DYNAMICS IN A CENTRAL POTENTIAL WITH AN AXISYMMETRIC PERTURBATION

Ladislav Šubr[1,3], Vladimír Karas[2], Jaroslav Haas[3]

Argelander Institut für Astronomie, University of Bonn

Astronomical Institute, Czech Academy of Sciences, Prague

Astronomical Institute, Charles University, Prague

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Brief outline

- rising the Kozai resonance. . .
- . . . and damping it
- dissipative drag of an accretion disc
- selfconsistent model (a.k.a. unbelievable scenario for S-stars)

Kozai–Lidov mechanism

- evolution of a hiearchical triple system $M_1 > M_2 > M_3$

  Lidov 1961: Earth > Moon > satellite

  Kozai 1962: Sun > Jupiter > asteroid

- secular evolution of the orbital elements e, i and ω
- “averaging” technique of the Hamiltonian perturbation theory allows to get rid of “fast” variable (mean anomaly)
- integrals of motion: a, $C_1 \equiv \sqrt{1 - e^2}\,\cos i$ and $\bar{V}_{\mathrm{d}}$
- usable also for motion of a test particle in the compound field of the central mass and an axisymmetric perturbation (ring, torus, disc. . .)

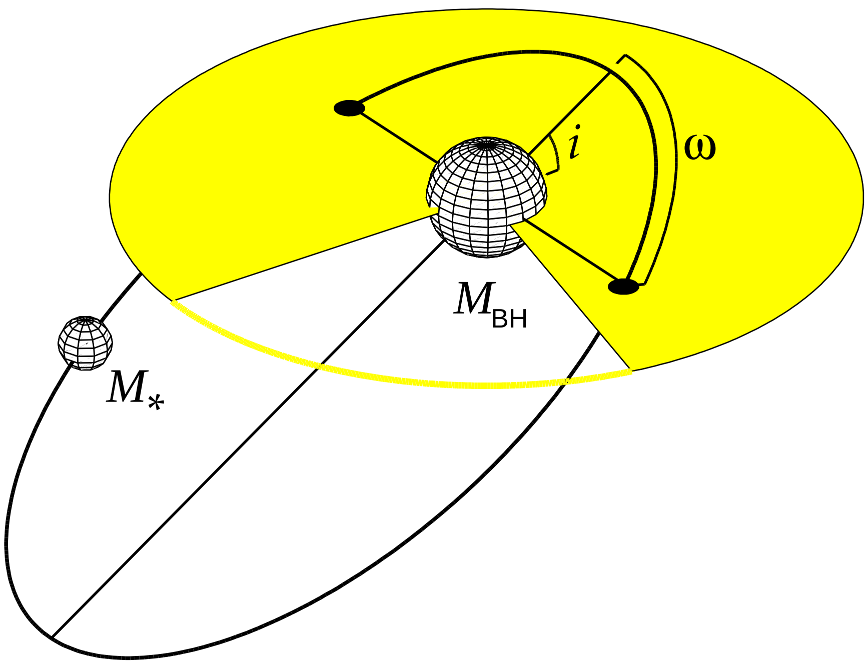
2-body Hamiltonian

Carthesian coordinates:

$$\mathcal{H} = \frac{1}{2}\left(v_1^2 + v_2^2 + v_3^2\right) - \frac{\mathcal{G}\,(m_0 + m_1)}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$

Delaunay variables:

$$\mathcal{H} = -\frac{\mathcal{G}^2(m_0 + m_1)^2}{2L^2}$$



$$L = \sqrt{\mathcal{G}(m_0 + m_1)a} \qquad l = M$$

$$G = L\sqrt{1 - e^2} \qquad g = \omega$$

$$H = G\cos i \qquad h = \Omega$$

Sophus Lie

- Lie series:

$$\mathcal{S}_{\mathcal{H}}^{t} f \equiv f(0) + \sum_{1}^{\infty} \frac{t^n}{n!} \mathcal{L}_{\mathcal{H}}^{n} f\,, \quad \mathcal{L}_{\mathcal{H}}^{n} f \equiv \mathcal{L}_{\mathcal{H}}^{1} \mathcal{L}_{\mathcal{H}}^{n-1} f\,, \quad \mathcal{L}_{\mathcal{H}}^{1} f \equiv \{f, \mathcal{H}\} = \frac{\mathrm{d}f}{\mathrm{d}t}$$

- Lie's criterion:

Transformation $(p, q) \rightarrow (p', q')$ is canonical if $\exists$ *generating* Hamiltonian $\chi(p', q')$: $\dot{p}' = -\partial\chi/\partial q' \,\wedge\, \dot{q}' = \partial\chi/\partial p'$ and parameter ϵ:

$$p = p' + \int_0^{\epsilon} \dot{p}' \mathrm{d}t \equiv p'(\epsilon)\,, \quad q = q' + \int_0^{\epsilon} \dot{q}' \mathrm{d}t \equiv q'(\epsilon)$$

In terms of Lie series: $p = \mathcal{S}_{\chi}^{\epsilon} p'\,, \quad q = \mathcal{S}_{\chi}^{\epsilon} q'$

$\mathcal{H}(p,q) = \mathcal{H}_0(p) + \epsilon\mathcal{H}_1(p,q)$

(?) Canonical transformation $p = p' + \epsilon f(p',q')\,,\ q = q' + \epsilon g(p',q')$:

$$\mathcal{H}(p,q) \rightarrow \mathcal{H}'(p',q') = \mathcal{H}_0(p') + \epsilon\bar{\mathcal{H}}_1(p') + \epsilon^2\mathcal{H}_2(p',q')$$

⬇

$$p = \mathcal{S}_\chi^\epsilon p'\ ,\quad q = \mathcal{S}_\chi^\epsilon q'$$

$$\mathcal{H}(p,q) = \mathcal{H}\left(p'(\epsilon), q'(\epsilon)\right) \equiv \mathcal{H}'(p',q') = \mathcal{S}_\chi^\epsilon\mathcal{H}\left(p'(0), q'(0)\right)$$

$$\mathcal{H}' = \mathcal{H}_0 + \epsilon\mathcal{H}_1 + \epsilon\{\mathcal{H}_0,\chi\} + \epsilon^2\{\mathcal{H}_1,\chi\} + \frac{\epsilon^2}{2}\{\{\mathcal{H}_0,\chi\},\chi\} + \mathcal{O}(\epsilon^3)$$

⬇

(?) $\exists\ \chi(p',q'),\ \bar{\mathcal{H}}_1(p')$: $\mathcal{H}_1 + \{\mathcal{H}_0,\chi\} = \bar{\mathcal{H}}_1$

Yes

q' are angles and Hamiltonian $\mathcal{H}$ is periodic in $q' \longrightarrow$ Fourier:

$$\mathcal{H}_1(p', q') = \sum_{k \in \mathrm{Z}^n} c_k(p') \exp[i\, k.q']$$

...and look for χ in the same form:

$$\chi(p', q') = \sum_{k \in \mathrm{Z}^n} d_k(p') \exp[i\, k.q']$$

$$\{\mathcal{H}_0, \chi\} = -i \sum_{k \in \mathrm{Z}^n} d_k(p')\, k.\omega_0(p') \exp[i\, k.q'] \;, \quad \omega_0 = \nabla_{p'} \mathcal{H}_0$$

solution:

$$d_0 = 0 \;, \quad d_k(p') = -i \frac{c_k(p')}{k.\omega_0(p')} \quad \forall k \neq 0 \;, \quad \bar{\mathcal{H}}_1(p') = c_0(p')$$

3-body problem

$$\mathcal{H} = \mathcal{H}_{\mathrm{K}}(p^1, q^1) + \mathcal{H}_{\mathrm{K}}(p^2, q^2) + \frac{\mathcal{G} m_2 (m_0 + m_1)}{|r_2|} - \frac{\mathcal{G} m_0 m_2}{|r_2|} - \frac{\mathcal{G} m_1 m_2}{|r_{12}|}$$

$$\mathcal{H}_1 = -\frac{\mu}{2} \frac{|r_{01}|^2}{|r_2|^3} \left(\frac{3 r_{01} . r_2}{|r_{01}||r_2|} - 1 \right) + \frac{\mu}{|r_2|} \mathcal{O}\left(\frac{|r_{01}|^3}{|r_2|^3} \right) , \quad \mu \equiv \frac{\mathcal{G} m_0 m_1 m_2}{m_0 + m_1}$$

$$\bar{\mathcal{H}} = -\frac{\mu a_1^2}{8 a_2^3 (1 - e_2^2)^{3/2}} \left((2 + 3e_1^2)(3\cos^2 I - 1) + 15 e_1^2 \sin^2 I \cos 2\omega \right)$$

Kozai equations

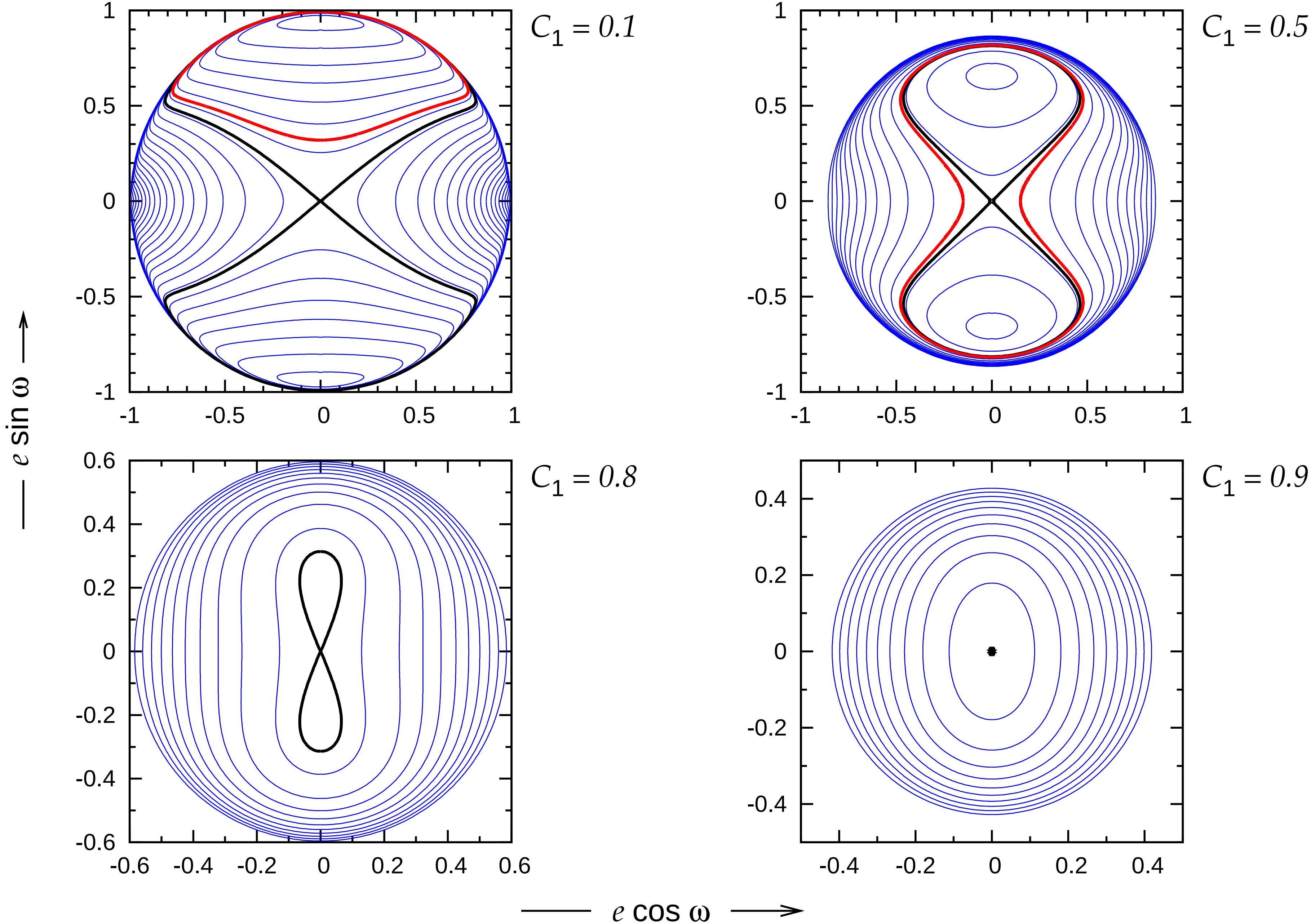
$$T_{\mathrm{K}}\sqrt{1-e^2}\,\frac{\mathrm{d}i}{\mathrm{d}t} = -5e^2 \sin i \cos i \sin\omega \cos\omega$$

$$T_{\mathrm{K}}\sqrt{1-e^2}\,\frac{\mathrm{d}e}{\mathrm{d}t} = 5e(1-e^2)\sin^2 i \sin\omega \cos\omega$$

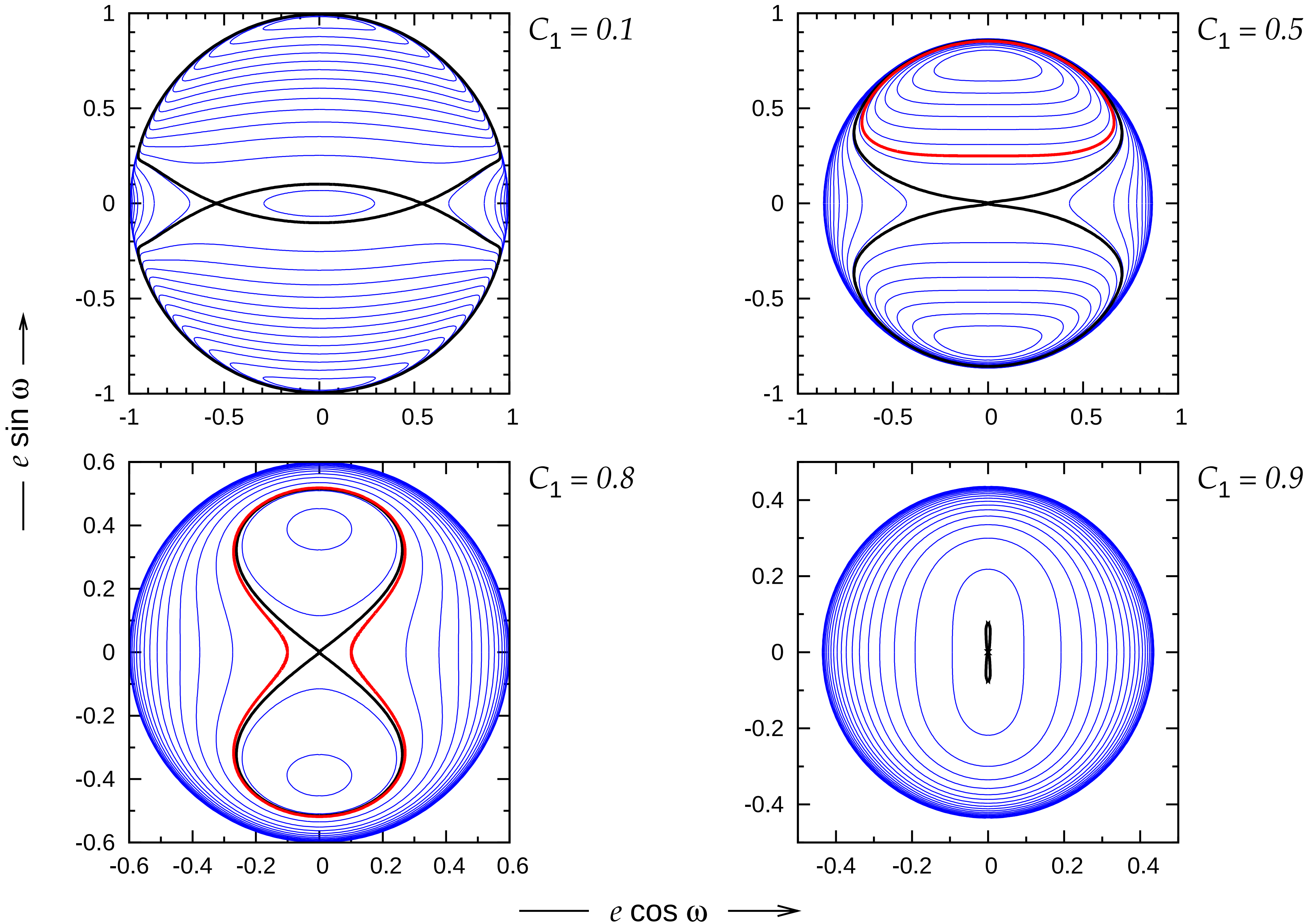
$$T_{\mathrm{K}}\sqrt{1-e^2}\,\frac{\mathrm{d}\omega}{\mathrm{d}t} = 2(1-e^2) + 5(e^2 - \sin^2 i)\sin^2\omega$$

$$T_{\mathrm{K}} \equiv \frac{4}{3}\frac{M_{\mathrm{BH}}}{M_{\mathrm{d}}}\left(\frac{R_{\mathrm{d}}}{a}\right)^3 P$$

$\bar{V}_{\mathrm{d}} = \mathrm{const.}$ *contours for a ring*



$\overline{V}_\mathrm{d}$ = const. *contours for a disc*



Damping effect of the relativistic pericentre advance

- characteristic timescales:

$$T_{\mathrm{K}} = \frac{4}{3}\frac{M_{\mathrm{BH}}}{M_{\mathrm{d}}}\left(\frac{R_{\mathrm{d}}}{a}\right)^3 P \qquad \text{vs.} \qquad T_{\mathrm{E}} = \frac{1}{3}\frac{a(1-e^2)}{R_{\mathrm{g}}} P$$

- Kozai oscillations are suppressed for

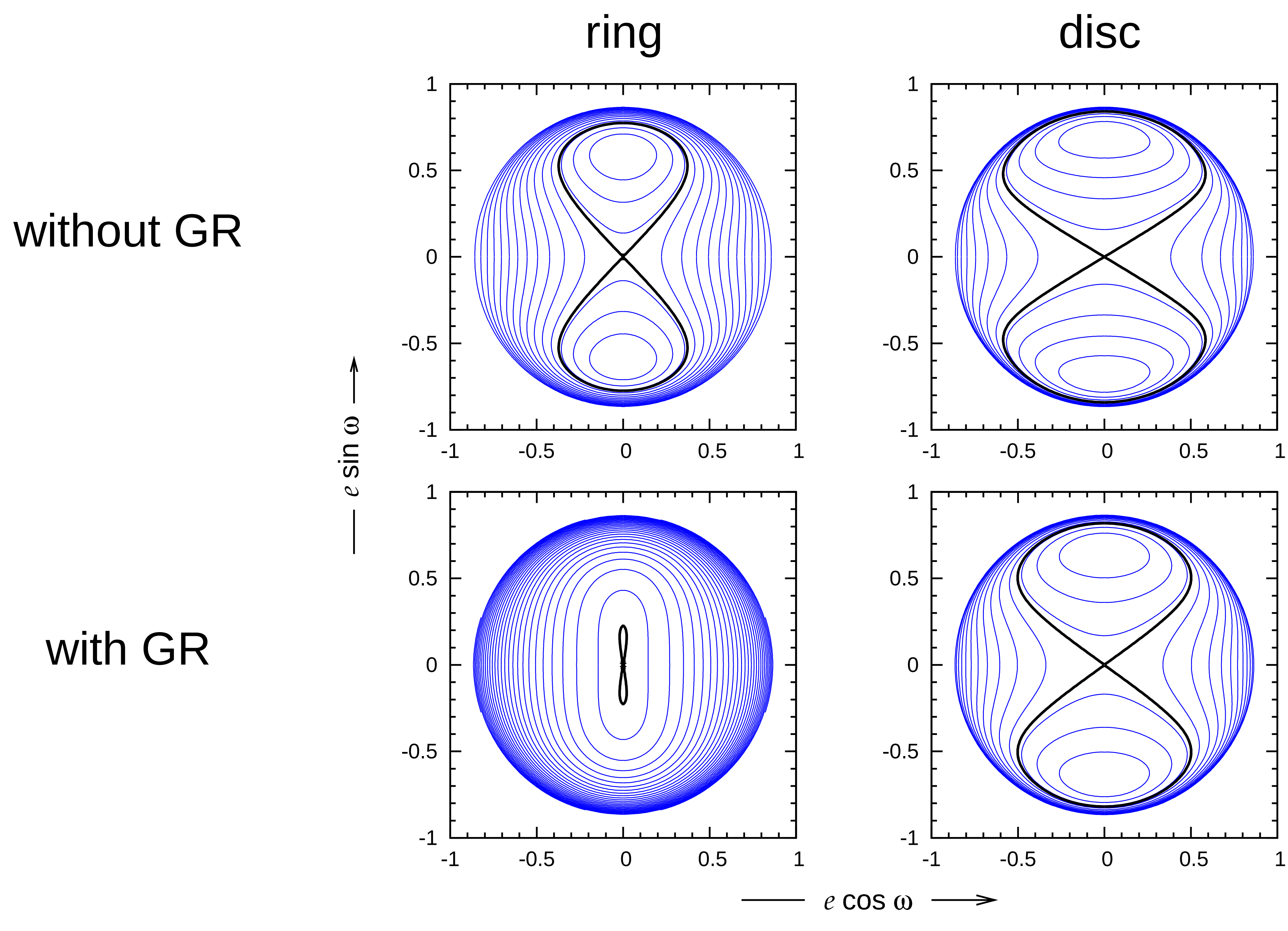
$$a < a_{\mathrm{min}} \approx \left(\frac{M_{\mathrm{BH}}}{M_{\mathrm{d}}}\right)^2 \left(\frac{R_{\mathrm{d}}}{R_{\mathrm{g}}}\right)^{6/7} \left(\frac{R_{\mathrm{min}}}{R_{\mathrm{g}}}\right)^{-1/7} R_{\mathrm{g}}$$

(Karas & Šubr, 2006, A&A submitted)

- Paczyǹski-Wiita description of the central mass potential:

$$V_{\mathrm{PW}} = -\frac{GM_{\mathrm{BH}}}{r-2R_{\mathrm{g}}} = -\frac{GM_{\mathrm{BH}}}{r} - \frac{2GM_{\mathrm{BH}}R_{\mathrm{g}}}{r(r-2R_{\mathrm{g}})}$$

Damping effect of the relativistic pericentre advance



Effect of the extended star cluster

Stellar cusp in the sphere of influence of the central black hole

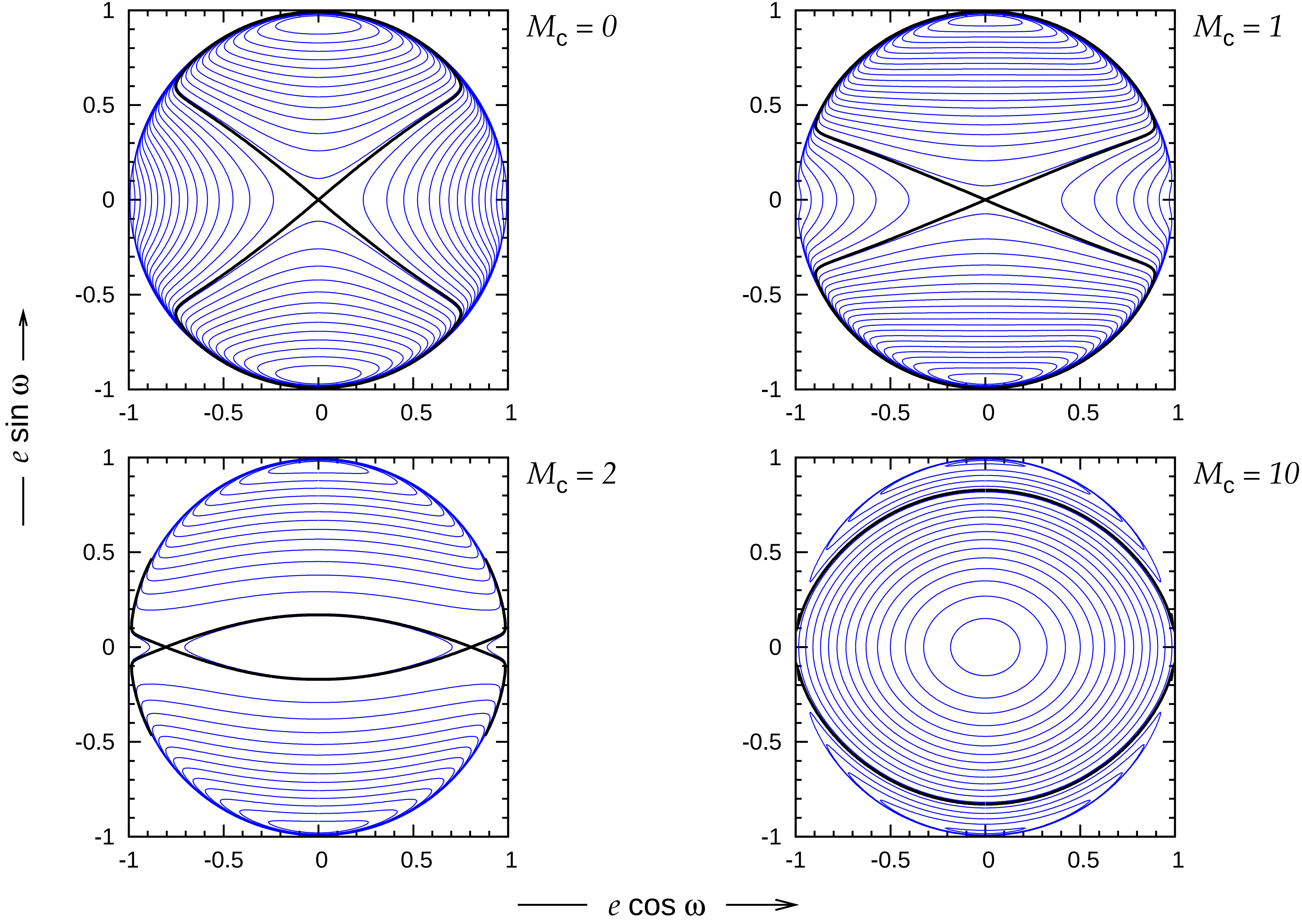
- Bahcall & Wolf (1976): $\rho(r) \propto r^{-7/4}$
- Galactic centre:

$$\rho(r) \approx 1.2 \times 10^6 \left(\frac{r}{0.4\mathrm{pc}}\right)^{-\alpha} M_\odot\,\mathrm{pc}^{-3}$$

$$\alpha = \begin{cases} 1.4 & r \lesssim 0.4\mathrm{pc} \\ 2.0 & r \gtrsim 0.4\mathrm{pc} \end{cases}$$

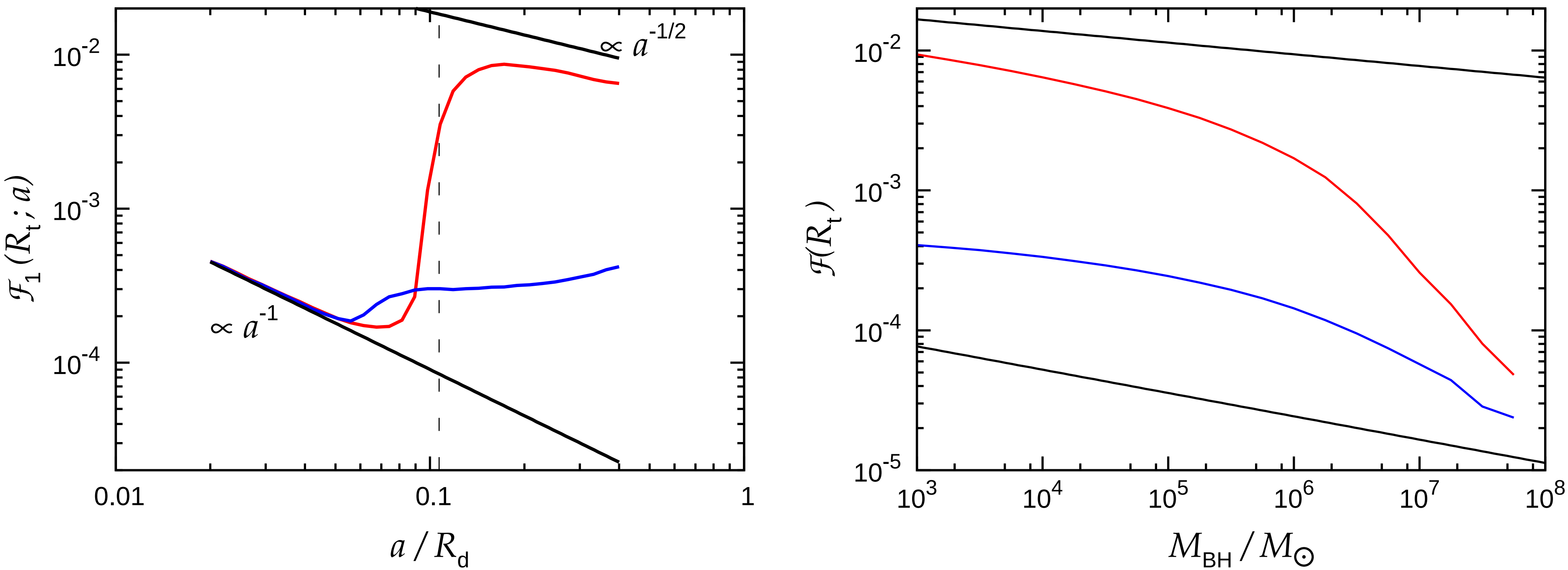
$$V_c(r) \propto \begin{cases} r^{2-\alpha} & \alpha < 2 \\ \ln(r) & \alpha = 2 \end{cases}$$

Effect of the extended star cluster



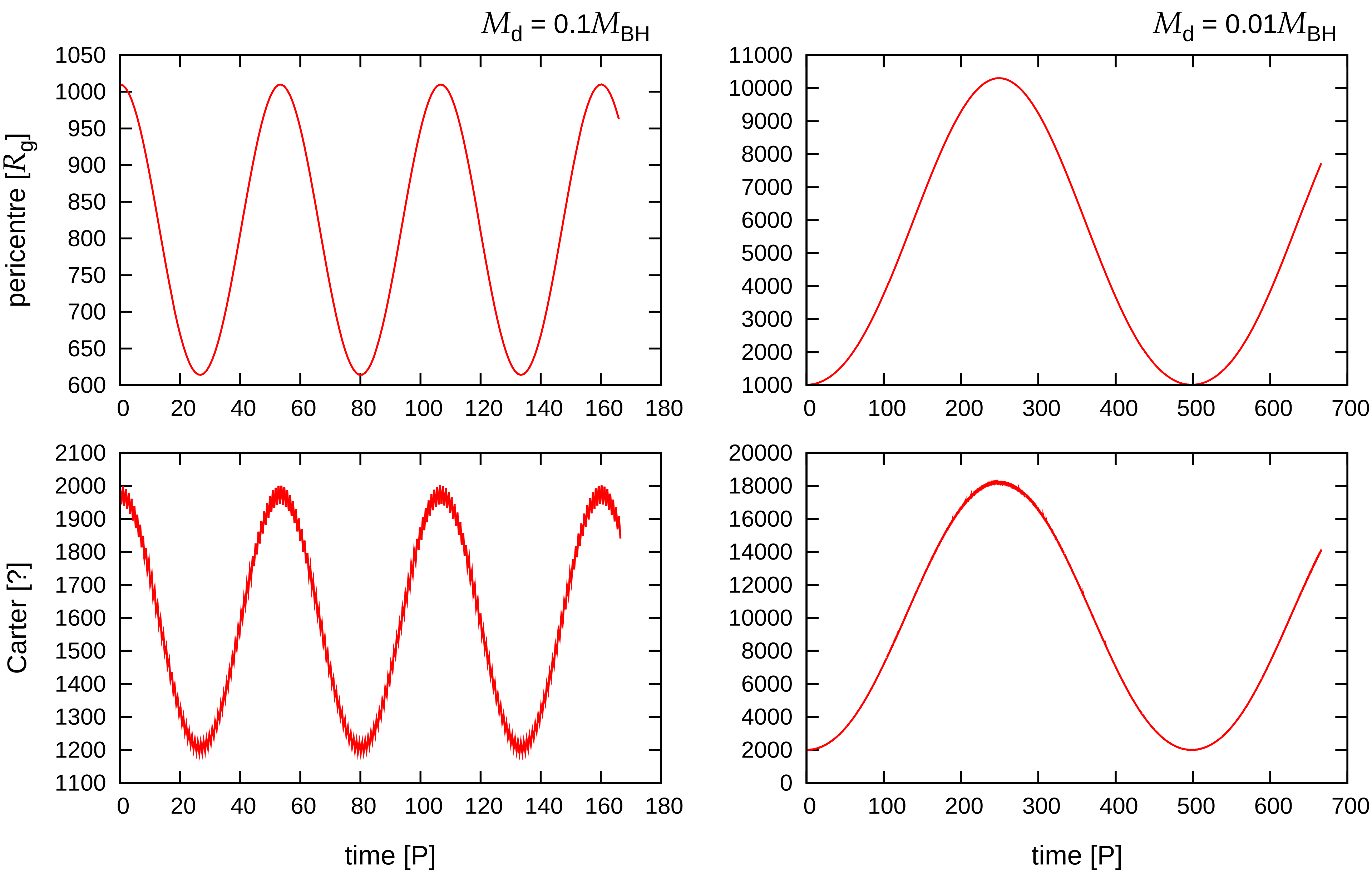
Tidal disruption rate

- single parameter model ($M_{\rm BH}$)



(Karas & Šubr, 2006, A&A submitted)

Schwarzschild + Bach-Weyl = Kozai



Sub-conlusions

- AGNs — likely to produce EMRIs?
  - accretion disc $\Longrightarrow$ axisymmetric perturbation $\Longrightarrow$ eccentricity oscillations
  - accretion disc $\Longrightarrow$ energy dissipation
- post-Newtonian corrections important for the stellar dynamics even at $a \sim 10^5 R_\mathrm{g}$
- Kozai oscillations in exact solution spacetimes — worth to be done (things may qualitatively differ from Newtonian ones; e.g. boundary between chaos and regularity)

Dissipative drag of the disc

- supersonic passages
  (e.g. Syer, Clarke & Rees 1991)

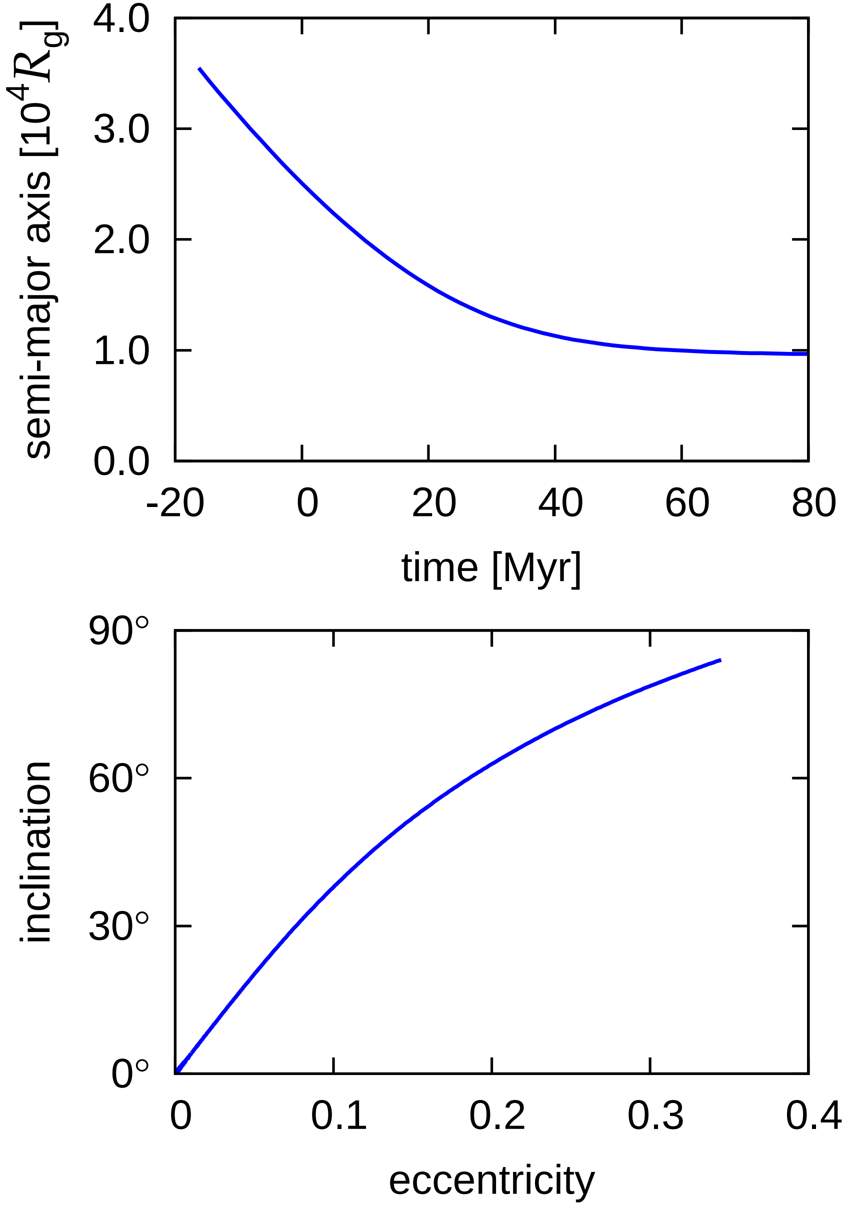
- parametric solution:

$$C_1 = a(1-e^2)(1+\cos^2 i)$$
$$C_2 = e^2(1+\cos i)^3/(1-\cos i)$$

- late phase:

$$e(t) \approx \sqrt{C_2}/4\, i(t)$$
$$i(t) \propto \exp[-t/\tau]$$

(Šubr, Karas & Huré 2004)

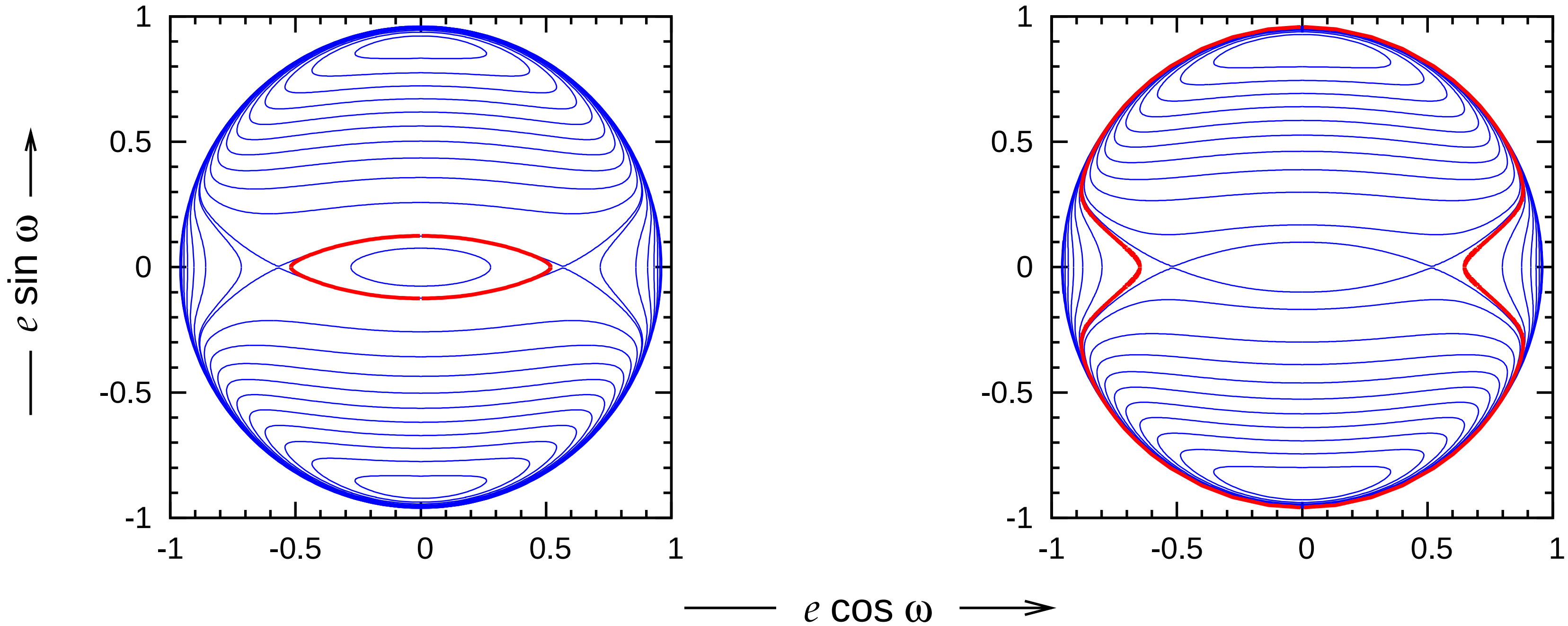


Push it through the separatrix

self-consistent model $\Longleftrightarrow$ accretion disc:

i) has suitable $\bar{V}_{\mathrm{d}}$ isocontours topology

ii) produces hydrodynamical drag



(Šubr & Karas, 2005, A&A 433, 405)

Temporal evolution