Kinetic equations

Abstract

Kinetic theory is a way of describing the time evolution of a system consisting of a large number of elementary objects which may be called particles. The mathematical study of such systems leads to a class of partial differential equations called kinetic equations. Equations of this type have many applications in physics and other sciences. The 'particles' could for instance be electrons, stars, gas molecules or microorganisms. This course presents general theory of kinetic equations together with a discussion of a variety of their applications.

1 Introduction

Kinetic theory is a way of describing the time evolution of a system consisting of a large number of elementary objects. In the following these will be referred to as particles. The mathematical study of such systems leads to a class of partial differential equations called kinetic equations.

Example 1 Plasma physics
A plasma consists of a large collection of fast-moving charged particles. Consider for instance the case where the particles are electrons each with charge -1 and protons each with charge $+1$. No quantum effects are taken into account in the description. The particles are described not individually (which would be too complicated) but as a continuous cloud of a certain density. Let $f_-(t, x, v)$ be the number density of electrons (number per unit volume) at position $x = (x^1, x^2, x^3)$ with velocity $v = (v^1, v^2, v^3)$ at time t. The function f_- is real-valued and non-negative. Similarly, let f_+ be the number density of protons. Each electron repels each other electron by means of an electrostatic force. Similarly each proton repels each other proton. Finally each proton attracts each electron and conversely. The laws of physics should allow us to predict the value of the functions f_- and f_+ at all times if we know their values at an initial time t_0. Mathematically this leads to the initial value problem or Cauchy problem for the kinetic equation.

Now a kinetic equation will be written down. To make things as simple as possible assume that there is only one type of particle involved (not two as in the example above) and that each particle has unit charge and unit mass. Assume that this charge is positive. In this case there is just one distribution function f. An individual particle satisfies Newton's law of motion

$$\dot{x} = v, \quad \dot{v} = \nabla U \tag{1}$$

where U is the electrostatic potential. It is a function of x for each fixed value of t. When the particles are described statistically it is required that the density of particles does not change along the path of each individual particle. Under this condition the function f satisfies the following equation:

$$\frac{\partial f}{\partial t} + v^i \frac{\partial f}{\partial x^i} + \delta^{ij} \frac{\partial U}{\partial x^i} \frac{\partial f}{\partial v^j} = 0 \tag{2}$$

Here δ^{ij} is the Kronecker delta and the Einstein summation convention is used. In a more concise notation, using the standard inner product in $\mathbf{R}^3$, this becomes

$$\partial_t f + v \cdot \nabla_x f + \nabla_x U \cdot \nabla_v f = 0 \tag{3}$$

This equation is known as the Vlasov equation. The electrostatic potential is supposed to satisfy the Poisson equation.

$$\Delta U = \delta^{ij} \frac{\partial^2 U}{\partial x^i \partial x^j} = \rho \tag{4}$$

where the charge density ρ is given by

$$\rho(t, x) = \int_{\mathbf{R}^3} f(t, x, v) dv \tag{5}$$

These equations together form what is called the Vlasov-Poisson system. It is one of the simplest examples of a kinetic equation (or, more precisely, system of kinetic equations). In this system there is no direct interaction between the particles. The motion of each individual particle is affected by a field which is generated collectively by all particles together. Note that while both the Vlasov equation and the Poisson equation are linear in the corresponding unknown, the coupling between them makes the whole system nonlinear. For a fixed potential the Vlasov equation is what is known as a transport equation. Kinetic equations result by coupling a transport equation to some other effects, such as that of the electric field in the example just presented.

Example 2 Galactic dynamics.
Stars form a major component of our universe. They tend to be grouped together into galaxies. The aim in galactic dynamics is to describe these by a kinetic equation where the 'particles' are the stars. Stars are large objects by everyday standards but their diameters are very small compared to the typical distances between them. This is the justification for treating them like point particles. Galaxies can be roughly classified into spiral and elliptical galaxies. In the spiral galaxies the stars interact with clouds of gas and dust not belonging to stars and so a kinetic description of the stars alone is not very appropriate

for modelling their structure. It must be augmented by hydrodynamics. On the other hand elliptical galaxies are free from this complication and are ideal for the application of kinetic equations. The number of 'particles' is of the order 10^7 and so is really large. A continuum description is reasonable. Spiral galaxies typically have a central bulge where a kinetic description is also appropriate. In the case of galactic dynamics the interaction of the particles is via the Newtonian gravitational potential. Each star attracts each other star. Another class of object which is well described by the Vlasov-Poisson system are the globular clusters.

For simplicity assume that all particles have unit mass. For galactic dynamics the relevant equation is the Vlasov equation

$$\partial_t f + v \cdot \nabla_x f - \nabla_x U \cdot \nabla_v f = 0. \tag{6}$$

The quantity U is the Newtonian gravitational potential. It satisfies the Poisson equation as written in Example 1. The mass density is defined in terms of f as in (5). The equations (6) and (4) form a system which is once again referred to as the Vlasov-Poisson system. It differs from the system of the same name for the plasma only by a sign. When it is necessary to distinguish these two systems it is specified that it is the plasma physics case or the galactic dynamics case which is being considered.

Example 3 Dilute gases.
In the models introduced so far each particle only feels the field which is generated collectively by all the other particles. There is no direct interaction between particles. A relatively simple kind of direct interaction is a collision. Two particles come together, their velocities are changed and then they move apart again. The Vlasov equation is augmented by a collision term. Suppose that there are no long-range interactions so that particles move freely between collisions. Then the kinetic equation reads

$$\partial_t f + v \cdot \nabla_x f = Q(f). \tag{7}$$

Various forms of the collision term $Q(f)$ are possible. This equation for electrically neutral 'particles' gives a kinetic description of a gas, with the particles being the molecules of the gas. This kind of model can be applied when the gas is dilute, i.e. the density of particles is low enough so that they are not continually undergoing collisions. The best known form of the collision term gives rise to the Boltzmann equation. In that case

$$Q(f)(v) = \int_C k(v, w, v', w')[f(v')f(w') - f(v)f(w)] \tag{8}$$

Here C is the collision manifold defined by the laws of conservation of momentum, $v + w = v' + w'$, and energy, $|v|^2 + |w|^2 = |v'|^2 + |w'|^2$. The dependence of f on t and x has been suppressed in the notation. The function k, which is supposed non-negative, will be discussed in more detail later. It describes the rate at which incoming particles with velocities v and w give rise to outgoing particles with velocities v' and w'. The collision term is the sum of two terms which are quadratic in f. One of these (the gain term) is positive while the other (the loss term) is negative.

Example 4 Chemotaxis
Chemotaxis is the process by which living cells move in response to the concentration gradient of a chemical substance. The behaviour of populations of cells can be described by kinetic equations. In the absence of a chemical gradient one relevant equation is (see [12])

$$\partial_t f + v \cdot \nabla_x f = -\lambda f + \lambda \int T(v, v') f(v') dv'. \tag{9}$$

Here λ is a positive constant and $\int T(v, v')dv = 1$. The dependence of f on t and x has been suppressed in the notation. In this model the particles (the cells) change velocity with a given probability described by T between periods where they move with constant velocity. The first term on the right hand side of (9) describes how cells with velocity v are lost by this process and the integral on the right hand side describes how particles with velocity v are gained by coming from other velocities. The normalization condition on T ensures conservation of particles. The inclusion of the influence of a chemical gradient will be discussed later.

Many of the detailed calculations carried out in the lectures are not reproduced in these notes. In many cases more details can be found in the book of Glassey [6].

2 The Vlasov-Poisson system

2.1 Introductory remarks

The Vlasov-Poisson system was already written down in the introduction. In this course when we talk about solutions of this system we assume that f is continuously differentiable (C^1) and U is twice continuously differentiable (C^2) so that all the derivatives occurring in the equations are defined in an elementary sense and continuous. In the following solutions will be considered which correspond to an isolated system. This means that the particle density is localized in space. Either it is assumed that it is confined to a compact set (support of the solution) in space on any given finite time interval or it is assumed to tend to zero sufficiently fast at infinity in space at any given time. In this section the first of these alternatives will be considered. Distribution functions f are considered which have compact support in (x, v)-space at each fixed time. It is also part of the idea of an isolated system that the forces on the particles should become small for large values of the spatial coordinates. In the case of the Vlasov-Poisson system this is generally achieved by assuming that U tends to zero as $x \to \infty$ for each fixed t.

As mentioned in the introduction, the Vlasov-Poisson system describes a deterministic system and solutions should be uniquely determined by the value of f at a fixed time t_0. With suitable assumptions this is true. The following theorem holds:

Theorem Let f_0 be a non-negative C^1 function of compact support defined on $\mathbf{R}^6$. Then there is a non-negative C^1 function f on $\mathbf{R}^7$ and a C^2 function U on $\mathbf{R}^4$, tending to zero at infinity for each fixed t, which satisfy the Vlasov-Poisson system and the initial condition $f(0,x,v)=f_0(x,v)$. For each fixed t the function $f(t,x,v)$ has compact support. The solution is determined uniquely by the initial datum f_0.

Note that the Vlasov-Poisson system is invariant under time reversal. Replacing t by $-t$ and v by $-v$ preserves the form of the equation. Thus a statement about existence in the future automatically gives one about existence in the past. The first proofs of this theorem, by two different methods, are due to Pfaffelmoser [1] in 1992 and Lions and Perthame [2] in 1991. The following discussion should provide some insight into the question, why it took so long to prove this theorem. Notice that this theorem applies both to the case of plasma physics and that of galactic dynamics. For convenience introduce a parameter γ which is equal to $+1$ in the plasma physics case and equal to -1 in the galactic dynamics case. Most of the steps in the proof of global existence apply equally well to both signs of γ. There is just one exception which has to do with the energy of solutions and will be described next.

For a solution of the Vlasov-Poisson system define the kinetic energy by

$$\mathcal{E}_{\mathrm{kin}}(t)=\frac{1}{2}\int_{\mathbf{R}^6} f(t,x,v)v^2dxdv \tag{10}$$

and the potential energy by

$$\mathcal{E}_{\mathrm{pot}}(t)=\frac{1}{2}\int_{\mathbf{R}^3}\gamma|\nabla U|^2dx \tag{11}$$

The total energy is equal to $\mathcal{E}(t)=\mathcal{E}_{\mathrm{kin}}(t)+\mathcal{E}_{\mathrm{pot}}(t)$. For sufficiently smooth solutions of the Vlasov-Poisson system with sufficient fall-off at infinity a computation shows that $\mathcal{E}(t)$ is independent of time. This is an expression of the physical law of energy conservation for this model. In the case $\gamma=1$ both $\mathcal{E}_{\mathrm{kin}}$ and $\mathcal{E}_{\mathrm{pot}}$ are manifestly non-negative. Thus the fact that their sum is bounded implies that each of them is bounded. For the galactic dynamics case, on the other hand, the potential energy is non-positive and it is a priori conceivable that the modulus of $\mathcal{E}_{\mathrm{kin}}$ and $\mathcal{E}_{\mathrm{pot}}$ could tend to infinity after a finite time while their sum remains bounded. It is an important step of the global existence proof in the case $\gamma=-1$ to show that in fact this does not happen. The proof uses an estimate which was first applied to the study of the Vlasov-Poisson system by Horst [7].

For a solution of the Vlasov-Poisson system which has compact support at each fixed time an important quantity is

$$P(t)=\sup\{|v|:f(t,x,v)\neq 0\ \ \text{for some}\ \ x\} \tag{12}$$

This is the maximum velocity of any particle at time t. It turns out that $P(t)$ provides a continuation criterion for solutions of the Vlasov-Poisson system: if a solution exists on the interval $[t_0, t_1)$ for some real numbers t_0 and t_1 with $t_1 > t_0$ and $P(t)$ is bounded on this interval then the solution can be extended to a time interval $[t_0, t_2)$ with $t_2 > t_1$. Thus if it can be shown that for prescribed initial data $P(t)$ is bounded on any interval $[t_0, t_1)$ where a solution exists then global existence follows. To prove this it suffices to consider the maximal interval of existence.

2.2 Some important estimates

For a fixed potential U the Vlasov equation is a scalar hyperbolic equation and can be solved by the method of characteristics. This will now be worked out explicitly. Let $(X(s,t,x,v), V(s,t,x,v))$ be the solution of the system of ordinary differential equations

$$\frac{dX}{ds}(s,t,x,v) = V(s,t,x,v) \tag{13}$$

$$\frac{dV}{ds}(s,t,x,v) = \nabla U(s, X(s,t,x,v)) \tag{14}$$

with initial data $X(t,t,x,v) = x$ and $V(t,t,x,v) = v$. This is called the characteristic system. The solution of the Vlasov equation is constant along the solutions of the characteristic system and so can be expressed in the following way:

$$f(t,x,v) = f_0(X(0,t,x,v), V(0,t,x,v)) \tag{15}$$

where f_0 is the initial datum. From this is can be concluded that

$$\|f(t)\|_{L^\infty} = \sup\{f(t,x,v) : (x,v) \in \mathbf{R}^6\} \tag{16}$$

is independent of time. Next it will be shown that for any constant $p \geq 1$ the L^p norm

$$\|f(t)\|_{L^p} = \left[\int_{\mathbf{R}^6} (f(t,x,v))^p dxdv\right]^{\frac{1}{p}} \tag{17}$$

is independent of time. This is a consequence of the fact that the Newtonian equations of motion can be written as a Hamiltonian system so that they preserve the volume form on phase space (Liouville's theorem). Concretely this means that if we define $X = X(0,t,x,v)$ and $V = V(0,t,x,v)$ for a fixed time t then the Jacobian of the mapping $(x,v) \mapsto (X,V)$ is equal to one.

This will now be looked at from a slightly different point of view. Define a vector field on $\mathbf{R}^6$ with coordinates $(x^a, p^a) = z^A$ by $(v^a, \nabla^a U) = Z^A$. Then the Vlasov equation can be written in the form $\partial_t f + Z^A \partial_A f = 0$. Hence

$$\begin{aligned} \frac{d}{dt}\left(\int_{\mathbf{R}^6} f(x,v)dxdv\right) &= \int_{\mathbf{R}^6} \partial_t f dxdv \\ &= -\int_{\mathbf{R}^6} Z^A \partial_A f dz = -\int_{\mathbf{R}^6} (\partial_A Z^A) f dz \end{aligned} \tag{18}$$

It follows that if the divergence of the vector field Z^A is zero then the L^1 norm of f is conserved. Evidently the divergence of the vector field occurring in the Vlasov-Poisson system vanishes since the first three components do not depend on x^a and the last three do not depend on v^a. The function f^p satisfies the same equation as f itself and so all L^p norms of f are conserved under the evolution.

Integrating the Vlasov equation with respect to v gives the conservation law $\partial_t\rho + \nabla \cdot j = 0$ where

$$j(t,x) = \int_{\mathbf{R}^3} vf(t,x,v)dv. \tag{19}$$

This is known as the continuity equation. Integrating it with respect to x shows that $\|\rho(t)\|_{L^1}$ is independent of time. The physical interpretation of this is that the total number of particles is conserved. Multiplying the Vlasov equation with $|v|^2$ and integrating with respect to x and v gives the conservation of energy.

It is well known that a solution of the Poisson equation $\Delta U = \rho$ tending to zero at infinity can be expressed in terms of the integral formula

$$U(t,x) = -\frac{1}{4\pi}\int_{\mathbf{R}^3} \frac{\rho(y)}{|x-y|}dy \tag{20}$$

The electric or gravitional field is, up to a sign, $E = \nabla U$ and has a corresponding representation

$$E(t,x) = \frac{1}{4\pi}\int_{\mathbf{R}^3} \frac{\rho(y)(x-y)}{|x-y|^3}dy \tag{21}$$

This formula can be used to obtain pointwise estimates for U in terms of the function ρ. Choose a positive parameter R.

$$\begin{aligned} |U| &\le \frac{1}{4\pi}\left(\int_{|x-y|\le R}\frac{\rho(y)}{|x-y|}dy + \int_{|x-y|\ge R}\frac{\rho(y)}{|x-y|}dy\right) \\ &\le \frac{1}{4\pi}\left(\|\rho\|_{L^\infty}\int_{|x-y|\le R}\frac{1}{|x-y|}dy + \|\rho\|_{L^1}R^{-1}\right) \\ &\le C(\|\rho\|_{L^\infty}R^2 + \|\rho\|_{L^1}R^{-1}) \end{aligned} \tag{22}$$

Here and in the following C is used to denote a positive constant whose exact value is not important. The next step is to optimize in R. A systematic way of doing this would be to calculate the minimum of this expression as a function of R by computing its derivative with respect to R, In fact there is an easier approach. The overall multiplicative constant in the final estimate is in many cases not of interest. Only the dependence on the norms is important. An empirical approach is to choose R so that the terms in brackets on the right hand side of the inequality are equal. This gives an inequality of the form $\|U\|_{L^\infty} \le C\|\rho\|_{L^\infty}^{1/3}\|\rho\|_{L^1}^{2/3}$. Note that the L^1 norm of ρ is independent of time and so for fixed initial data the factor involving $\|\rho\|_{L^1}$ can be absorbed into the constant C. What remains is an estimate for U in terms of $\|\rho\|_{L^\infty}$.

The first derivatives of U can be estimated in a similar way.

$$|\nabla U| \leq \frac{1}{4\pi}\left(\int_{|x-y|\leq R}\frac{\rho(y)|x-y|}{|x-y|^3}dy + \int_{|x-y|\geq R}\frac{\rho(y)|x-y|}{|x-y|^3}dy\right)$$
$$\leq \frac{1}{4\pi}\left(\|\rho\|_{L^\infty}\int_{|x-y|\leq R}\frac{1}{|x-y|^2}dy + \|\rho\|_{L^1}R^{-2}\right)$$
$$\leq C(\|\rho\|_{L^\infty}R + \|\rho\|_{L^1}R^{-2}) \tag{23}$$

Optimizing gives $\|\nabla U\|_{L^\infty} \leq C\|\rho\|_{L^\infty}^{2/3}\|\rho\|_{L^1}^{1/3}$. This estimate is important for determining the asymptotics of solutions with small initial data. An alternative estimate for ∇U which is important in other situations begins in the same way but then estimates the integral for large $|x-y|$ in a different way using Hölder's inequality. Recall that this says that if $f \in L^p$, $g \in L^q$ and $p^{-1} + q^{-1} = 1$ then $fg \in L^1$ and $\|fg\|_{L^1} \leq \|f\|_{L^p}\|g\|_{L^q}$. In the present estimate choose $f = \rho$, $g = |x-y|^{-2}$, $p = \frac{5}{3}$ and $q = \frac{5}{2}$.

$$\|\nabla U\|_{L^\infty} \leq \frac{1}{4\pi}\left(\|\rho\|_{L^\infty}\int_{|x-y|\leq R}\frac{1}{|x-y|^2}dy + \|\rho\|_{L^{5/3}}\left(\int_{|x-y|\geq R}|x-y|^{-5}\right)^{2/5}\right)$$
$$\leq C(\|\rho\|_{L^\infty}R + \|\rho\|_{L^{5/3}}R^{-4/5}) \tag{24}$$

Optimizing gives $\|\nabla U\|_{L^\infty} \leq C\|\rho\|_{L^\infty}^{4/9}\|\rho\|_{L^{5/3}}^{5/9}$

The second derivative cannot be estimated so easily since it is known that when ρ is C^2 the second derivatives of U are not necessarily bounded. Starting from the integral formula for U and trying to differentiate under the integral leads to integrals where the integrand is not locally integrable. Define

$$K_{ij}(x,y) = -\frac{3(x_i-y_i)(x_j-y_j)}{|x-y|^5} + \frac{\delta_{ij}}{|x-y|^3} \tag{25}$$

Then for any positive constant R the second derivatives of U satisfy the identity

$$\nabla_i\nabla_j U(t,x) = \frac{1}{3}\rho(t,x)\delta_{ij} + \frac{1}{4\pi}\int_{|x-y|\leq R}(\rho(t,y)-\rho(t,x))K_{ij}(x,y)dy$$
$$+\frac{1}{4\pi}\int_{|x-y|\geq R}\rho(t,y)K_{ij}(x,y)dy \tag{26}$$

This allows these derivatives to be estimated in the following way

$$|\nabla_i\nabla_j U(t,x)| \leq C\left[\|\rho\|_{L^\infty} + \int_{|x-y|\leq R}|x-y|^{-(3-\theta)}\frac{|\rho(x)-\rho(y)|}{|x-y|^\theta}dy\right.$$
$$\left.+\int_{|x-y|\geq R}|x-y|^{-3}\rho(y)dy\right] \tag{27}$$

Hence $|\nabla_i\nabla_j U(t,x)| \le C(\|\rho\|_{L^\infty} + \|\rho\|_\theta R^\theta + \|\rho\|_{L^1} R^{-3})$ where $\| \ \|_\theta$ denotes the Hölder seminorm

$$\|\rho\|_\theta = \sup_{x,y\in R^3} \frac{|\rho(x)-\rho(y)|}{|x-y|^\theta}. \tag{28}$$

where $\theta \in (0,1)$. Optimizing gives

$$|\nabla_i\nabla_j U(t,x)| \le C(\|\rho\|_{L^\infty} + \|\rho\|_\theta^{\frac{3}{\theta+3}} \|\rho\|_{L^1}^{\frac{\theta}{\theta+3}}) \tag{29}$$

Now $|\rho(x)-\rho(y)|^{1-\theta} \le 2\|\rho\|_{L^\infty}^{1-\theta}$ and hence

$$\|\rho\|_\theta \le 2\|\rho\|_{L^\infty}^{1-\theta} \|\nabla_x\rho\|^\theta \tag{30}$$

Combining these estimates gives the inequality

$$|\nabla_i\nabla_j U(t,x)| \le C(\|\rho\|_{L^\infty} + \|\rho\|_{L^\infty}^{\frac{3(1-\theta)}{3+\theta}} \|\nabla_x\rho\|^{\frac{3\theta}{3+\theta}} \|\rho\|_{L^1}^{\frac{\theta}{3+\theta}}) \tag{31}$$

An alternative estimate goes as follows. Introduce a positive constant $d < R$ and replace R by d in (26). Write the integral over the exterior of the ball of radius d as the sum of that over the annulus where $|x-y|$ is between d and R and that over the exterior of the ball of radius R. The integral over the ball of radius d can be estimated in terms of the Lipschitz constant L of the function ρ, i.e. the analogue of $\|\rho\|$ with θ replaced by one. This leads to an expression of the form CLd. The integral over the annulus can be bounded by an expression of the form $C\|\rho\|_{L^\infty}\log(R/d)$. The integral over the exterior region can be estimated as before. Choosing $d = L^{-1}$ gives

$$|\nabla_i\nabla_j U(t,x)| \le C[1 + \|\rho\|_{L^\infty}(1 + \log L + \log R)) + R^{-3}\|\rho\|_{L^1}] \tag{32}$$

Define

$$\begin{aligned} \log^*(s) \quad &= s \qquad\qquad , 0 \le s \le 1 \\ &= 1 + \log s \quad , s > 1 \end{aligned} \tag{33}$$

It follows that

$$\|\nabla_i\nabla_j U(t,x)\|_{L^\infty} \le C(1 + \|\rho\|_{L^\infty})(1 + \log^*(\|\rho\|_{C^1})). \tag{34}$$

Next the inequality of Horst relating the kinetic and potential energies will be discussed. To start with the following inequality is required. The solution U of the Poisson equation satisfies

$$\left(\int_{\mathbf{R}^3} |\nabla U(t)|^2 dxdv\right)^{\frac{1}{2}} \le C\|\rho\|_{L^{6/5}} \le C\|\rho\|_{L^1}^{\frac{7}{12}} \|\rho\|_{L^{\frac{5}{3}}}^{\frac{5}{12}} \tag{35}$$

The first of these inequalities follows from the Hardy-Littlewood-Sobolev inequality, which will not be proved here. For anyone who wants to find out more about this inequality, one place in which the proof is discussed is in [15]. It is related to Young's inequality, which says that if

$$h(x) = \int_{\mathbf{R}^n} g(y)f(y-x)dy \tag{36}$$

and p, q and r are numbers greater than or equal to one satisfying $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$ then $\|h\|_{L^r} \le \|f\|_{L^p}\|g\|_{L^q}$. The inequality (35) is not quite covered by this since in $\mathbf{R}^3$ the function $|x|^{-2}$ just fails to be in $L^{3/2}$. The second inequality in (35) is a consequence of the generalized Hölder inequality

$$\|fg\|_{L^r} \le \|f\|_{L^p}\|g\|_{L^q} \tag{37}$$

if $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. To get the second inequality in (35) choose $f = \rho^{\frac{7}{12}}$, $g = \rho^{\frac{5}{12}}$, $r = \frac{6}{5}$, $p = \frac{12}{7}$ and $q = 4$.

The last norm on the right hand side of (35) can be estimated by splitting the domain of integration in the definition of ρ into the regions where $|v|$ is less than or greater than some positive number R.

$$\begin{aligned}\rho(t,x) &= \int_{|v|\le R} f(t,x,v)dv + \int_{|v|\ge R} f(t,x,v)dv \\ &\le \frac{4\pi}{3}\|f\|_{L^\infty}R^3 + R^{-2}\int_{|v|\ge R} f(t,x,v)|v|^2dv \\ &\le \frac{4\pi}{3}\|f\|_{L^\infty}R^3 + 2R^{-2}\bar{e} \end{aligned} \tag{38}$$

where $\bar{e}(t,x) = \frac{1}{2}\int_{\mathbf{R}^3} f(t,x,v)|v|^2dv$. Optimizing in R shows that

$$|\rho(t,x)| \le C\bar{e}^{3/5}\|f\|_{L^\infty}^{2/5} \tag{39}$$

for a positive constant C. Since $\|f\|_{L^\infty}$ is known to be bounded it can be absorbed into the constant. Raising this inequality to the power $\frac{5}{3}$ and integrating in x allows the the power of the $L^{5/3}$ norm of ρ in (35) to be estimated by a constant multiple of $\mathcal{E}_{\rm kin}^{1/4}$. This leads to an inequality of the form $|\mathcal{E}_{\rm pot}| \le C\mathcal{E}_{\rm kin}^{1/2}$. Since the exponent on the right hand side of this inequality is less than one this can be combined with the conservation of energy to see that both $\mathcal{E}_{\rm kin}$ and $\mathcal{E}_{\rm pot}$ are bounded.

The fact that all the estimates above were carried out in three dimensions is motivated by the fact that physical space is three-dimensional. From a mathematical point of view it can be useful to relax this restriction and see things in a wider context. Consider then the analogue of the Vlasov-Poisson system is n space dimensions with $n \ge 3$. The solution formula for the Poisson equation generalizes to

$$U(t,x) = -\frac{1}{(n-2)\sigma_n}\int_{\mathbf{R}^3} \frac{\rho(y)}{|x-y|^{n-2}}dy \tag{40}$$

where σ_n is the volume of the unit sphere in $\mathbf{R}^n$. Using the same method as above gives estimates of the form

$$\|U\| \leq C\|\rho\|_{L^1}^{\frac{2}{n}}\|\rho\|_{L^\infty}^{\frac{n-2}{n}}, \quad \|\nabla U\| \leq C\|\rho\|_{L^1}^{\frac{1}{n}}\|\rho\|_{L^\infty}^{\frac{n-1}{n}} \tag{41}$$

The estimate for the second derivatives of U remains essentially unchanged in higher dimensions. In n dimensions the Hardy-Littlewood-Sobolev inequality allows the square root of potential energy to be estimated by the $L^{\frac{2n}{n+2}}$ norm of ρ as in the first inequality of (35). The exponent $\frac{3}{5}$ in (39) is replaced by $\frac{n}{n+2}$. In order to directly extend the argument in three dimensions to general n it would be necessary to bound the $L^{\frac{2n}{n+2}}$ in terms of the L^1 and the $L^{\frac{n+2}{n}}$ norms. This is only possible if $\frac{2n}{n+2} \leq \frac{n+2}{n}$, an inequality which fails for $n > 4$. Thus for $n > 4$ it cannot be shown that the potential energy is bounded by a power of the kinetic energy, at least not by the method introduced above. For $n = 3, 4$ the estimate $|\mathcal{E}_{\rm pot}| \leq |\mathcal{E}_{\rm kin}|^{\frac{n-2}{2}}$ is obtained. For $n = 4$ the exponent in this inequality is equal to one and so this estimate cannot be applied as it was in the case $n = 3$ to bound the potential and kinetic energies. In the case $n > 4$ the argument breaks down at an even earlier stage.

In fact global existence fails for the Vlasov-Poisson system in dimension ≥ 4. This can be proved by considering the evolution of the quantity

$$J = \int_{\mathbf{R}^n} \rho(t,x)|x|^2 dx. \tag{42}$$

It satisfies

$$\frac{dJ}{dt} = 2\int\int (x \cdot v) f dx dv \tag{43}$$

and

$$\frac{d^2J}{dt^2} = 2\int\int |v|^2 f dx dv - 2\gamma\int\int (x\cdot\nabla U) f dx dv \tag{44}$$

The second integral can be written as $\int (x\cdot\nabla U)\rho dx$. Replacing ρ by ΔU using the Poisson equation and integrating by parts gives

$$\frac{d^2J}{dt^2} = 4\mathcal{E} - \gamma\frac{4-n}{2}\int|\nabla U|^2. \tag{45}$$

In the case $\gamma = -1$ it can be concluded that $\frac{d^2J}{dt^2} \leq \mathcal{E}$ when $n \geq 4$. Integrating in time gives

$$J(t) \leq J(0) + t\frac{dJ}{dt}(0) + 2t^2\mathcal{E}. \tag{46}$$

If the energy $\mathcal{E}$ is negative then the expression on the right hand side of this inequality becomes negative for large t. For a global solution this is in contradiction to the fact that J is manifestly positive.

In the case of four space dimensions there is an explicit solution where a singularity forms from smooth data [9]. It will now be presented. The first step is to look for solutions of a special form. Let $y^i = \frac{x^i}{1-t}$ and $w^i = (1-$

$t)v^i + x^i$. Consider a distribution function of the form $f(t, x^i, v^i) = g(y^i, w^i)$ and a potential of the form $U(t, x^i) = (1-t)^{-2}V(y^i)$. Then when the spatial dimension n is four the Vlasov-Poisson system for (f, U) is equivalent to the same equation for (g, V). Note that g is independent of time so that it is a static solution. On the other hand it is clear that (f, U) has a singularity as $t \to 1$. It follows that any static solution of the Vlasov-Poisson system in four space dimensions which is C^1 and of compact support leads to a dynamical solution which becomes singular in the way just described. The existence of appropriate static solutions is discussed in the next section.

2.3 Static solutions

By a static solution of the Vlasov-Poisson system we mean one which is independent of t. In this section only the case $\gamma = -1$ is considered. The energy of a particle, $E = \frac{1}{2}|v|^2 + U$, is conserved along the characteristics of the Vlasov equation. This implies that if Φ is an arbitrary function a solution of the Vlasov equation is defined by

$$f(t, x^i, v^i) = \Phi(E) = \Phi\left(\frac{1}{2}|v|^2 + \gamma U(x^i)\right) \tag{47}$$

If this expression is substituted into the Poisson equation a nonlinear integrodifferential equation is obtained. Solving this with suitable boundary conditions can be used to produce models for equilibrium configurations of collisionless matter. This construction with $\gamma = -1$ is used in practice to produce models for galaxies or globular clusters.

The choice of the free function Φ which has been studied most is $\Phi(E) = (E_0 - E)_+^k$. Here E_0 is a constant and k is a positive real number with $\frac{1}{2} < k < \frac{7}{2}$ and for any function F the function F_+ is defined to be equal to F whenever F is positive and zero otherwise. Now restrict to the case of spherical symmetry. This means by definition that the unknowns are invariant under the natural action of the group of rotations of $\mathbf{R}^n$. More specifically, if $A \in SO(n)$ then $U(Ax) = U(x)$ and $f(t, Ax, Av) = f(t, x, v)$. The spherically symmetric static solutions of the Vlasov-Poisson system with the above choice of Φ are known as polytropic models. Define the pressure p by

$$p(t, x) = \int_{\mathbf{R}^n} |v|^2 f(t, x, v)\, dv \tag{48}$$

A computation shows that in this case (with $n = 3$) the density ρ is proportional to $(E_0 - U)^{k+\frac{3}{2}}$ and the pressure p is proportional to $(E_0 - U)^{k+\frac{5}{2}}$. Hence $p = K\rho^{\frac{2k+5}{2k+3}}$ for a constant K. In this way the models with collisionless matter can be related to models where the matter is described by a perfect fluid with a certain equation of state. This leads to a connection with the theory of stellar models which has been studied for more than a century. Introducing a new variable by $\theta = \rho^{\frac{1}{2k+3}}$ gives rise to an ordinary differential equation called the Lane-Emden equation. Much is known about its solutions. Here only solutions which are smooth at the centre are discussed. These will be called regular. Among other things it is known that for $-\frac{1}{2} < k < \frac{7}{2}$ the unique regular solution is such that the density decreases outwards and becomes zero at a finite radius. For $k = \frac{7}{2}$ the density only tends to zero in the limit $r \to \infty$ but the configuration has finite total mass $\|f\|_{L^1}$. In stellar dynamics this is known as the Plummer model. For $k > \frac{7}{2}$ the mass of the configuration is infinite and so it cannot usefully be applied in practise.

In applications an important question is that of the stability of static solutions. If the initial data for the static solution is perturbed a little, does the solution determined by the perturbed data stay close to the original solution in a suitable sense? It turns out that the answer to that question is different for the fluid model and the kinetic model. For a fluid it is classical in the astrophysics literature that solutions are stable for $k < \frac{3}{2}$ and unstable otherwise. In the case of kinetic theory the solutions are stable for $k \leq \frac{7}{2}$. This statement has been in the astrophysics literature for a long time and has relatively recently been proved rigorously.

Consider now the case of general dimension $n \geq 3$. In this case ρ is proportional to $(U - E_0)^{k+\frac{n}{2}}$, p is proportional to $(U - E_0)^{k+\frac{n}{2}+1}$ and $p = K\rho^{\frac{2k+n+2}{2k+n}}$. For $n = 4$ there also exist static solutions for a suitable range of k.

2.4 The relativistic Vlasov-Poisson system

The Vlasov-Poisson system has a clear interpretation within Newtonian physics and is invariant under the Galilei group. In certain areas of plasma physics particles may occur whose velocities are not negligible in comparison to the speed of light. Then it is desirable to include relativistic effects within the model. A minimal way of doing this is what is called the relativistic Vlasov-Poisson system. On a mathematical level the stellar dynamics case may be generalized in exactly the same way although it is not clear that the resulting system has any interesting physical applications. Define $\hat{v} = \frac{v}{\sqrt{1+|v|^2}}$. This is sometimes called the relativistic velocity. The relativistic Vlasov-Poisson system is based on the equations of motion

$$\dot{x} = \hat{v}, \quad \dot{v} = \gamma \nabla U \tag{49}$$

where as before the parameter γ is $+1$ or -1. The corresponding Vlasov equation reads

$$\partial_t f + \hat{v} \cdot \nabla_x f + \gamma \nabla_x U \cdot \nabla_v f = 0 \tag{50}$$

This is coupled to the Poisson equation with source ρ defined just as before. This system is not a fully relativistic description, i.e. it is not invariant under the Lorentz group. The Poisson equation and the coupling between the two equations retain non-relativistic aspects. The system as a whole is a kind of hybrid which is invariant under neither the Galilei nor the Lorentz group. Note that all L^p norms of f are independent of time by the same argument as in the case of the classical Vlasov-Poisson system.

There is a conserved energy which is the sum of a potential energy which is as in the case of the ordinary Vlasov-Poisson system and a kinetic energy which is defined by

$$\mathcal{E}_{\rm kin} = \frac{1}{2}\int\int \sqrt{1+|v|^2} f(t,x,v)dxdv. \tag{51}$$

The main reason for introducing this system here is to show that the global existence theorem which holds for the Vlasov-Poisson system should not be taken for granted. For the relativistic Vlasov-Poisson system (in three space dimensions) global existence fails in the stellar dynamics case. There are smooth initial data for which the corresponding solution becomes singular after finite time. In the plasma physics case the global existence question is still open. There is no global existence theorem and no proof that singularities occur for any choice of smooth initial data. To get a better idea of the difference between the classical and relativistic models it is useful to look at Horst's estimate. If we try to derive an analogue of (38) then the power $\frac{3}{5}$ is replaced by $\frac{3}{4}$ since the R^{-2} in the preceding estimate is replaced by R^{-1}. To make use of this it is necessary choose different exponents in Hölder's inequality when estimating $\|\rho\|_{L^{\frac{6}{5}}}$. In this case take $f=\rho^{\frac{1}{3}}$, $g=\rho^{\frac{2}{3}}$, $r=\frac{6}{5}$, $p=3$, $q=2$. The resulting inequality is

$$\|\rho\|_{L^{\frac{6}{5}}} \le \|\rho\|_{L^1}^{\frac{1}{3}} \|\rho\|_{L^{\frac{4}{3}}}^{\frac{2}{3}} \tag{52}$$

This leads to $|\mathcal{E}_{\rm pot}| \le C|\mathcal{E}_{\rm kin}|$. This is similar to what is obtained for the Vlasov-Poisson system in four space dimensions.

The fact that global existence fails for the relativistic Vlasov-Poisson system in the stellar dynamic case has been known for many years. The proof was by contradiction and is similar to that for the non-relativistic Vlasov-Poisson system given above, but a bit more complicated. Moreover the argument which will now be presented only applies in the spherically symmetric case. Consider a solution for which the energy $\mathcal{E}$ is negative. Computations give the identities

$$\frac{d}{dt}\left(\int\int (x\cdot v) f dxdv\right) = 2\mathcal{E} - \int\int \frac{f}{\sqrt{1+|v|^2}} \tag{53}$$

and

$$\frac{d}{dt}\left(\int\int |x|^2\sqrt{1+|v|^2} f dxdv\right) = 2\int\int (x\cdot v) f dxdv - \int |x|^2 E\cdot j dx \tag{54}$$

where

$$j = \int \hat{v} f dv \tag{55}$$

Integrating (53) in time shows that

$$\int\int (x\cdot v) f(t,x,v)dxdv \le \int\int (x\cdot v) f(0,x,v)dxdv + 2\mathcal{E}t \tag{56}$$

Now the spherically symmetric nature of the solution will be used to estimate the second term on the right hand side of (54). In that case the field can be written as

$$\nabla U(t,x) = \frac{x}{r^3}\int_0^r s^2\rho(t,s)ds \tag{57}$$

Let

$$M(t,x) = \frac{x}{r}\int_0^r s^2\rho(t,s)ds \tag{58}$$

The quantity M can be bounded uniformly by constant M_0 due to the conservation of the L^1 norm of ρ. Hence

$$\left|\int |x|^2 E\cdot\hat{j}dx\right| \le M_0\int|\hat{j}|dx \le M_0\int\rho dx \le M_0^2. \tag{59}$$

Using this in (54) and integrating in time gives

$$\int\int |x|^2\sqrt{1+|v|^2}f \le C_1 + C_2 t + \mathcal{E}t^2 \tag{60}$$

where C_1 and C_2 are constants which only depend on the initial data. For $\mathcal{E} < 0$ this once again shows that assuming global existence of a solution leads to a contradiction. More recently a rather explicit singular solution was found [10] and this gives some more insight into what happens. In fact this is a solution of the equation for massless particles, which means that the expression $\sqrt{1+|v|^2}$ is replaced by $|v|$. Its relevance comes from the fact that the particles are moving at close to the speed of light as the singularity is approached.

2.5 Local existence for the Vlasov-Poisson system

Let f_0 be a smooth function of compact support on $\mathbf{R}^6$. The aim of this section is to sketch how the existence of a solution of the Vlasov-Poisson system with initial data f_0 can be proved. The discussion here concerns existence locally in time, which is the first step towards proving global existence. The basic strategy is to define a sequence (f_n, U_n) by solving the Vlasov and Poisson equations alternately and then to show that this sequence converges on a suitable (possibly short) time interval to a limit (f, U). If the convergence takes place in a strong enough norm it follows that the limiting quantities satisfy the Vlasov-Poisson system with the desired initial datum.

This procedure will now be described in some more detail. The aim here is not to give a complete proof, but to explain some of the main ideas which play a role in a proof of this type. First extend the initial datum f_0 to $(0,\infty)$ in a time independent way. Define $\rho_0 = \int f_0 dv$ and let U_0 be the unique solution of $\Delta U_0 = \rho_0$ which vanishes at infinity. This function U_0 is also independent of time. Define f_1 to be the unique solution of the equation

$$\partial_t f_1 + v\cdot\nabla_x f_1 + \gamma\nabla_x U_0\cdot\nabla_v f_1 = 0 \tag{61}$$

with initial datum f_0. More generally if f_n and U_n are known define f_{n+1} and U_{n+1} to be the solutions of the equations

$$\partial_t f_{n+1} + v \cdot \nabla_x f_{n+1} + \gamma \nabla_x U_n \cdot \nabla_v f_{n+1} = 0 \tag{62}$$

and

$$\Delta U_{n+1} = \rho_{n+1} \tag{63}$$

Here it is assumed that f_n has initial data f_0 and that U_n tends to zero at infinity while ρ_n is defined by integrating f_n with respect to v. In this way a sequence (f_n, U_n) is defined recursively. For each value of n the regularity statements are obtained that f_n is C^1 and the restriction of U_n to each fixed time is C^2.

Let $P_n(t)$ be the maximum velocity in the support of f_n on the interval $[0, t]$. Note that $\|f_n(t)\|_{L^\infty}$ is known to be bounded independently of n and t, as is $\|\rho_n(t)\|_{L^1}$. Now $\|\rho_n(t)\|_{L^\infty} \leq CP_n(t)^3$ and so using an estimate for $\|\nabla U_n\|_{L^\infty}$ derived previously shows that it can be estimated by a quantity of the form $CP_n(t)^2$. Substituting this information into the characteristic system gives the inequality

$$P_n(t) \leq P(0) + \int_0^t P_n(s)^2 ds. \tag{64}$$

Now compare this with the solution of the ODE $\dot{z} = z^2$ with initial datum $P(0)$. A comparison theorem for solutions of integral inequalities shows that $P_n(t) \leq z(t)$ as long as the solution z exists. The solution of the equation for z exists on some interval $[0, T_*]$ and so a bound is obtained for the iterates on that interval. This then also leads to uniform bounds for $\|\rho_n(t)\|_{L^\infty}$ and $\|\nabla U_n\|_{L^\infty}$ on that interval.

In order to show convergence of the iterates in a strong enough sense to prove that the limit satisfies the VP system it is necessary to obtain estimates of derivatives of the quantities estimated up to now. Differentiating the Vlasov equation with respect to x gives

$$\partial_t(\nabla_x f_{n+1}) + v \cdot \nabla_x(\nabla_x f_{n+1}) + \gamma \nabla_x U_n \cdot \nabla_v(\nabla_x f_{n+1}) = -\nabla_x^2 U_n \nabla_v f_{n+1} \tag{65}$$

It can be concluded that

$$\|\nabla_x f_{n+1}\|_{L^\infty} \leq C + \int_0^t \|\nabla_v f_{n+1}(s)\|_{L^\infty} \|D^2 U_n(s)\|_{L^\infty} ds \tag{66}$$

The derivative $\nabla_v f_{n+1}$ can be estimated in a similar way. Putting these estimates together gives

$$\|f_{n+1}(t)\|_{C^1} \leq C \left(1 + \int_0^t (1 + \|D^2 U_n\|_{L^\infty}) \|f_{n+1}(s)\|_{C^1} ds\right) \tag{67}$$

Using an estimate previously obtained for the second derivatives of U gives

$$\|f_{n+1}(t)\|_{C^1} \leq C \left(1 + \int_0^t (1 + \log^*(\|f_n(s)\|_{C^1}) \|f_{n+1}(s)\|_{C^1} ds\right) \tag{68}$$

In this inequality both n and $n+1$ occur. To get an integral inequality for a single quantity define $N_n = \max_{0\le m\le n}\|f_m\|_{C^1}$. Then, using the fact that the function $\log^*$ is monotone increasing

$$N_{n+1}(t) \le C\left(1+\int_0^t (1+\log^*(N_{n+1}(s)))N_{n+1}(s)ds\right) \tag{69}$$

Now compare with the solution of the corresponding ODE, $\dot z = Cz(1+\log^* z)$. This easily gives boundedness of $\|f_{n+1}\|_{C^1}$ on an interval independent of n. In fact it gives more since the solution of the ODE exists globally in time. Hence $\|f_{n+1}\|_{C^1}$ is uniformly bounded on the time interval $[0,T^*]$ introduced previously. To show that the solution of the ODE exists globally it suffices to show that it is bounded on any finite time interval. Note that z is an increasing function so that attention can be confined to a region where $z\ge 1$. Thus the equation can be replaced by $\dot z = 2Cz\log z$. By rescaling the time coordinate it can be assumed without loss of generality that $C=\frac{1}{2}$. Now use the fact that the general solution of the equation $\dot z = z\log z$ with positive initial value is $z(t)=e^{e^{t+C}}$. Once it is known that $\|f_{n+1}\|_{C^1}$ is bounded it can be concluded that $\|\rho_{n+1}\|_{C^1}$ is bounded and that $\|U_n\|_{C^2}$ is bounded.

It has now been shown that the iteration is bounded but what is needed is some kind of convergence. One way of proceeding, which will not be elaborated on here, is to use compactness arguments. It then remains to show that it is justified to pass to the limit in the equations and that the limit obtained has the desired regularity. An alternative method, which is more elementary but requires more calculations, is to obtain estimates for differences of iterates and use the contraction mapping principle (Banach fixed point theorem) in a suitable Banach space.

When this argument is carried through it gives not only local existence but also a useful continuation criterion - as long as $P(t)$ remains bounded the solution continues to exist.

2.6 Global existence for small data

The subject of this section is a proof of global existence of solutions of the Vlasov-Poisson system for small initial data. For that system global existence is known for general initial data. There are nevertheless reasons for studying the small data proof. The first is that analogues of this proof work in cases where global existence for general data is not known or not even true. The second is that this proof gives more than just existence. It also gives rather precise information about the late-time asymptotics.

Consider first a solution of the Vlasov equation with $U=0$. This is sometimes known as 'free streaming'.

$$\rho(t,x)=\int f_0(t,x-vt,v)dv = t^{-3}\int f_0\left(\xi,\frac{x-\xi}{t}\right)d\xi \tag{70}$$

Hence $\|\rho(t)\|_{L^\infty}\le Ct^{-3}$. If this is put into the estimates previously derived for U then good control of the coefficients in the Vlasov equation is obtained.

It turns out to be useful to assume a little bit less. Assume then that we know that on some time interval $\|\rho(t)\|_{L^\infty} \leq Ct^{-3+\delta}$ for some $\delta > 0$. Then it follows that $\|\nabla U\|_{L^\infty} \leq Ct^{-2+2\delta/3}$. For δ sufficiently small this means that $\|\nabla U\|_{L^\infty} \leq Ct^{-1-\epsilon}$ for some $\epsilon > 0$. Hence the force on a particle is integrable in time and the paths of the particles are asymptotically linear. Thus we have asymptotically free streaming in a weak sense. The aim is now to recover the estimate for $\|\rho\|_{L^\infty}$ with the power -3. It turns out that for this it is also necessary to estimate one derivative more. The idea is to change variables in the integral defining ρ from (x, v) to $(X(0, t, x, v), v)$ and it is necessary to control the Jacobian of this transformation. For this information is needed on the first derivatives of the solution of the characteristic system and hence on the second derivatives of U. The Jacobian can be estimated from below by an expression of the form Ct^3. It then follows that $\|\rho(t)\|_{L^\infty} \leq Ct^{-3}$ as in the case of free streaming. It also follows that $P(t)$ is bounded.

It may sound as if the argument just sketched is circular since it was assumed at the beginning that the solution exists globally and has certain decay properties. It is, however, the case that the arguments just presented can form the basis for the proof of a theorem where it is shown that solutions exist globally for small data and that these solutions have the type of asymptotics considered above. One approach to this is to set up an iteration where the first iterate is a free-streaming solution and obtain the relevant estimates for each iterate. This is done in detail in [1]. Another approach to use what is called a bootstrap argument. This is similar to mathematical induction but uses a continuous parameter (such as time) instead of a discrete index. It requires an improvement in the asymptotics as in the replacement of $-3+\delta$ by -3 in the example above.

2.7 Global existence

The proof of global existence of solutions of the Vlasov-Poisson system with general initial data proceeds in a different way to the above. Existence is obtained without precise information on the asymptotic behaviour of the solution. In the method introduced by Pfaffelmoser and developed further by Schaeffer an important element is that instead of estimating the maximum value of ∇U (maximum force on the particle) the integral of this quantity on a suitable short time interval is considered. The resulting integral is written as a sum of three parts, which Schaeffer calls the good, the bad and the ugly. These have the following intuitive interpretation. The aim is to understand the total effect which one particle (call it the source particle) has on another (call it the target particle) on the given (short) time interval. The good set corresponds to the situation where either the velocity of the source particle is small or the velocity of the source particle is not too much different from that of the target particle. Let a fixed characteristic corresponding to the target particle be denoted by $(\hat{X}(s), \hat{V}(s))$. Let the characteristic flow describing the source particles be denoted by $(X(s, t, x, v), V(s, t, x, v)$ as before. For some constant parameters

Δ and Q to be chosen appropriately the good set is defined by

$$G = \left\{(s,x,v) : t - \Delta < s < t \text{ and } (|v| < Q \text{ or } |v - \hat{V}(t)| < Q)\right\} \tag{71}$$

This set is good because it is not too large. In the case of the bad set both the velocity of the source particle and its velocity relative to the target particle are large. At the same time the distance between the two particles is small, in comparison to one of these velocities. For some appropriate positive constant R the definition is

$$\begin{aligned} B = \{(s,x,v) : t - \Delta < s < t, |v| > Q, |v - \hat{V}(t)| > Q, \\ |X(s,t,x,v) - \hat{X}(s)|, R \max\{|v|^{-3}, |v - \hat{V}(t)|^{-3}\}\} \end{aligned} \tag{72}$$

This set too is in a sense also not too large, although this is more difficult to use than in the case of the good set. The ugly set is the complement of the good and bad sets. It is in estimating the contribution from the ugly set that the integration in time is crucial. In this case the target particle cannot remain too long in region where the density is very large or where the distance to the source particle is too small, due to its high velocity.

The alternative proof of global existence for the Vlasov-Poisson system, due to Lions and Perthame, makes extensive use of moments like $\int\int |v|^p f dx\, dv$. It is more flexible in the context of more general systems but usually gives less precise asymptotics.

2.8 Asymptotic behaviour

This global existence theorem provides only limited information on the asymptotic behaviour of the solution for t large. This is not surprising since the proof is valid for both signs of γ while the asymptotic behaviour can be expected to be quite different in these two cases. In the case $\gamma = -1$ it is known that static solutions exist. Thus it cannot be the density tends to zero in any sense. On the other hand it has been proved that in the plasma physics case the $L^{5/3}$ norm of ρ tends to zero like $t^{-\frac{3}{5}}$ as $t \to \infty$. It might be guessed that in the case $\gamma = 1$ all solutions enter the small data regime at late times but this seems to be an open question at present. In the stellar dynamics case it would be interesting to know if the density ρ remains bounded at late times. This is apparently also not known. A known estimate which appears far from optimal is $P(t) \leq C(1+t)\log(2+t)$. In the plasma physics case this has been improved to $P(t) \leq C(1+t)^{\frac{2}{3}}$.

3 The Vlasov-Maxwell system

3.1 Introductory remarks

In some situations the description of a plasma by the Vlasov-Poisson system is not sufficient. It may, for instance, be important to include magnetic fields. In order to do this the Poisson equation of electrostatics is replaced by the Maxwell equations of electrodynamics. The unknowns in the Maxwell equations are two vector-valued functions E and B, the electric and magnetic fields, respectively. The Maxwell equations are

$$\tfrac{1}{c}\tfrac{\partial E}{\partial t} = \nabla \times B - \tfrac{4\pi}{c} j, \quad \nabla \cdot E = 4\pi\rho \tag{73}$$

$$\tfrac{1}{c}\tfrac{\partial B}{\partial t} = -\nabla \times E, \qquad \nabla \cdot B = 0 \tag{74}$$

where c is the speed of light. The quantities on the right hand sides of these equations other than the electric and magnetic fields are the charge and current densities ρ and j defined by a distribution of charged particles. The charge density ρ is defined just as in the case of the Vlasov-Poisson system. The current density is given by

$$j(t,x) = \int_{\mathbf{R}^3} \hat{v} f(t,x,v) dv \tag{75}$$

In this case the Vlasov equation reads

$$\partial_t f + \hat{v} \cdot \nabla_x f + \left(E + \frac{\hat{v}}{c} \times B \right) \cdot \nabla_v f = 0 \tag{76}$$

To be clear, the system which is considered here is what is sometimes called the relativistic Vlasov-Maxwell system. It shares the property of invariance under Lorentz transformations which holds for the source-free Maxwell equations. This contrasts with the invariance of the Vlasov-Poisson system under the Galilei group. In plasma physics sometimes a 'non-relativistic Vlasov-Maxwell system' is considered which is a hybrid like the relativistic Vlasov-Poisson system. It will not be mentioned further in this course. The Vlasov-Maxwell system is invariant under the Lorentz group and it can be written in rather concisely in terms of spacetime notation. For this purpose it is convenient to choose physical units such that the speed of light has the numerical value unity. Define $F_{0i} = E^i$ for $i = 1,2,3$, $F_{32} = B^1$, $F_{13} = B^2$, $F_{21} = B^3$ and require that $F_{\alpha\beta} = -F_{\beta\alpha}$. Here the Greek indices take the values $0,1,2,3$. Define $j^0 = \rho$. Then the Maxwell equations as written above, with $c = 1$ are equivalent to the equations $\nabla^\alpha F_{\alpha\beta} = 4\pi j_\beta$ and $\nabla_\alpha F_{\beta\gamma} + \nabla_\gamma F_{\alpha\beta} + \nabla_\beta F_{\gamma\alpha} = 0$. Here the convention is used that the value of an object with indices does not depend on the position (upper or lower) of a spatial index $(1,2,3)$ but is multiplied by -1 when the position of the time index is changed. In this notation the Vlasov equation can be written in the form

$$v^\alpha \frac{\partial f}{\partial x^\alpha} - F^a{}_\beta v^\beta \frac{\partial f}{\partial v^a} = 0 \tag{77}$$

Here the notation is used that $x^0 = t$ and $v^0 = \sqrt{1 + |v|^2}$. In the following the notation with electric and magnetic fields will mostly be used.

The L^p norms of the distribution function f are conserved by the evolution defined by the Vlasov-Maxwell system. The only part which is slightly more complicated than in the Vlasov-Poisson system is to treat the term containing the magnetic field. For example

$$\partial_{v^1}(v^2 B^3 - v^3 B^2) = 0. \tag{78}$$

Two of the four Maxwell equations contain no time derivatives and thus represent restrictions on the initial data. They are known as the constraint equations. Initial data for the Vlasov-Maxwell system consist of a set (f, E, B) which satisfies the constraints. The other two Maxwell equations are known as the evolution equations. Usually the constraint equations play only a minor role in the analysis of solutions of the Vlasov-Maxwell system because of the following fact, known as 'propagation of the constraints'. If the constraint equations are satisfied on some initial hypersurface $t = t_0$ and if the Maxwell evolution equations and the Vlasov equation are satisfied on a time interval $[t_0, t_1)$ then the constraints are satisfied on the interval $[t_0, t_1)$. This is proved by differentiating the constraint quantities (whose vanishing constitutes the constraints) with respect to time and noting that the result vanishes.

In the spherically symmetric case the Vlasov-Maxwell system reduces to the relativistic Vlasov-Poisson system with $\gamma = 1$. To see this, note first that the spherical symmetry implies that $B = b(r)x$ for some function b. Then the field equation $\nabla \cdot B = 0$ and the condition of regularity at $x = 0$ show that $B = 0$. Similarly, spherical symmetry shows that $E = e(r)x$ for some function e and this implies that E is the gradient of a function U. Making these substitutions in the Vlasov-Maxwell system brings us back to the relativistic Vlasov-Poisson system with $\gamma = 1$. The evolution equation for E is solved automatically. It would in principle also be possible to consider the Vlasov-Maxwell system with a different sign corresponding to the case $\gamma = -1$ of the relativistic Vlasov-Poisson system. However it appears that this system has no physical applications and it has also not been considered in the mathematics literature.

There is no general global existence theorem for solutions of the Vlasov-Maxwell system available comparable with that which has been proved for the Vlasov-Poisson system. A theorem of this type is known in the spherically symmetric case. As far as the case without symmetry is concerned there is apparently not even an intuitive argument why global existence should fail for this system.

There are global existence theorems for solutions of the Vlasov-Maxwell equations with small initial data, for solutions with certain symmetry properties and for solutions in two space dimensions. It is perhaps not obvious what the Maxwell equations should be in two dimensions. There is a natural definition which is most easily seen from the spacetime formulation of the Maxwell equations. When the equations are written in that way they can be taken to hold in any dimension, at least up to the multiplicative constant relating the fields to ρ and j, which is a question of the choice of physical units. In two space dimensions the electric field is a vector but the magnetic field is a scalar. This is because the magnetic field corresponds to a two-index object which is antisymmetric and therefore only has one independent component in two dimensions.

The Maxwell equations in two dimensions read

$$\frac{1}{c}\frac{\partial E}{\partial t} = \nabla \times B - \frac{2\pi}{c} j, \quad \nabla \cdot E = 2\pi\rho, \tag{79}$$

$$\frac{1}{c}\frac{\partial B}{\partial t} = -\nabla \times E. \tag{80}$$

The divergence equation for B familiar from three space dimensions is not present in two dimensions. It is necessary to explain the meaning of the operator $\nabla\times$ as applied to a scalar or a vector.

$$(\nabla \times B)^i = \epsilon^{ij}\nabla_j B, \quad \nabla \times E = \epsilon^{ij}(\nabla_i E_j) \tag{81}$$

where ϵ^{ij} is the alternating symbol. In this case the Vlasov equation reads

$$\frac{\partial f}{\partial t} + \hat{v}^1 \frac{\partial f}{\partial x^1} + \hat{v}^2 \frac{\partial f}{\partial x^2} + (E^1 + \hat{v}^2 B)\frac{\partial f}{\partial v^1} + (E^2 - \hat{v}^1 B)\frac{\partial f}{\partial v^2} = 0. \tag{82}$$

There is also a three-dimensional model for this. Consider a solution of the Vlasov-Maxwell system in three space dimensions and assume that $E_3 = 0$, $B_1 = B_2 = 0$, $j^3 = 0$ and that the unknowns in the equations do not depend on the coordinate x^3. Then, setting $B = B_3$ the three-dimensional Maxwell equations reduce to the two-dimensional case presented above. Note that the conditions on the electric and magnetic fields correspond to $F_{03} = F_{13} = F_{23} = 0$. From this it can be seen that all the extra assumptions are consequences of the assumptions that translations and the reflection in x^3 are symmetries of the solution. Under these symmetry assumptions the three-dimensional Vlasov-Maxwell system does not quite reduce to the two-dimensional case. A discrepancy arises from the fact that the unknown f in the Vlasov equation depends on three velocity variables in the one case and only on two in the other. Formally this problem can be removed by setting

$$f_{(3)}(t, x^1, x^2, x^3, v^1, v^2, v^3) = f_{(2)}(t, x^1, x^2, v^1, v^2)\delta(v^3). \tag{83}$$

Another interesting symmetry assumption is as follows. Consider the Vlasov-Maxwell system in two space dimensions and suppose that translations in the x^2 direction in $\mathbf{R}^2$ are symmetries, i.e. that E, B and f are invariant under these transformations. Thus E and B depend only on t and x^1 while f depends only on t, x^1 and the velocity variables. The reduced system of equations obtained is

$$\frac{1}{c}\frac{\partial E^1}{\partial t} = -\frac{2\pi}{c} j^1, \quad \frac{1}{c}\frac{\partial E^2}{\partial t} = -\partial_1 B - \frac{2\pi}{c} j^2, \tag{84}$$

$$\partial_1 E^1 = 2\pi\rho, \quad \frac{1}{c}\frac{\partial B}{\partial t} = -\partial_1 E^2, \tag{85}$$

It has been studied by Glassey and Schaeffer, who called it the one-and-one-half dimensional Vlasov-Maxwell system. They proved a global existence theorem for it. Later they also proved a global existence theorem for the full two-dimensional Vlasov-Maxwell system. Finally they studied the two-and-one-half dimensional

Vlasov-Maxwell system. This can be obtained by assuming that in a solution of the Vlasov-Maxwell system in three space dimensions translations x^3 and the reflection in x^3 are symmetries. This is the system discussed previously where the dependence of f on all three velocity variables is kept. Glassey and Schaeffer proved global existence for this system in 1997. Finally, it should be mentioned that global existence is known for almost spherically symmetric solutions of the Vlasov-Maxwell system.

As in the case of the Vlasov-Poisson system the Vlasov-Maxwell system has a conserved energy. It is defined by

$$\mathcal{E} = \int\int \hat{v} f(t,x,v)dxdv + \frac{1}{2}\int |E(t,x)|^2 + |B(t,x)|^2 dx. \qquad (86)$$

There is also a good continuation criterion for this system. A solution can be continued as long as $P(t)$ remains bounded, where $P(t)$ is defined just as the case of the Vlasov-Poisson system.

We have now seen two models, one Newtonian and one relativistic, for the same physical system, namely a plasma. What is the relation between the two? Introduce a parameter λ in the Vlasov-Maxwell system by the relation $c^{-1} = \lambda^2$. Then the non-relativistic limit $c \to \infty$ corresponds to the limit $\lambda \to 0$. Consider now expansions of the form

$$E = E_0 + \lambda^{1/2}E_1 + \lambda E_2 + \lambda^{3/2}E_3 + \ldots \qquad (87)$$

$$B = B_0 + \lambda^{1/2}B_1 + \lambda B_2 + \lambda^{3/2}B_3 + \ldots \qquad (88)$$

$$\rho = \rho_0 + \lambda^{1/2}\rho_1 + \lambda\rho_2 + \lambda^{3/2}\rho_3 + \ldots \qquad (89)$$

$$j = j_0 + \lambda^{1/2}j_1 + \lambda j_2 + \lambda^{3/2}j_3 + \ldots \qquad (90)$$

and insert them into the Vlasov-Maxwell system. This is done in a purely formal way without worrying about existence of a solution with this kind of expansion or convergence of the expansion if one exists. It will be assumed that the expansion coefficients are defined on $\mathbf{R}^3$ and vanish at infinity. Comparing coefficients shows that $\nabla \cdot B_0 = 0$ and $\nabla \times B_0 = 0$ and these together imply that $B_0 = 0$. The coefficient of order zero of the electric field satisfies $\nabla \cdot E_0 = \rho$ and $\nabla \times E_0 = 0$. The second of these equations implies that E_0 is of the form ∇U for a function U and the first implies that U satisfies the Poisson equation with source ρ_0. It follows that f_0 and U satisfy the Vlasov-Poisson system. This shows that the Vlasov-Poisson system is formally a limit of the Vlasov-Maxwell system. What is really desirable is a statement comparing solutions of the two systems under suitable conditions. This follows if it can be shown that there exist parameter-dependent solutions having expansions of the above type. It is not obvious that expansions of this type exist since the limit is singular. The type of the equations changes from hyperbolic to partly elliptic in the limit. It can be shown that expansions of this type do exist for low orders. For sufficiently high orders this simple type of expansion breaks down and something more sophisticated must be done.

It turns out that it is consistent to set the odd half-integer powers in the expansion of E, ρ and j and the integer coefficients in the expansion of B to zero and that this is enough for many physical applications. Continuing to compare coefficients leads to $\Delta B_1 = -4\pi\nabla \times j_0$. Thus we can produce a self-consistent truncated model containg only a potential and this contribution to the magnetic field. The next order expansion (one half order higher in the expansion) leads to the so-called Darwin approximation to Maxwell's equations.

3.2 The Vlasov-Nordström system

From the point of view of physics the Vlasov-Maxwell system is the natural relativistic generalization of the Vlasov-Poisson system in the plasma physics case. What is the corresponding system in the stellar dynamic case? It is much more complicated. A fully relativistic description of the gravitational field requires general relativity and this goes beyond the scope of this course. It should be noted however that the resulting Einstein-Vlasov system does have interesting physical applications. One is the formation of a massive black hole at the centre of a galaxy by collapse of a large collection of stars. Another is cosmology, which is the study of the structure of the universe on the largest scales we can observe. In the latter case the galaxies could be treated as 'particles' in the sense of a kinetic model.

It is very hard to prove theorems about the Einstein-Vlasov system and it would be nice to have a simpler model system which is fully relativistic and describes self-gravitating collisionless matter. There is a model of this kind, which is known as the Vlasov-Nordström system. It is based on the Nordström's theory of gravitation which was introduced early in the 20th century. It is not physically realistic and does not correctly reproduce the bending of starlight by the sun. It does nevertheless share some of the features of models which are of physical importance while being simpler. Interestingly a general global existence proof has been given for this system [3].

In the Vlasov-Nordström system the gravitational field is described by a real-valued function ϕ. The equation describing its evolution is

$$\partial_t^2\phi - \Delta\phi = -\mu \tag{91}$$

where

$$\mu(t,x) = \int f(t,x,v)\frac{dv}{\sqrt{1+|v|^2}}. \tag{92}$$

Let S denote the operator $\partial_t + \hat{v}\cdot\partial_x$. Then the Vlasov equation is

$$Sf - [(S\phi)v + (1+|v|^2)^{-1/2}\nabla_x\phi]\cdot\nabla_v f = 4fS\phi. \tag{93}$$

In contrast to the other equations of Vlasov type introduced above this equation does not imply that f is constant along the characteristics. In fact the particle density is given by $e^{-4\phi}f(t,x,e^{\phi}v)$. The function f used here turns out to be convenient for the analytic theory.

3.3 The source-free Maxwell equations

In the Vlasov-Poisson system if f is identically zero then the same is true of U. In contrast there can be non-zero fields in the Vlasov-Maxwell system even when there are no particles. These describe freely propagating electromagnetic waves. In the case of solutions of the Vlasov-Poisson system with small initial data the L^∞ norm of ∇U falls off like t^{-2} as $t \to \infty$. The free Maxwell field does not fall off that fast. In fact its fall-off is anisotropic, with different components falling off at different rates. A Maxwell field coming from a solution of the Vlasov-Maxwell system cannot be expected to fall off faster than a free field. To describe this in more detail it is convenient to use the spacetime description given by the tensor $F_{\alpha\beta}$. Let l^α and n^α be the vector fields with components $(1, x^1/r, x^2/r, x^3/r)$ and $(1, -x^1/r, -x^2/r, -x^3/r)$ and let e_2^α and e_3^α be an orthonormal basis (necessarily local) of vectors tangent to the unit sphere. Define the quantities $\alpha_A = F_{\mu\nu}e_A^\mu n^\nu$, $\bar{\alpha}_A = F_{\mu\nu}e_A^\mu l^\nu$ for $A = 2, 3$. Let $\rho = F_{\mu\nu}l^\mu n^\mu$ and $\sigma = F_{\mu\nu}e_2^\mu e_3^\nu$. These quantities contain all the information about $F_{\mu\nu}$. Let $\tau_- = [1 + (r-t)^2]^{1/2}$. The region where there is slow fall-off limiting the rate of fall-off of the L^∞ norm of the field in space is the exterior region defined by $r \geq 1 + \frac{t}{2}$. In this region

$$|\bar{\alpha}_A(t,x)| \leq Cr^{-1}\tau_-^{-3/2} \tag{94}$$

$$|(\rho,\sigma)(t,x)| \leq Cr^{-2}\tau_-^{-1/2} \tag{95}$$

$$|\alpha_A(t,x)| \leq Cr^{-5/2} \tag{96}$$

The consequences of this for the L^∞ norms of $\bar{\alpha}_A$, ρ, σ and α_A is that they fall off like t^{-1}, t^{-2}, t^{-2} and $t^{-5/2}$ respectively. For all quantities except α_A these powers are optimal. Taking derivatives in the directions l^α, e_2^α and e_3^α improves the fall-off by one order. On the other hand, taking a derivative in direction n^α gives no improvement whatsoever. In particular, taking spatial derivatives gives no improvement. This contrasts with what happens for the Vlasov-Poisson system. This means that an attempt to adapt the proof of small data global existence for the Vlasov-Poisson system to the Vlasov-Maxwell system, using L^∞ norms of the fields in space is doomed to failure. Instead it is necessary to use norms weighted by the factors included in the estimates above. It can be seen that the fall-off of the Maxwell field is anisotropic, both in terms of the different components of the field and in terms of different directions in spacetime. The physical phenomenon behind this is radiation.

3.4 Global existence for small data

To try to prove global existence for small initial data in the case of the Vlasov-Maxwell system it is natural to try to follow the proof of the corresponding result in the case of the Vlasov-Poisson system. There a couple of obstructions to doing this. One is that the Maxwell equations, as a hyperbolic system, have poorer regularity properties than the Poisson equation, which is elliptic. It is possible to obtain a small data global existence theorem for the Vlasov-Maxwell system based on pointwise estimates but it is hard work. The other difficulty has to do with the fall-off of the solution at infinity. In a solution of the Vlasov-Poisson system with small initial data the L^∞ norm of the force decays like t^{-2}. Unfortunately, as discussed in the last section, in the case of the Maxwell equations (even with small data and without sources) the L^∞ norms of the electric and magnetic fields fall off no faster than t^{-1}.

Concerning the regularity question the issue is the following. The natural function spaces for studying the initial value problem for hyperbolic equations are Sobolev spaces, i.e. spaces whose norms are defined by the L^2 norms of a function and its derivatives up to some order. Data in one of these spaces gives rise to solutions in the same space. To get pointwise information it is necessary to use the Sobolev embedding theorem, which relates Sobolev estimates to pointwise estimates. It involves some loss of derivatives. The standard proof of small data global existence for the Vlasov-Maxwell system is based on pointwise estimates and so is affected by this loss of derivatives. The fact that the proof nevertheless works shows that there is some room to spare - wasting derivatives does not lead to disaster. The same remark applies to the Vlasov-Poisson system but in a weaker form. To get optimal regularity estimates for an elliptic equation like the Poisson equation pointwise estimates must be avoided. It is necessary to use Hölder or Sobolev spaces. However when solving elliptic equations derivatives are gained in Hölder or Sobolev spaces to an extent which is greater than can be achieved for hyperbolic equations. It may be noted that the is a variant of the small data global existence result for the Vlasov-Maxwell system in the case of two species. The individual densities of the two types of particles in the initial data is allowed to be large as long as the total charge density is small. This is known as 'nearly neutral' data.

The proof of global existence is based essentially on the continuation criterion provided by $P(t)$. In getting good estimates for $P(t)$ there is a device which plays a crucial role. Here, instead of describing this device in the context of the full complexity of general solutions of the Vlasov-Maxwell system it will be explained in the simpler context of the one-and-one-half dimensional Vlasov-Maxwell system. It is essential in the proof of global existence even in that relatively simple case. Adding and subtracting the equations for E_2 and B gives propagation equations for these along the $t+x$- and $t-x$-directions with j_2 as source. This gives pointwise bounds for E_2 and B in terms of their initial data and the integrals of j^2 along certain lines. (In fact these lines are the characteristics of the Maxwell equations.) For instance

$$\begin{aligned} E^2(t,x) &= \frac{1}{2}[E_0^2(x-t)+B_0(x-t)+E_0^2(x+t)-B_0(x+t)] \\ &- 2\pi\int_0^t [j^2(\tau,x-t+\tau)+j^2(\tau,x+t-\tau)]d\tau. \end{aligned} \tag{97}$$

These integrals can be bounded by integrating the identity

$$\partial_t e+\partial_x(E^2B+4\pi\int f\sqrt{1+|v|^2}dv)=0 \tag{98}$$

where e is the energy density

$$e = \frac{1}{2}(|E|^2 + B^2) + 4\pi \int f\sqrt{1+|v|^2}dv \tag{99}$$

over a suitable triangle. E^1 can be bounded by integrating the equation for it in space and using charge conservation. Thus in this way pointwise bounds for the electric and magnetic fields are obtained. Putting this in the Vlasov equation shows that $P(t)$ is bounded and this in turn implies pointwise bounds on ρ and j. At this point we do not want to assume that the continuation criterion in terms of P has been proved. The intention is rather to show the essential element in its proof in the given situation.

The essential thing is to bound one more derivative. How can first order spatial derivatives of the electric and magnetic fields be estimated. To do this the Maxwell equations must be differentiated once with respect to the spatial variables. This gives rise to first order derivatives of ρ and j which can in turn be estimated in terms of the spatial derivative of f. To estimate the last quantity it is necessary to differentiate the Vlasov equation with respect to x and this gives rise to terms like the product of first derivatives of f with first derivatives of E. Suppose we define a quantity Q to be the maximum of the first derivatives of E, B and f. Then the procedure just outlined, if followed in the obvious way, leads to a differential inequality of the form

$$Q(t) \le C(1 + \int_0^t Q^2(s)ds) \tag{100}$$

and this is no use for obtaining a global bound for Q. The estimate needs to be refined. Differentiating the expression for E_2 with respect to x gives rise to integrals of $\partial_x j^2$ along lines of constant $t-x$ or $t+x$. The quantity j^2 can be expressed in terms of its definition as an integral in v. Let S and T denote the vector fields $\partial_t + \hat{v}^1\partial_x$ and $\partial_t + \partial_x$ respectively.

$$Tf(\tau, x-t+\tau, v) = \frac{d}{d\tau}[f(\tau, x-t+\tau, v)]. \tag{101}$$

This means that in an integral of the type under consideration here Tf can be estimated straightforwardly by using the fundamental theorem of calculus to eliminate the derivative with respect to τ. On the other hand Sf can be written as a divergence in the v variables. As a consequence its contribution to an integral of the form being considered here can be freed from derivatives by partial integration in v. Finally note that the derivative to be estimated, $\partial_x f$ can be expressed in terms of Sf and Tf, since $\partial_x = (1-\hat{v}^1)^{-1}(S-T)$.

4 The Boltzmann equation

The Boltzmann equation was written down in the introduction but the collision term was only discussed schematically. Now more details will be given. If an interaction law between point particles is given then it is possible to study the corresponding scattering problem. In other words, if two particles come in from infinity with asymptotically constant velocities, interact and go out to infinity again with asymptotically constant velocities, how do the final velocities depend on the initial velocities? It is this information which determines the kernel k in the collision term of the Boltzmann equation. Often, however, instead of doing this type of calculation a plausible form is chosen for k, ('Plausible' could mean that it is qualitatively similar to what is obtained in cases where the calculation has been done.)

Another issue which has to be addressed is the integration over the collision manifold. What is the measure used to do this integration? The measure can be characterized by invariance and normalization conditions. To work with the collision term in practise it is necessary to have a more explicit analytical expression for the integral. This means, in particular, that a convenient parametrization of the collision manifold must be chosen. This will be discussed next. By conservation of momentum it follows that $v'^i - v^i = -(w'^i - w^i)$. Write this quantity in the form $a\Omega^i$ where Ω^i is a unit vector. It follows that

$$\begin{aligned} v' &= v + a\Omega \\ w' &= w - a\Omega \end{aligned} \tag{102}$$

These relations ensure that conservation of momentum is satisfied automatically. The factor a depends on v^i, w^i and Ω^i. The explicit expression is obtained by imposing the condition of conservation of energy. The result is

$$a(v, w, \Omega) = \Omega \cdot (w - v). \tag{103}$$

Turning to the form of the collision cross-section, the usual assumption is that the measure used for the integration can be written in the form $q(\Omega, |v - w|)d\sigma$ where $d\sigma$ is the measure on the unit sphere. Putting all these facts together leads to the following explicit form for the collision term:

$$C(f)(w) = \int_{\mathbf{R}^3} \int_{|\Omega|=1} q(\Omega, |v - w|)(f(v')f(w') - f(v)f(w))d\Omega dv \tag{104}$$

where v' and w' are to be expressed in terms of v, w and Ω as above. A concrete example of a collision cross-section is given by that of hard spheres. In that case

$$q(\Omega, |v - w|) = b_0 |v - w| |\cos \theta| \tag{105}$$

where b_0 is related to the size of the spheres and $\Omega \cdot (w - v) = |w - v| \cos \theta$. If the interaction between particles is chosen to be a power law with a potential proportional to r^{-s} for some exponent s then the dependence of the collision cross-section on the argument $|v - w|$ is also power-law. A special case is where $s = 4$. In that case, known as Maxwell molecules, the cross-section is independent of $|v - w|$. This leads to mathematical simplifications and Maxwell molecules are a favourite test case in the mathematical study of the Boltzmann equation. It represents the boundary between soft interactions ($s < 4$) and hard interactions ($s > 4$). Hard spheres correspond formally to the limit $s \to \infty$.

Let ϕ be a function on $\mathbf{R}^3$. It is said to be a collisional invariant if $\phi(v') + \phi(w') = \phi(v) + \phi(w)$ for all (v, w, v', w') satisfying conservation of momentum and energy. In this case a computation shows that $\int_{\mathbf{R}^3} \phi(v) Q(f)(v) dv = 0$ for any f. With the choices 1, v^i and $|v|^2$ for ϕ the following identities are obtained:

$$\int \int Q(f) dx dv = 0 \tag{106}$$

$$\int \int v^i Q(f) dx dv = 0 \tag{107}$$

$$\int \int |v|^2 dx dv = 0 \tag{108}$$

Multiplying the Boltzmann equation by $\phi(v)$, integrating with respect to x and v and using one of these identities gives a conservation law for solutions of the Boltzmann equation. The three choices listed above give conservation of mass, momentum and energy respectively. In order to justify this statement it is necessary to know that the solutions have sufficient decay properties at infinity so that the relevant integrals converge. Under appropriate technical assumptions on the function ϕ it can be shown that the only collisional invariants are linear combinations of those listed above with constant coefficients. Assuming that $f > 0$ a similar calculation leads to the inequality

$$\frac{d}{dt} \int \int (-f \log f) dx dv \geq 0 \tag{109}$$

This is known as the Boltzmann H-theorem. Its physical interpretation is that entropy increases for solutions of the Boltzmann equation.

The case of equality in this inequality is given by $\int \int Q(f) \log f dx dv = 0$. It can be shown that the integrand is non-positive and only vanishes when $\log f$ is a collisional invariant. Hence $f(v) = \exp(a + b \cdot v + c|v|^2)$. Here a, b and c can depend on t and x. This is what is known as a (local) Maxwellian and represents a situation where at each spatial point the particles are in local equilibrium. Substituting this expression back into the left hand side of the Boltzmann equation leads to restrictions on the functions a, b and c. With suitable boundary conditions it may be possible to conclude that they are constant. Without further restrictions they do not need to be constant and solutions of this kind were found by Boltzmann himself. For a (local) Maxwellian the collision term $Q(f)$ vanishes identically and hence these are solutions of the free Vlasov equation which for which $\log f$ depends quadratically on v. The function $\log f$ itself satisfies the Vlasov equation. Consider for instance the initial datum $f_0(x, v) = \exp(-|x - t_0 v|^2)$ for a negative constant t_0. Then $f(t, x, v) = \exp(-|x - (t + t_0)v|^2)$. At time $-t_0$ the density ρ blows up. This bad dynamical behaviour is only possible because of the badly behaved initial data.

The Boltzmann equation is not invariant under time reversal and it cannot be expected that the Cauchy problem can be solved towards the past. Only existence in the future of the initial hypersurface is usually considered. Sometimes the irreversibility of the Boltzmann equation is seen as a philosophical problem, since it is derived, at least formally, from the dynamics of particles which is invariant under time reversal. In fact a derivation of the Boltzmann equation from particle dynamics requires an input where a decision is made to ignore certain information. This is what leads to the loss of reversibility.

A convenient property of the Vlasov equation is that if the initial data have compact support the support of the solution at any later time is also compact. This property is unfortunately not shared by the Boltzmann equation. Even if the support is initially compact it immediately becomes non-compact. Thus in this case initial data corresponding to an isolated system must be defined in terms of fall-off conditions. There is also another situation of physical interest. The Boltzmann equation has equilibrium solutions which are independent of time and spatially homogeneous. An interesting class of solutions are those for which the data are a small perturbation of a spatially homogeneous equilibrium solution.

There is no global existence theorem known for smooth solutions of the Boltzmann equation without symmetry or size restrictions. There is a famous result of DiPerna and Lions [5] where they proved the existence of so-called 'renormalized' solutions. These are solutions in a very weak sense and nothing is known about their uniqueness in terms of initial data. It is known that all these solutions converge to equilibrium in a certain sense as $t \to \infty$. There are global existence theorems for classical solutions in the cases of small initial data and the initial data close to that of an equilibrium solution. Without a small data assumption the only known global existence theorems are for the spatially homogeneous case, i.e. for the case where f does not depend on the space variables x^i. This kind of statement can be proved for a large class of collision kernels.

There is a relativistic generalization of the Boltzmann equation. In that case even less is known about existence of solutions than in the case of the classical Boltzmann equation. The global existence for data close to equilibrium has been generalized to the relativistic case but remarkably the small data case has turned out to be very difficult. The relativistic Boltzmann equation has interesting applications in astrophysics. An example is in supernova explosions. Here it plays an important role that huge quantities of neutrinos are produced - these have even been detected on earth. The density is so high that collisions of neutrinos are important and the process is modelled using the Boltzmann equation. In this case the particles are taken to be massless. The Boltzmann equation is also used to model the behaviour of particles in early epochs of the universe. Here the so-called quantum Boltzmann equation is important. This is a partial differential equation where quantum mechanics is not directly visible. In some places in the collision integral there are extra factors of the form $1 + \tau f$ where τ is plus or minus one. The first sign applies to bosons and the second to fermions. The first models the process of stimulated emission while the second expresses the Pauli exclusion principle.

It has been mentioned that the Boltzmann equation can be used to model a gas. Another common way of modelling a gas is as a fluid described by the compressible Euler equations. What is the relation between these two descriptions? Starting from a solution f of the Boltzmann equation it is possible to define the moments

$$\rho(t,x) = \int f(t,x,v)dv \tag{110}$$

$$j^i(t,x) = \int v^i f(t,x,v)dv \tag{111}$$

$$e(t,x) = \frac{1}{2}\int |v|^2 f(t,x,v)dv \tag{112}$$

These are the mass, momentum and energy densities. Provided the mass density does not vanish it is possible to define a velocity V^i by the relation $j^i = \rho V^i$. It is the quantities ρ, V^i and e which can be related to the corresponding quantities for a fluid. The starting point for doing this is called the Hilbert expansion. The idea of this is to introduce a parameter ϵ into the Boltzmann equation to get the equation

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\epsilon}Q(f). \tag{113}$$

Doing a suitable formal expansion in the limit $\epsilon \to 0$ shows that in leading order the distribution function f takes the form of a local Maxwellian and that the corresponding moments satisfy the Euler equations of hydrodynamics. There are also theorems which give a rigorous justification of this expansion in some cases and show that the Euler equations can be obtained as a limit of the Boltzmann equation as $\epsilon \to 0$. This procedure can be continued further to obtain the Navier-Stokes equations for a viscous heat-conducting fluid.

5 Kinetic models in biology

5.1 Introductory remarks

Chemotaxis is the process by which cells move in response to chemical gradients. Some models of this have been constructed using kinetic equations. Before coming to the differential equations some biological background will be presented. It is convenient here to distinguish between prokaryotic cells, which have no nucleus, and eukaryotic cells, which do have a nucleus. The former include bacteria while the latter include many unicellular organisms and most cells in higher plants and animals, including humans. For the mechanism of chemotaxis it is important that eukaryotic cells are usually much larger than prokaryotic cells. How can a cell measure a chemical gradient? One strategy is to measure the concentration at two different points of the cell and compute the difference. It can be shown that this cannot work for a typical bacterium. Due to the small size of the cell, diffusion from one end of the cell to the other is so fast that the cell cannot measure and compare the concentrations fast enough so as to extract useful information. An alternative strategy which is open to bacteria is the following. The cell moves fast in some random direction and compares the concentrations at two different times. If the concentration of the chemical substance to be detected is essentially constant at a given spatial point on the time scale of this measurement then this procedure can be used to provide information about the gradient.

A typical example of chemotaxis in a eukaryotic cell is that of neutrophils (a type of white blood cells). These cells move through tissues in order to check for the presence of foreign organisms such as bacteria. The bacteria produce small peptides with a formyl group attached. Since this type of molecule is not produced by eukaryotic cells it is a sign of an invader. The neutrophils move up the gradient of this substance in order to increase the probability of encountering the bacteria. When they do they ingest the bacterium by phagocytosis and destroy it. Experimentally neutrophils are observed to undergo chemotaxis due to gradients of fLMP. This is a peptide consisting of three amino acids and a formyl group and looks like the substances typically produced by bacteria. Detection of the substance is based on the fact that it binds to receptors on the surface of the cell. Under certain circumstances a eukaryotic cell is polarized. This means that certain parts of the cell are identified as 'front' and 'back'. Suppose that the cell measures the density of occupied receptors at the front and back of the cell. This provides information about the concentrations ϕ_f and ϕ_b of the substance at the front and back of the cell. The signal which drives chemotaxis is the ratio $\frac{\phi_f-\phi_b}{\phi_b}$.

A prokaryote has to be more ingenious to measure gradients. A case which has been studied in great detail is that of the common gut bacterium *Escherichia coli* [2]. It uses what is known as the 'run and tumble' strategy. The cell moves for a certain time in a straight line in some direction ('run'). Then it stops and changes its orientation randomly ('tumble'), after which it does a run in the new direction. The information about the gradient of a substance is implemented as follows. The bacterium monitors the concentration as a function of time. If the change in concentration is less than or equal to zero the run has a standard length. If the change in concentration is greater than zero then the length of the run is increased. This leads to a biased random walk which means that on average the bacterium moves up the gradient. The motion of the bacterium is caused by rotating flagella which have a helical form. When they turn with one orientation this leads to a run and when they turn with the other orientation a tumble results. The activity of the motor which causes the rotation of a flagellum is controlled by the perceived concentration of the chemical.

In these examples the concentration of the chemical driving chemotaxis is not influenced by the organisms themselves. It is however common that there is an influence. For instance, bacteria are attracted towards high concentrations of a nutrient which they consume as they move. Another important situation is that where the cells produce the chemoattractant as a form of signalling. The most famous example of this is the case of the cellular slime mould *Dictyostelium discoideum*. When food is plentiful the cells of this organism live as independent amoebae. During times of starvation the cells come together in a complex process which eventually results in a mass of the cells migrating as a slug and, if no better conditions are encountered, finally forming a fruiting body which can wait for better times. The initial stages of this process are influenced by chemotaxis. The cells produce a signalling molecule (cAMP) and move up the gradient of that substance. This organism is a popular model for understanding embryonic development in higher organisms.

5.2 Mathematical models

In Sect. 1 a kinetic model for the motion of cells was introduced. In that case the cells were moving without any external influence. A type of external influence of particular interest is that occurring in chemotaxis and some information on this will be given in this section. Before doing so, it will be noted that the equation (9) bears a certain resemblance to the Boltzmann equation. It is possible to introduce a small parameter into the equation by rescaling the variables and to carry out an expansion similar to the Hilbert expansion. Consider the scaled version of the equation given by

$$\epsilon^2 \partial_t f + \epsilon v \cdot \nabla_x f = -\lambda f + \lambda \int T(v, v') f(v') dv'. \tag{114}$$

In the limit $\epsilon \to 0$ comparing the leading order coefficients gives the diffusion equation

$$\partial_t p_0 = \nabla \cdot (D \nabla p_0) \tag{115}$$

where the diffusion coefficient D can be computed in terms of λ and T. It is assumed that $f(t, x, v)$ has an expansion in a parameter ϵ where the coefficient of order zero is $p_0(t, x)$. This type of equation is also known outside biology where it describes the deflection of some type of particles (e.g. photons) by fixed scatterers.

The discussion which follows is based primarily on the paper [13] of Othmer and Hillen. To incorporate the influence of the concentration S of a chemical substance let us replace the constant λ and the turning kernel $T(v, v')$ by quantities $\lambda(v, \hat{S})$ and $T(v, v', \hat{S})$. Here the notation $\hat{S}$ indicates that not only pointwise dependence on the value of S is allowed. Dependence on ∇S or non-local dependence on S are not excluded. Consider now an analogue of the diffusion approximation considered above with

$$T(v, v', \hat{S}) = T_0(v, v') + \epsilon^k T_1(v, v', \hat{S}), \tag{116}$$

$$\lambda(v, \hat{S}) = \lambda_0 + \epsilon^l \lambda_1(v, \hat{S}). \tag{117}$$

Here ϵ is the parameter in the diffusion approximation and k and l are integers. It turns out that the simple choice $k = 0$ and $\lambda_1 = 0$ does not lead to a useful model incorporating chemotaxis. A better choice is given by $k = 1$ and $\lambda_1 = 0$. An equation is obtained which is of the form

$$\partial_t p_0 = \nabla \cdot (D \nabla p_0 - u_c p_0) \tag{118}$$

where the chemotactic velocity u_c depends on $\hat{S}$. Suppose now that λ_1 depends linearly on ∇S, $\lambda_1(v, v, \hat{S}) = Q_1(v, v', S) \cdot \nabla S$ for some vector-valued function

Q_1. Then it can be shown that $u_c = \chi(S)\nabla S$ for a matrix-valued chemotactic sensitivity $\chi(S)$. If it is further assumed that $Q_1(v, v', S)$ is of the form $k_1(v', S)v$ then the $\chi(S)$ reduces to a scalar. Thus we get the equation

$$\partial_t p_0 = \nabla \cdot (D\nabla p_0 - p_0\chi(S)\nabla S). \tag{119}$$

This is one of the equations in the Keller-Segel system which is a very well-known model for chemotaxis. Often χ is chosen for simplicity to be a constant. The other equation in the Keller-Segel system is a diffusion equation whose source is proportional to the density of organisms

$$\partial_t S = D_1\Delta S + K_1 p_0 - K_2 S. \tag{120}$$

In the simplest case K_1 and K_2 are constants which represent the production rate of chemical by the cells and degradation of the chemical in the medium, respectively. In the case of positive chemotaxis (where the cells move up the gradient of the chemical) the constant χ is positive. The constant K_2 is always positive. Sometimes it is assumed that the diffusion constant D_1 is much larger than D and then it is tempting to make a steady state assumption in the second equation. This essentially means dropping the time derivative. A similar analysis can be done with the alternative assumptions that $T_1 = 0$ and $l = 1$.

Next a particular model will be presented which incorporates experimental data on the bacteria *E. coli* and *Salmonella typhimurium.* The turning kernel only depends on the angle θ between the old and the new velocities, i.e.

$$\theta = \cos^{-1}\left(\frac{v \cdot v'}{|v||v'|}\right). \tag{121}$$

It was found that the experimental results could be fitted by choosing

$$T(v, v') = \frac{f(\theta)}{\sin\theta} \tag{122}$$

for a certain sixth order polynomial f. Note, in particular, that the kernel does not depend on S. The other important coefficient is given by

$$\lambda(v, \hat{S}) = \lambda_0 \exp\left(\frac{c_1 K_D}{(K_D + S)^2}(S_t + v \cdot \nabla S)\right) \tag{123}$$

Here λ_0 is the turning rate in the absence of the chemical, c_1 is a constant and K_D is the dissociation constant for the binding of the chemical to the receptor. The diffusion limit gives the Keller-Segel type of equation with $\chi(S)$ proportional to $\frac{K_D}{(K_D+S)^2}$.

5.3 Remarks on the Keller-Segel model

The Keller-Segel model is a system of nonlinear parabolic equations, or a parabolic-elliptic limit thereof. In the original work of Keller and Segel they considered a homogeneous solution of the equations and linearized about it. They found that there are linearized solutions which grow in time, which is a sign that the homogeneous solution is unstable. They presented this as a model of part of the life cycle of *D. discoideum*. It may be noted that this procedure bears a formal resemblance to the way the formation of galaxies is described by astrophysicists. According to recent ideas this is initiated by gravitational collapse of dark matter. This is a form of matter with only very weak interactions which can reasonable be considered as collisionless. Since a non-relativistic description is probably sufficient the Vlasov-Poisson system is an appropriate model for this. In the partly elliptic form of the Keller-Segel system a Poisson-like equation occurs and for positive chemotaxis the sign in the source term corresponds to the stellar dynamic case of the Vlasov-Poisson system.

The Keller-Segel model has been subject to much more intensive mathematical study than the kinetic models discussed in the last subsection. Now some of the known mathematical results on the Keller-Segel system will be discussed briefly. One of the features of the evolution is that solutions tend to blow up and that the particles may concentrate at a point. This looks a bit like what happens for the higher-dimensional Vlasov-Poisson system and the relativistic Vlasov-Poisson system. More specifically, the Keller-Segel system in two space dimensions is analogous to the Vlasov-Poisson system in four space dimensions. These are borderline cases. It is not at all obvious that even when a system obtained as a diffusion limit of a kinetic model has solutions which blow up this will also be true of the kinetic model itself. Examples are known where there is global existence for the kinetic model but blow-up for solutions of the limiting system. It is not clear whether this can happen in cases where the kinetic model includes dependence on the gradient of the chemoattractant. Of course a solution which blows up must become inappropriate for describing the biological system before the singularity is reached. Nevertheless this type of solution can indicate that the microorganisms have a tendency to form strong concentrations and may give information about how many of them will end up in one of these concentrations. For the Keller-Segel system it is found that the total mass in one of the concentrations, measured in suitable units, must be at least 8π. This statement applies to the case where solutions are considered in the whole space. It is also possible to consider a situation where the solution is defined on a compact region and Neumann boundary conditions are imposed. This means that if n is the normal vector to the boundary $n \cdot \nabla p_0 = 0$ and $n \cdot \nabla S = 0$. It corresponds to the conditions that the cells and the chemical cannot cross the boundary. In that case it is found that singularities can also form on the boundary and that the necessary mass is only 4π.

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