho_c/3 + 4\pi r^5\rho_2/5. \end{aligned}$$

Estimate the error in neglecting (uncalculated) the next term of each equation to be about equal to the square of the last calculated term, i.e., choose r such that the calculated terms are smaller than $\sqrt{(0.01)} \approx 3\%$. This means that

$$r^2 \leqslant |\rho_c/30\rho_1\Gamma_c|.$$

19 (a) $1.12 \times 10^{42}\,\mathrm{kg\,m^2\,s^{-1}}$; (b) $1.4 \times 10^4\,\mathrm{s^{-1}}$; yes, by 50%; (c) 2.7×10^{-4}; (d) 5×10^9 Gauss.

Chapter 11

1 In Eq. 11.9, find the minimum approach distance by setting $\frac{dr}{d\tau} = 0$, and calling this radial distance b. If $M = 0$ solving for b gives the result. Note that if $m = 0$ the result is the same as if one had started with Eq. 11.10. The impact parameter is defined for any orbit (even if M is not zero) by the equation

$$b = L/[E^2 - m^2]^{1/2}.$$

It can be thought of as the offset of the "aim" of the trajectory from the center of the metric, when the trajectory is far away. If M is non-zero then the trajectory will approach the center more closely than b. For a photon orbit with $m = 0$, again even when M is not zero, one can replace in Eq. 11.12 L by bE. Then if one re-scales the affine parameter λ to a new one $\lambda' = E\lambda$, the resulting equation depends only on one parameter, b. Re-scaling the affine parameter is always allowed and does not change the path of the geodesic through the spacetime. So we learn that photons follow geodesics that depend only on their "aim", or offset, from the center. Massive particles do not: the degree to which they approach more closely than b depends on their velocity far away, which (unlike that of a photon) depends on the initial conditions.

4 The key radii are $3M$, the location of the unstable photon circular orbit, and $6M$, the location of the last stable circular orbit for massive particles. So if a star has a radius of $2.5M$, as in (a), then all possible orbits exist outside it, even photon circular orbits. If a star, as in (b), has a radius of $4M$, then it will not have any photon orbits, but the last stable massive-particle orbit will still be outside its surface, so all stable massive-particle orbits will exist. Finally, in case (c), if the star has a radius of $10M$ then there will be stable circular orbits down to its surface, and of course any quasi-elliptical and quasi-hyperbolic orbits that approach no closer than the surface radius.

5 (a) The question is badly worded, since the value $1-2M/R$ is the relativistic value of $-g_{00}$, not its Newtonian approximation. The function that is to be compared with the Newtonian potential is $\sqrt{(}-g_{00})-1 = -M/r - M^2/r^2 - \ldots$. This differs from the Newtonian potential $-M/r$ by the term M^2/r^2, which makes a 1% change when $M/r = 0.01$.

(b) For a $10^6 M_\odot$ black hole, this is at a distance of about 1.5×10^{11} m, just 15 time larger than the radius of a normal star. So nor more than of order 1000 normal stars can be entirely inside the highly relativistic zone around a black hole of that mass. But if the mass is $10^9 M_\odot$, then the radius is 1000 times larger, and the volume is 10^9 times larger, so that billions of normal stars could be in the highly relativistic region near such a mega-massive black hole.

7 (a) Use Eq. (11.24) in $\mathrm{d}s^2 = g_{00}\mathrm{d}t^2 + g_{\phi\phi}\mathrm{d}\phi^2$ to get $\tau = 20\pi\sqrt{7}\,M$.

(b) Same as coordinate time interval, Eq. (11.25): $20\pi\sqrt{10}\,M$.

(c) Integrate $\mathrm{d}s^2 = g_{00}\mathrm{d}t^2$ over the time in (b): $40\pi\sqrt{2}\,M$.

(d) 6.4 ms: innermost stable orbit.

(e) $40/\sqrt{2}$ yr, independent of M.

9 (a) Turning points are at $r = 12.5\,M$ and $5.47\,M$. (Notice that the high $\tilde{L}$ enables the particle to turn around *inside* $6M$. This is not a contradiction with is radius being the last stable orbit, because that applies only to *circular* orbits.) The orbit changes by $\Delta\phi = 7.4$ rad between these points. A full orbit has $\Delta\phi = 14.8$ rad, or a perihelion shift of 8.5 rad. The approximation Eq.11.37) gives only 2.3 rad. This shows that highly noncircular orbits can have much greater shifts.

10 (a) $(\tilde{E}/\tilde{L})^2 = [1 + 36M^2/\tilde{L}^2 + (1 - 12M^2/\tilde{L}^2)^{3/2}]/(54\,M^2)$.

14 In arc sec per orbit and per year: Venus (0.052, 0.085), Earth (0.038, 0.038), Mars (0.025, 0.013).

20 (a) The Christoffel symbols may be computed as a special case of those in the solution of Ex. 35 of Sec. 6.9.

(d) The quantity computed here is called the Riemann scalar. Note that it is perfectly well-behaved at the horizon and only singular at the center $r = 0$.

22 Approximately $10^5\,\mathrm{M}_\odot$.

24 The point $u = v = 0$ has $r = 2M$, so is located on the horizon. We expect it to be locally flat, but this can also be easily shown by expanding the metric in a Taylor series in u and v. It is clear from Eq. 11.68 that the expansion does not contain terms linear in u and v, and so with some rescaling of the coordinates by constant factors it can be brought into the form of a Minkowski metric with corrections of order u^2 and v^2. Of course, the metric is locally flat everywhere except at the singularity at $r = 0$, but we have chosen to look at the origin because it belongs to the extended Kruskal-Szekeres metric and is not easily examined in Schwarzschild coordinates, and also because it is relatively easy to do the computation there!

25 Estimate the total breakup force to be M/R^2, giving $R_\mathrm{T} \sim (M_\mathrm{H}/\rho)^{1/3}$. For the given density we find that the disruption mass is about $10^6\ \mathrm{M}_\odot$. This is an interesting mass considering that there is strong evidence that a black hole of about this mass resides in the center of our Galaxy.

31 The ZAMO has angular velocity ω so that $U^\phi/U^0 = \omega$. Denoting U^0 by the variable A, one can compute A from the normalization condition $U^\alpha U^\beta g_{\alpha\beta} = -1$ in the Kerr metric.

34 (a) $(2 - \sqrt{2})M$, (b) $E \leqslant m_1 + m_2 - \sqrt{(m_1^2 + m_2^2)} \approx m_1(1 - \frac{1}{2}m_1/m_2)$.

37 (a) The photon follows a null line along which only the coordinates t and ϕ change. So the relevant interval is

$$\mathrm{d}s^2 = 0 = g_{tt}\mathrm{d}t^2 + 2g_{t\phi}\mathrm{d}t\mathrm{d}\phi + g_{\phi\phi}\mathrm{d}\phi^2.$$

This can be solved to give the photon's angular velocity:

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \omega \pm \sqrt{\Omega^2 + \omega^2},$$

where ω is the frame-dragging angular velocity defined in Eq. 11.90, whose value in the equatorial plane ($\theta = \pi/2$) is

$$\omega = \frac{-g_{t\phi}}{g_{\phi\phi}} = \frac{2Ma/r}{r^2 + a^{+}2Ma^2/r},$$

and where Ω is the basic photon circular angular velocity and is defined (again in the equatorial plane) by

$$\Omega^2 = \frac{-g_{tt}}{g_{\phi\phi}} = \frac{1 - 2M/r}{r^2 + a^2 + 2Ma^2/r}.$$

The forward-going photon's angular velocity is $(\Omega^2 + \omega^2)^{1/2} + \omega$, while the backward-going angular velocity is $-[(\Omega^2 + \omega^2)^{1/2} - \omega]$. The backward-going photon travels more slowly than the forward-going one. This is an illustration of the action of the dragging of inertial frames, which drags in the forward direction.

Notice that, as r gets large, Ω approaches $1/r$ while ω falls off as $1/r^3$. So Ω dominates the angular velocity and the frame-dragging ω makes a correction. The limiting value of $\Omega = 1/r$ is just what one needs in order for the photon to travel around a distant circle of radius r in a time equal to the circumference of the circle, $2\pi r$.

The coordinate time that elapses is 2π divided by the absolute value of the angular velocity. In the forward direction this is

$$\Delta t_+ = \frac{2\pi}{\Omega}\left(\sqrt{1+\frac{\omega^2}{\Omega^2}} - \frac{\omega}{\Omega}\right).$$

In the backward direction this is

$$\Delta t_- = \frac{2\pi}{\Omega}\left(\sqrt{1+\frac{\omega^2}{\Omega^2}} + \frac{\omega}{\Omega}\right).$$

The difference in the round-trip times is

$$\Delta t_- - \Delta t_+ = \frac{4\pi\omega}{\Omega^2}.$$

There is no redshift or blueshift of the photons: the components of the photon's 4-momentum are constant in time as it moves around the circle, since the geometry is the same everywhere along the path.

(b) The ZAMO observer orbits with angular velocity ω. It follows that the angular velocity of a photons relative to him is $\mathrm{d}\phi/\mathrm{d}t - \omega$, which evaluates simply to $\pm(\Omega^2 + \omega^2)^{1/2}$: for this observer the two photons have the same (absolute value) angular velocity, so they take the same time to go once around and meet him again! If he launches the two photons simultaneously, they arrive back and meet him at the same time. From the point of view of the first observer at rest, the forward-going photon is going faster but has to travel further in order to meet him again, since he moves around the black hole in the meantime, while the backward-going photon is going slower but has less far to go since the moving observer's motion takes it towards this photon.

Notice that *all* the speeds of the photons we are talking about are coordinate angular velocities. The photons always travel at speed 1 relative to any local inertial frame, but the curvature of space and time makes them take different amounts of time to traverse their circular paths.

Chapter 12

8 (a) Use $\mathrm{d}\chi^2 = \mathrm{d}r^2/(1+r^2)$. Integrating leads to $r = \sinh\chi$.

(c) A Lorentz transformation leaves the metric and hence the hyperbola unchanged, but changes the origin of spatial coordinates. Any point on the hyperbola can be made into this origin by choosing the correct transformation.

9 (a) In the text it is pointed out that p_χ is conserved due to the homogeneity of the universe. A radially propagating photon has just two non-zero four-momentum components, p^0 and p^χ, so we have

$$0 = g_{00}(p^0)^2 + g_{\chi\chi}(p^\chi)^2,$$

which by the diagonality of the metric can also be written as

$$0 = g_{00}(p^0)^2 + g^{\chi\chi}(p_\chi)^2.$$

For each kind of universe $g_{\chi\chi} = R^2(t)$, so that $g^{\chi\chi} = 1/R^2(t)$. Since $g_{00} = -1$, we finally get that

$$p^0 = p_\chi/R(t).$$

The constancy of p_χ establishes the fact that the locally measured energy of the photon, p^0, decreases with time as $1/R(t)$. Notice that, since the location of the origin of coordinates in a homogeneous space is arbitrary, the radially-moving photon really can be *any* photon, so this law is quite general.

(b) The redshift relation follows from the fact that the wavelength of a photon is inversely proportional to its energy.

10 In Eq. 12.67, which is the result of the previous Exercise, expand $R(t_r)$ about $R(t_e)$:

$$R(t_r) = R(t_e) + \dot{R}(t_e)(t_r - t_e) + \ldots.$$

Now, for a photon traveling at the speed of light, $t_r - t_e = d_0$, the distance to the emitter as defined in Eq. 12.24. So we get

$$z = (\dot{R}/R)(t_e)d_0.$$

Now, to the accuracy we are working (first order in time intervals or d_0) we can replace t_e in this expression with t_r, which is the same as today's time t_0 as used in Eq. 12.24. This makes z from Eq. 12.67 identical to v in Eq. 12.24 at this order.

13 Make sure you use the definition of the luminosity distance d_{L} in Eq. 12.34.

15 The result follows from the fact that the energy density of black-body radiation is proportional to T^4.

18 Pressure forces depend on pressure *gradients,* and in a homogeneous universe the gradient is zero. So a cosmological tension does not lead to a pulling inward. Rather, the entire effect of the negative pressure is in the "active gravitational mass per unit volume" term $\rho + 3p$ in Eq. 12.55, where the large negative pressure overwhelms the density and causes gravity itself to become repulsive.

22 (a) The energy density of black-body radiation is $\rho_r = 4\sigma T^4/c$, where σ is the Stefan-Boltzmann constant, whose value in SI units is $5.67 \times 10^{-8}\mathrm{Wm^{-2}K^{-4}}$. Divide by another factor of c^2 to get this in mass-density units. When divided by ρ_m we get

$$\varepsilon = \frac{\rho_r}{\rho_m} \sim 4m^{-1} \times 10^{-4}.$$

Then we get the number of photons per baryon to be about $2m^{-1} \times 10^9$. In the early universe, if there were no baryon-antibaryon asymmetry, then one would have expected of order one baryon and one antibaryon per photon, because the baryons would have been in thermal equilibrium with the photons at temperatures where the photons had enough energy to create baryon-antibaryon pairs. What this large number tells us is that the excess of baryons over antibaryons is of order one part in 10^9.

23 $z = 3 \times 10^3$.

24 The density must exceed $\rho = 3H_0^2/8\pi = 10^{-26}\mathrm{kg\,m^{-3}}$. This is called the *closure density* or *critical density* and is denoted by ρ_c.