PN two-body dynamics for BHs in dense stellar systems

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Motivation

We are facing a series of problems that require the study of the interplay of collisional stellar dynamics and Post-Newtonian dynamics. A few examples:

1) Massive BH binaries and the last parsec problem;

2) Triple (stellar, intermediate, or massive) BH configurations: close encounters, Kozai cycles, mergers;

3) EMRIs and IMRIs: estimate event rates, triaxiality and binaries, rotation, resonant relaxation;

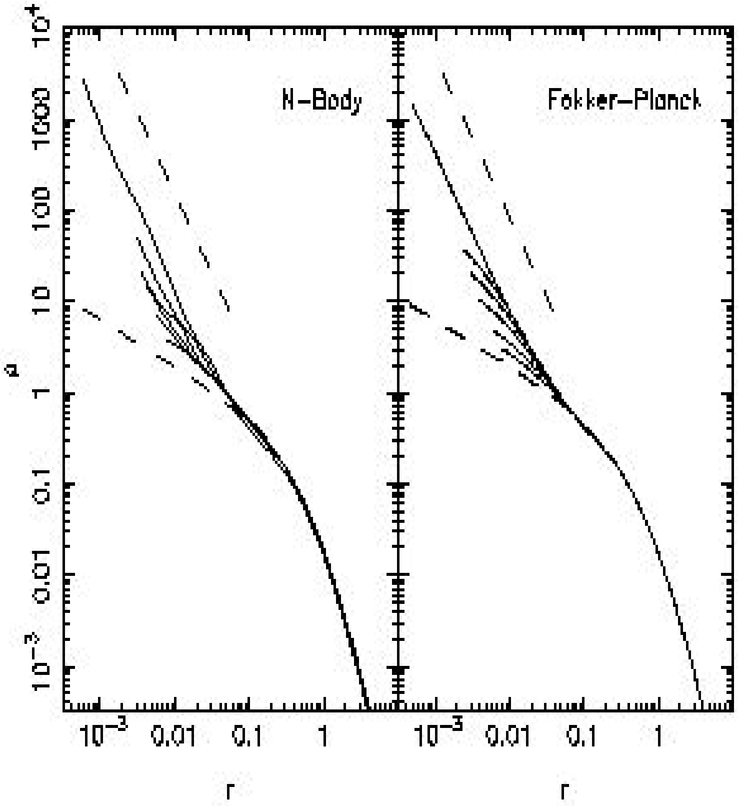
4) Compact binaries and stellar BH populations in globular clusters

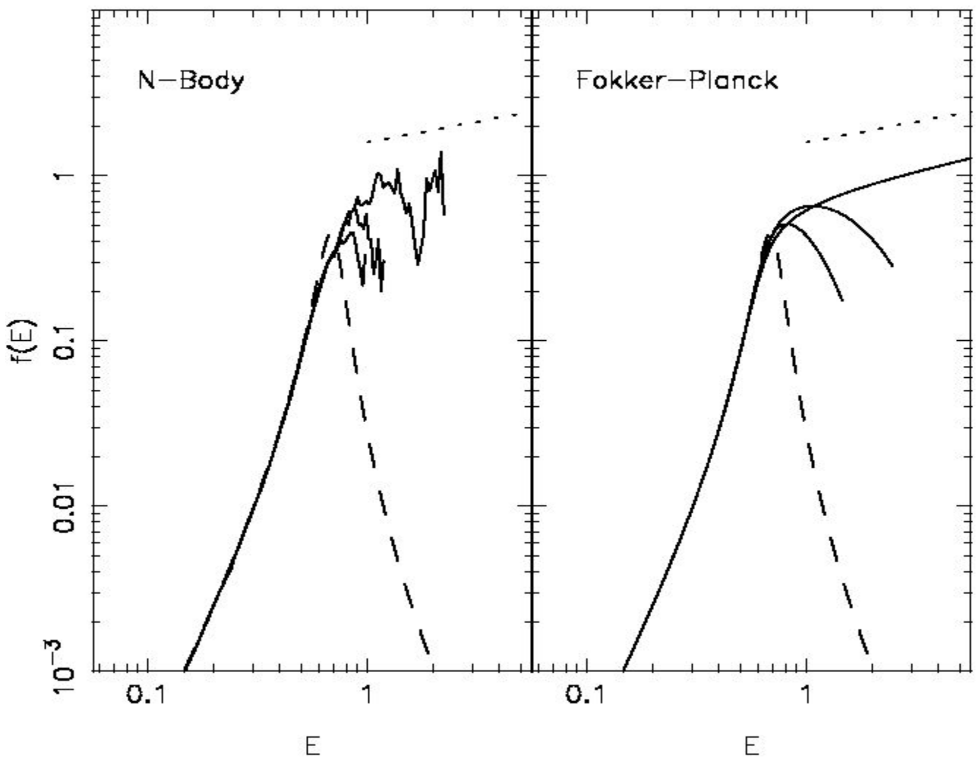
N-Body Growth of a Bahcall-Wolf cusp around a MBH

Single mass population:
(Bahcall & Wolf 1976)

$$\text{density cusp},\quad \rho(r) \propto r^{-7/4},\quad r<r_h$$

$$\text{distribution function},\quad f(E) \sim E^p,\quad p=1/4,\quad E>E_h$$

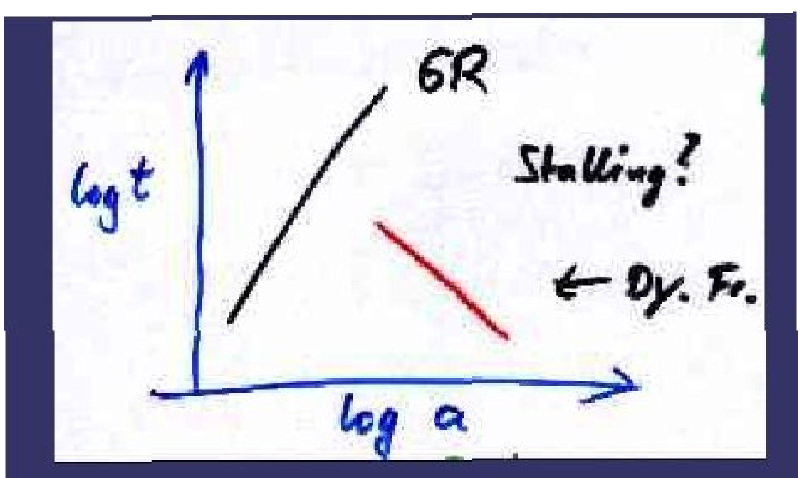


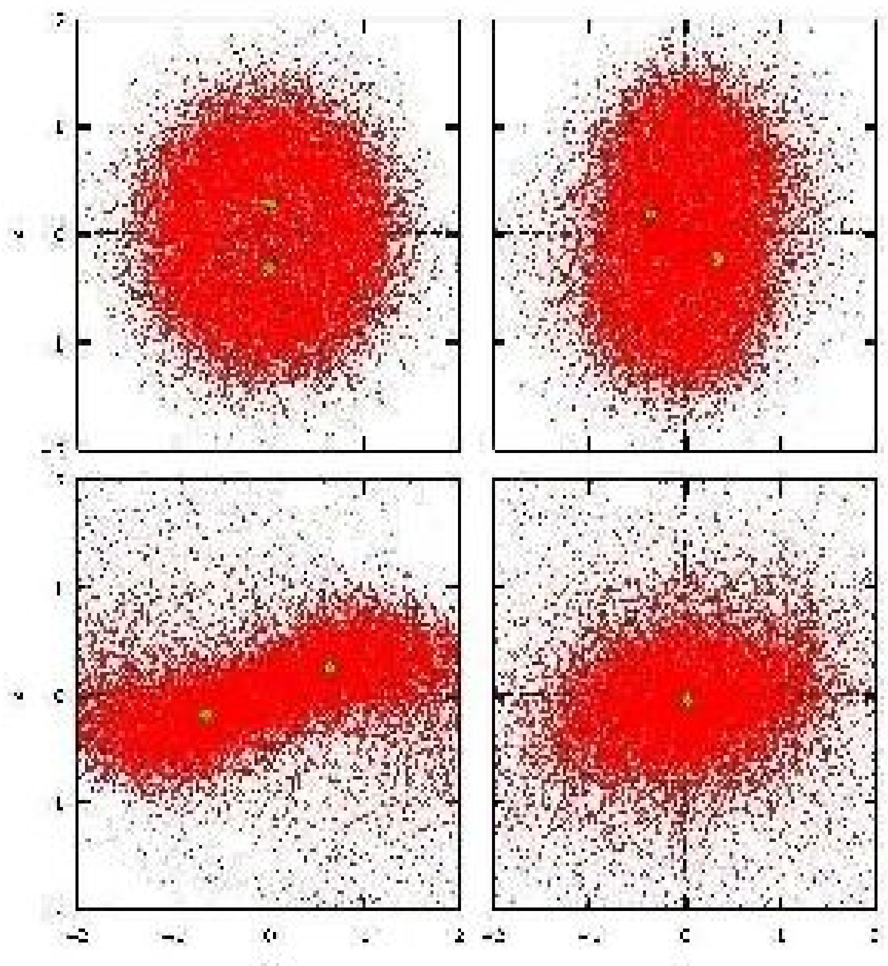


This type of N-body calculations may take several
weeks or months, and only possible with special-purpose
hardware GRAPE 6-A and micro-GRAPE!!

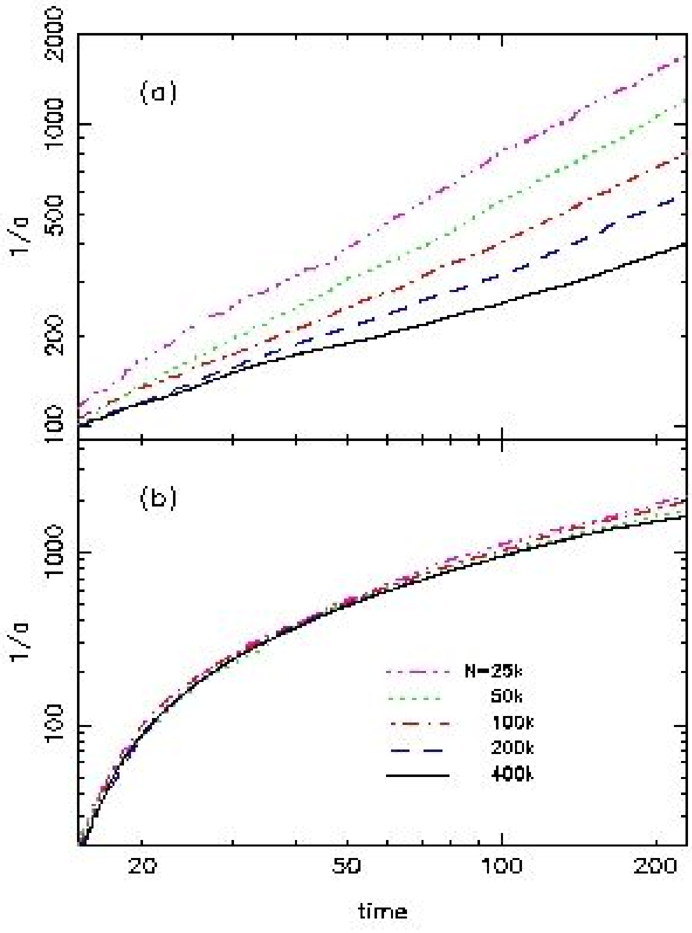
Preto et al. 2004

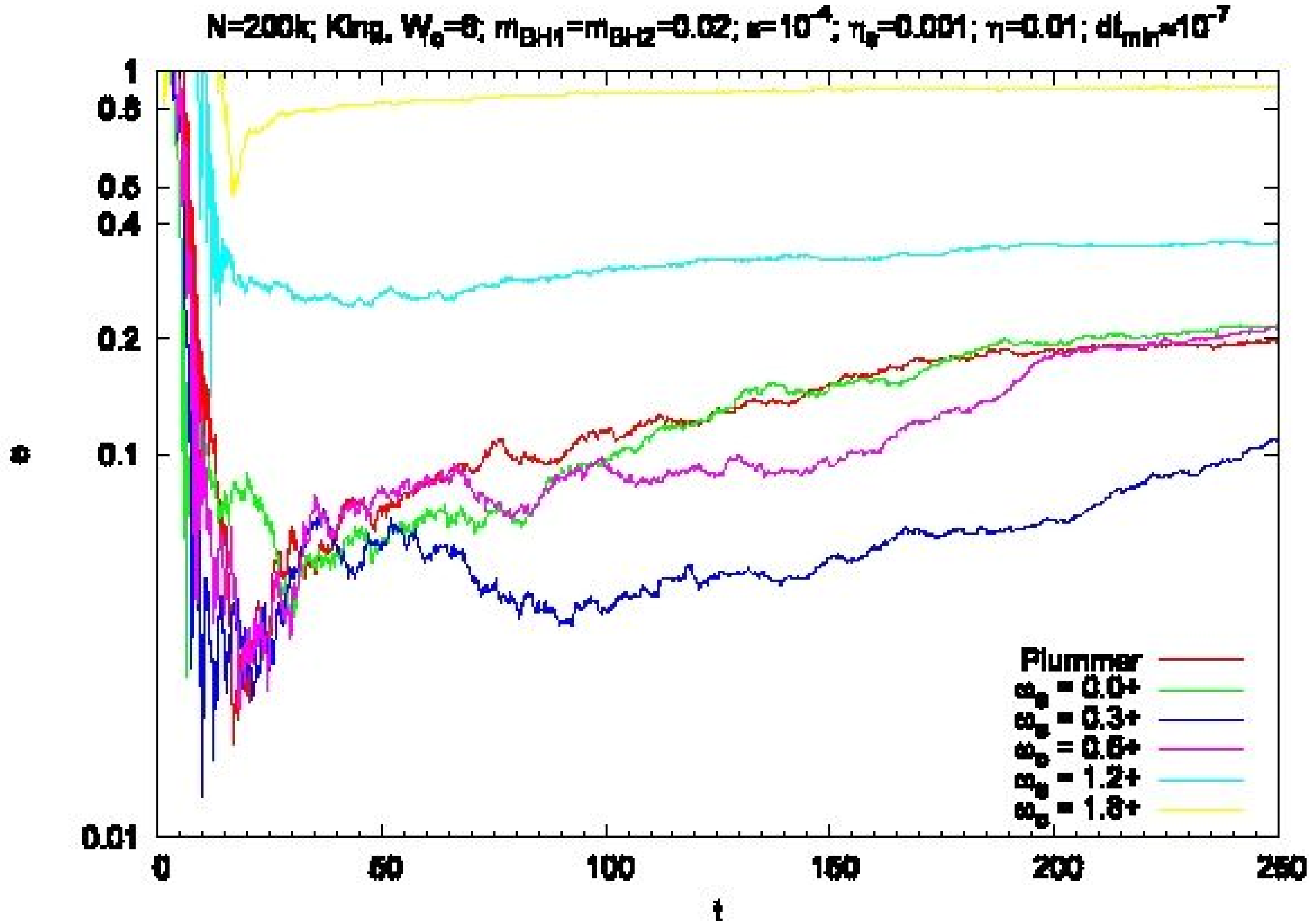
The last parsec problem





The last parsec problem: rotating galaxy models





$1/a \propto t/N$,   diffusive regime (empty loss cone)
$1/a \propto t$,   pinehole regime (full loss cone)

Berczik et al. 2006

PN Equations of Motion

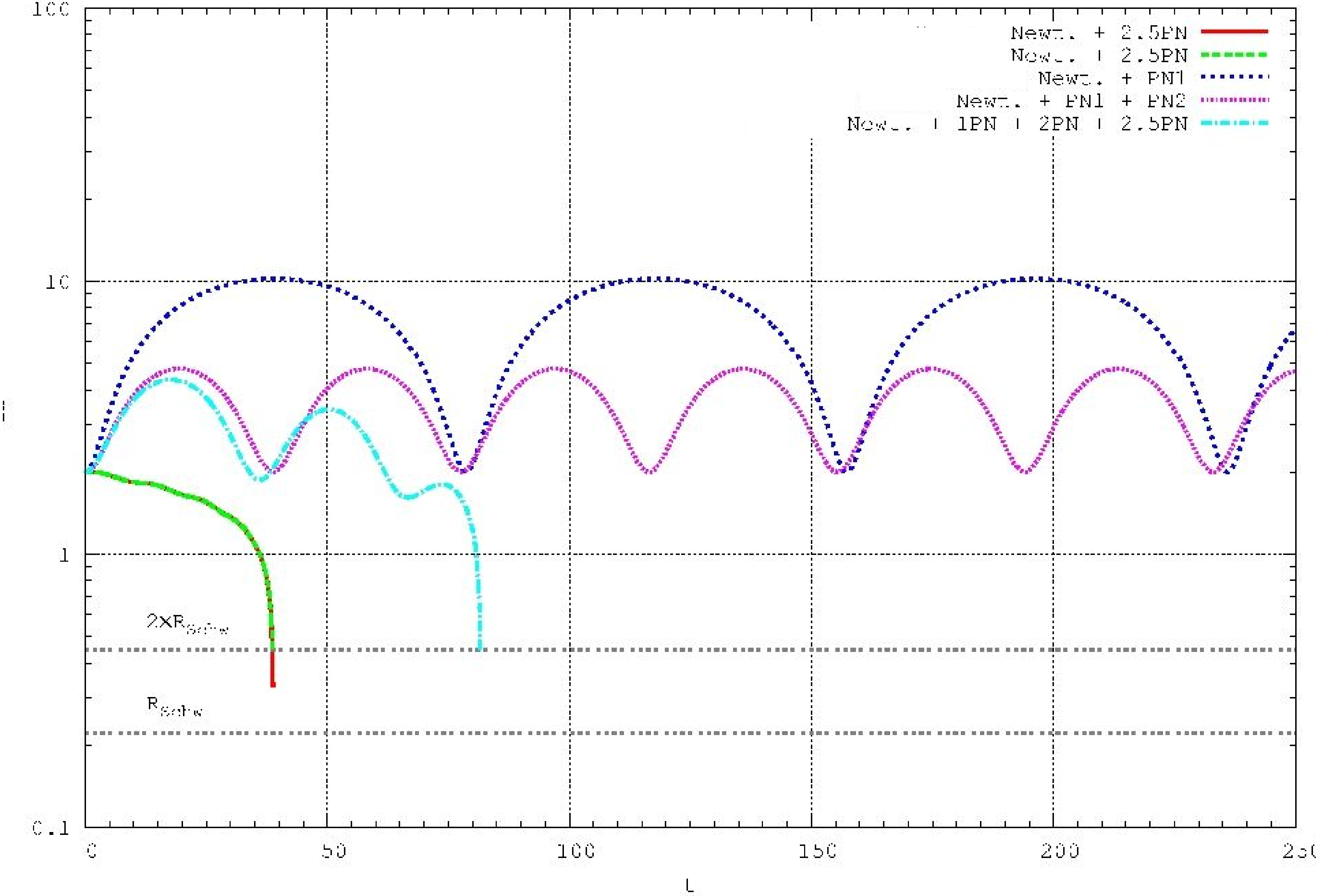
$$\frac{dv^i}{dt} = -\frac{Gm}{r^2}\left[(1+\mathcal{A})\,n^i + \mathcal{B}\,v^i\right] + \mathcal{O}\left(\frac{1}{c^8}\right),$$

$$\begin{aligned}
\mathcal{A} = {} & \frac{1}{c^2}\left\{-\frac{3\dot{r}^2\nu}{2} + v^2 + 3\nu v^2 - \frac{Gm}{r}(4+2\nu)\right\} \\
& + \frac{1}{c^4}\left\{\frac{15\dot{r}^4\nu}{8} - \frac{45\dot{r}^4\nu^2}{8} - \frac{9\dot{r}^2\nu v^2}{2} + 6\dot{r}^2\nu^2 v^2 + 3\nu v^4 - 4\nu^2 v^4\right. \\
& \qquad \left. + \frac{Gm}{r}\left(-2\dot{r}^2 - 25\dot{r}^2\nu - 2\dot{r}^2\nu^2 - \frac{13\nu v^2}{2} + 2\nu^2 v^2\right) + \frac{G^2m^2}{r^2}\left(9 + \frac{87\nu}{4}\right)\right\} \\
& + \frac{1}{c^5}\left\{-\frac{24\dot{r}\nu v^2}{5}\frac{Gm}{r} - \frac{136\dot{r}\nu}{15}\frac{G^2m^2}{r^2}\right\} \\
& + \frac{1}{c^6}\left\{-\frac{35\dot{r}^6\nu}{16} + \frac{175\dot{r}^6\nu^2}{16} - \frac{175\dot{r}^6\nu^3}{16} + \frac{15\dot{r}^4\nu v^2}{2} - \frac{135\dot{r}^4\nu^2 v^2}{4} + \frac{255\dot{r}^4\nu^3 v^2}{8}\right. \\
& \qquad - \frac{15\dot{r}^2\nu v^4}{2} + \frac{237\dot{r}^2\nu^2 v^4}{8} - \frac{45\dot{r}^2\nu^3 v^4}{2} + \frac{11\nu v^6}{4} - \frac{49\nu^2 v^6}{4} + 13\nu^3 v^6 \\
& \qquad + \frac{Gm}{r}\left(79\dot{r}^4\nu - \frac{69\dot{r}^4\nu^2}{2} - 30\dot{r}^4\nu^3 - 121\dot{r}^2\nu v^2 + 16\dot{r}^2\nu^2 v^2 + 20\dot{r}^2\nu^3 v^2 + \frac{75\nu v^4}{4}\right. \\
& \qquad\qquad \left. + 8\nu^2 v^4 - 10\nu^3 v^4\right) \\
& \qquad + \frac{G^2m^2}{r^2}\left(\dot{r}^2 + \frac{32573\dot{r}^2\nu}{168} + \frac{11\dot{r}^2\nu^2}{8} - 7\dot{r}^2\nu^3 + \frac{615\dot{r}^2\nu\pi^2}{64} - \frac{26987\nu v^2}{840} + \nu^3 v^2\right. \\
& \qquad\qquad \left. - \frac{123\nu\pi^2 v^2}{64} - 110\dot{r}^2\nu\ln\left(\frac{r}{r'_0}\right) + 22\nu v^2\ln\left(\frac{r}{r'_0}\right)\right) \\
& \qquad \left. + \frac{G^3m^3}{r^3}\left(-16 - \frac{437\nu}{4} - \frac{71\nu^2}{2} + \frac{41\nu\pi^2}{16}\right)\right\} \\
& + \frac{1}{c^7}\left\{\frac{Gm}{r}\left(\frac{366}{35}\nu v^4 + 12\nu^2 v^4 - 114 v^2\nu\dot{r}^2 - 12\nu^2 v^2\dot{r}^2 + 112\nu\dot{r}^4\right)\right. \\
& \qquad + \frac{G^2m^2}{r^2}\left(\frac{692}{35}\nu v^2 - \frac{724}{15}v^2\nu^2 + \frac{294}{5}\nu\dot{r}^2 + \frac{376}{5}\nu^2\dot{r}^2\right) \\
& \qquad \left. + \frac{G^3m^3}{r^3}\left(\frac{3956}{35}\nu + \frac{184}{5}\nu^2\right)\right\}, \qquad (182)
\end{aligned}$$

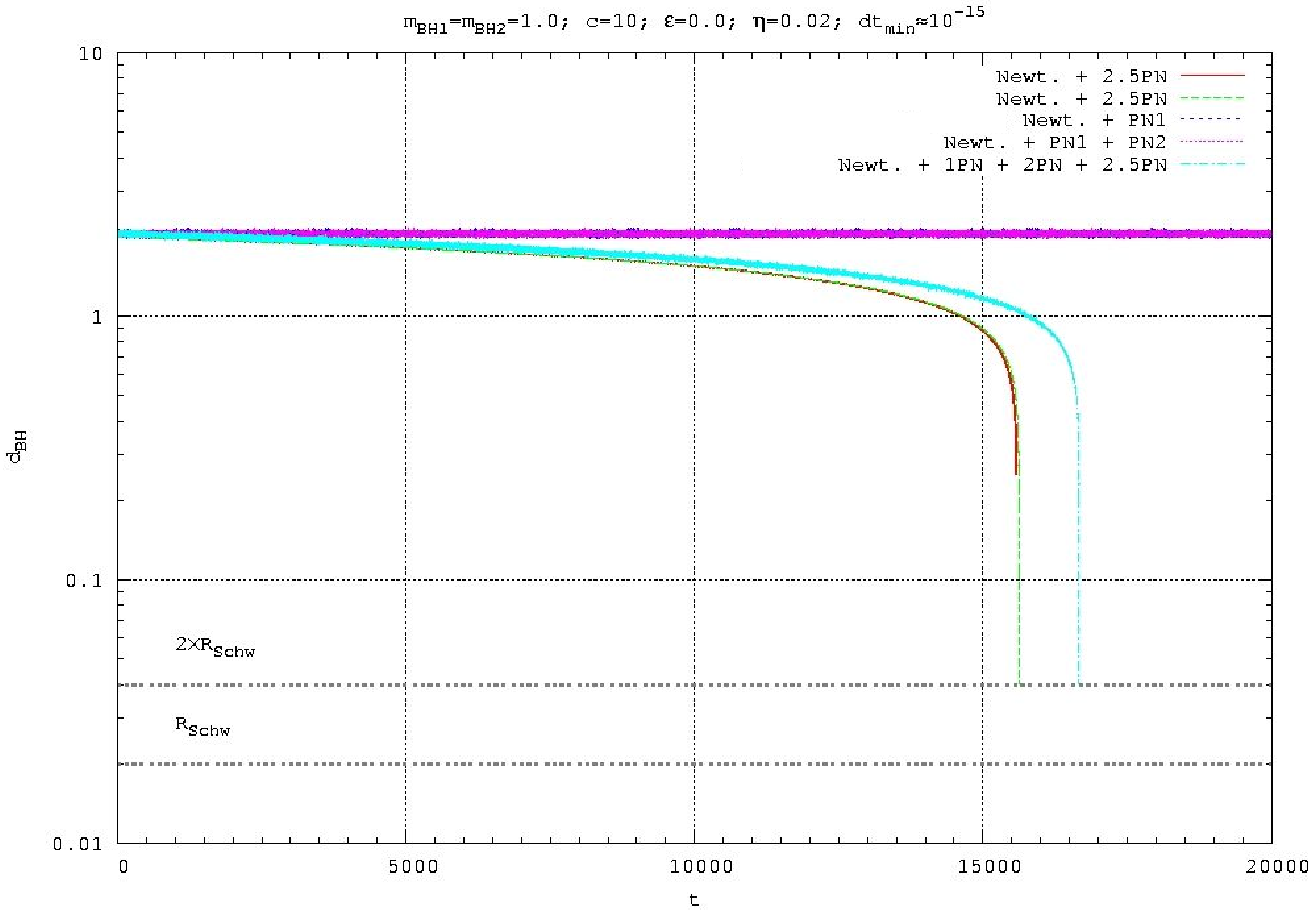
$$\begin{aligned}
\mathcal{B} = {} & \frac{1}{c^2}\left\{-4\dot{r} + 2\dot{r}\nu\right\} \\
& + \frac{1}{c^4}\left\{\frac{9\dot{r}^3\nu}{2} + 3\dot{r}^3\nu^2 - \frac{15\dot{r}\nu v^2}{2} - 2\dot{r}\nu^2 v^2 + \frac{Gm}{r}\left(2\dot{r} + \frac{41\dot{r}\nu}{2} + 4\dot{r}\nu^2\right)\right\} \\
& + \frac{1}{c^5}\left\{\frac{8\nu v^2}{5}\frac{Gm}{r} + \frac{24\nu}{5}\frac{G^2m^2}{r^2}\right\} \\
& + \frac{1}{c^6}\left\{-\frac{45\dot{r}^5\nu}{8} + 15\dot{r}^5\nu^2 + \frac{15\dot{r}^5\nu^3}{4} + 12\dot{r}^3\nu v^2 - \frac{111\dot{r}^3\nu^2 v^2}{4} - 12\dot{r}^3\nu^3 v^2 - \frac{65\dot{r}\nu v^4}{8}\right. \\
& \qquad + 19\dot{r}\nu^2 v^4 + 6\dot{r}\nu^3 v^4 \\
& \qquad + \frac{Gm}{r}\left(\frac{329\dot{r}^3\nu}{6} + \frac{59\dot{r}^3\nu^2}{2} + 18\dot{r}^3\nu^3 - 15\dot{r}\nu v^2 - 27\dot{r}\nu^2 v^2 - 10\dot{r}\nu^3 v^2\right) \\
& \qquad \left. + \frac{G^2m^2}{r^2}\left(-4\dot{r} - \frac{18169\dot{r}\nu}{840} + 25\dot{r}\nu^2 + 8\dot{r}\nu^3 - \frac{123\dot{r}\nu\pi^2}{32} + 44\dot{r}\nu\ln\left(\frac{r}{r'_0}\right)\right)\right\} \\
& + \frac{1}{c^7}\left\{\frac{Gm}{r}\left(-\frac{626}{35}\nu v^4 - \frac{12}{5}\nu^2 v^4 + \frac{678}{5}\nu v^2\dot{r}^2 + \frac{12}{5}\nu^2 v^2\dot{r}^2 - 120\nu\dot{r}^4\right)\right. \\
& \qquad + \frac{G^2m^2}{r^2}\left(\frac{164}{21}\nu v^2 + \frac{148}{5}\nu^2 v^2 - \frac{82}{3}\nu\dot{r}^2 - \frac{848}{15}\nu^2\dot{r}^2\right) \\
& \qquad \left. + \frac{G^3m^3}{r^3}\left(-\frac{1060}{21}\nu - \frac{104}{5}\nu^2\right)\right\}.
\end{aligned}$$

Blanchet 2006
Living Rev. Relativity, **9**, 4
http://www.livingreviews.org/lrr-2006-4

PN Binary Evolution



PN Binary Evolution



Conservative energy terms

$$
\begin{aligned}
\frac{E}{\mu} = \frac{v^2}{2} - \frac{m}{r} \quad & + \frac{1}{c^2}\left\{ \frac{3\,v^4}{8} - \frac{9\,\nu\,v^4}{8} + \frac{m}{r}\left(\frac{\dot r^2\,\nu}{2} + \frac{3\,v^2}{2} + \frac{\nu\,v^2}{2} \right) + \frac{m^2}{2r^2} \right\} \\
& + \frac{1}{c^4}\left\{ \frac{5\,v^6}{16} - \frac{35\,\nu\,v^6}{16} + \frac{65\,\nu^2\,v^6}{16} \right. \\
& \qquad + \frac{m}{r}\left(-\frac{3\,\dot r^4\,\nu}{8} + \frac{9\,\dot r^4\,\nu^2}{8} + \frac{\dot r^2\,\nu\,v^2}{4} - \frac{15\,\dot r^2\,\nu^2\,v^2}{4} + \frac{21\,v^4}{8} - \frac{23\,\nu\,v^4}{8} - \frac{27\,\nu^2\,v^4}{8} \right) \\
& \qquad + \frac{m^2}{r^2}\left(\frac{\dot r^2}{2} + \frac{69\,\dot r^2\,\nu}{8} + \frac{3\,\dot r^2\,\nu^2}{2} + \frac{7\,v^2}{4} - \frac{55\,\nu\,v^2}{8} + \frac{\nu^2\,v^2}{2} \right) \\
& \qquad \left. + \frac{m^3}{r^3}\left(-\frac{1}{2} - \frac{15\,\nu}{4} \right) \right\} \\
& + \frac{1}{c^6}\left\{ \frac{35\,v^8}{128} - \frac{413\,\nu\,v^8}{128} + \frac{833\,\nu^2\,v^8}{64} - \frac{2261\,\nu^3\,v^8}{128} \right. \\
& \qquad + \frac{m}{r}\left(\frac{5\,\dot r^6\,\nu}{16} - \frac{25\,\dot r^6\,\nu^2}{16} + \frac{25\,\dot r^6\,\nu^3}{16} - \frac{9\,\dot r^4\,\nu\,v^2}{16} + \frac{21\,\dot r^4\,\nu^2\,v^2}{4} \right. \\
& \qquad\quad - \frac{165\,\dot r^4\,\nu^3\,v^2}{16} - \frac{21\,\dot r^2\,\nu\,v^4}{16} - \frac{75\,\dot r^2\,\nu^2\,v^4}{16} + \frac{375\,\dot r^2\,\nu^3\,v^4}{16} \\
& \qquad\quad \left. + \frac{55\,v^6}{16} - \frac{215\,\nu\,v^6}{16} + \frac{29\,\nu^2\,v^6}{4} + \frac{325\,\nu^3\,v^6}{16} \right) \\
& \qquad + \frac{m^2}{r^2}\left(-\frac{731\,\dot r^4\,\nu}{48} + \frac{41\,\dot r^4\,\nu^2}{4} + 6\,\dot r^4\,\nu^3 + \frac{3\,\dot r^2\,v^2}{4} + \frac{31\,\dot r^2\,\nu\,v^2}{2} \right. \\
& \qquad\quad \left. - \frac{815\,\dot r^2\,\nu^2\,v^2}{16} - \frac{81\,\dot r^2\,\nu^3\,v^2}{4} + \frac{135\,v^4}{16} - \frac{97\,\nu\,v^4}{8} + \frac{203\,\nu^2\,v^4}{8} - \frac{27\,\nu^3\,v^4}{4} \right) \\
& \qquad + \frac{m^3}{r^3}\left(\frac{3\,\dot r^2}{2} + \frac{803\,\dot r^2\,\nu}{840} + \frac{51\,\dot r^2\,\nu^2}{4} + \frac{7\,\dot r^2\,\nu^3}{2} - \frac{123\,\dot r^2\,\nu\,\pi^2}{64} + \frac{5\,v^2}{4} \right. \\
& \qquad\quad - \frac{6747\,\nu\,v^2}{280} - \frac{21\,\nu^2\,v^2}{4} + \frac{\nu^3\,v^2}{2} + \frac{41\,\nu\,\pi^2\,v^2}{64} \\
& \qquad\quad \left. + 22\,\dot r^2\,\nu\,\ln\left(\frac{r}{r'_0}\right) - \frac{22\,\nu\,v^2}{3}\ln\left(\frac{r}{r'_0}\right) \right) \\
& \qquad \left. + \frac{m^4}{r^4}\left(\frac{3}{8} + \frac{2747\,\nu}{140} - \frac{11\,\lambda\,\nu}{3} - \frac{22\,\nu}{3}\ln\left(\frac{r}{r'_0}\right) \right) \right\}. \qquad (4.6)
\end{aligned}
$$

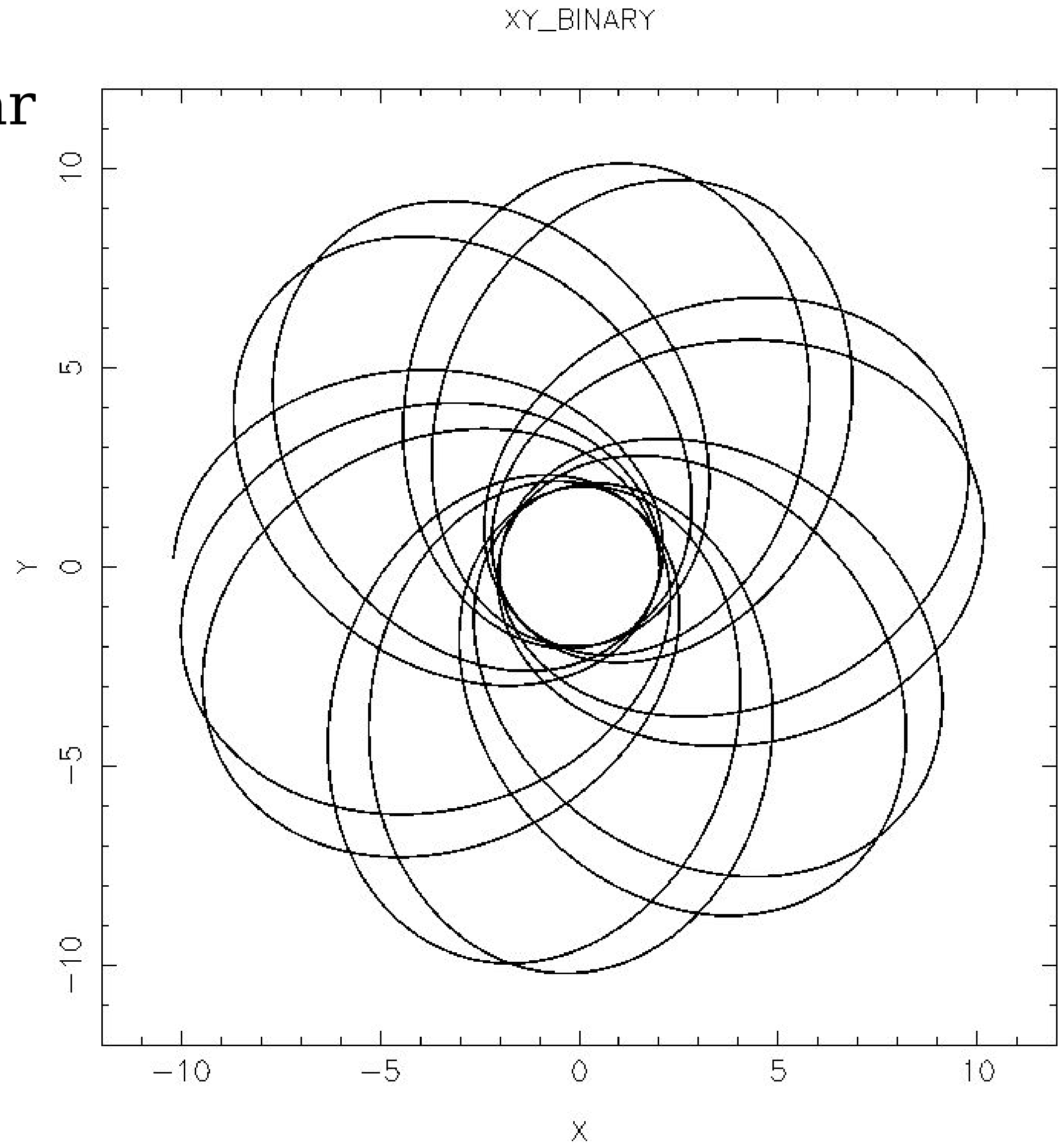
Circular orbits in 1PN dynamics

Initial conditions for a circular Keplerian orbit:

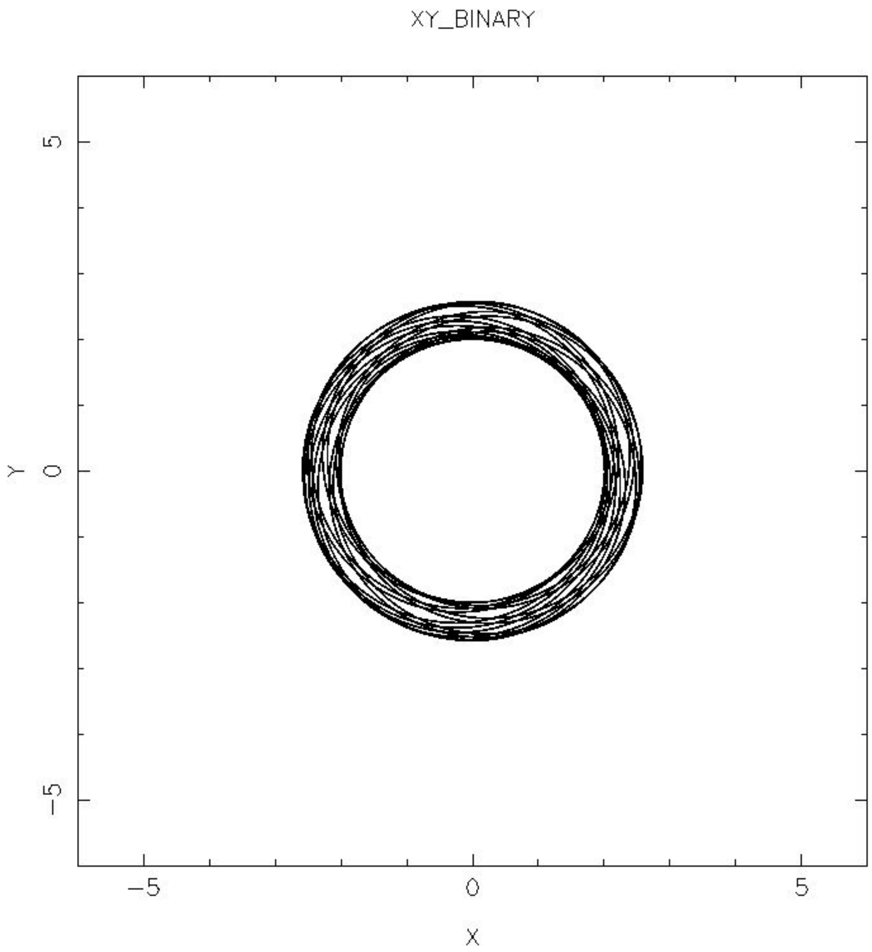
$M_1=M_2=1.0, \quad c=3$

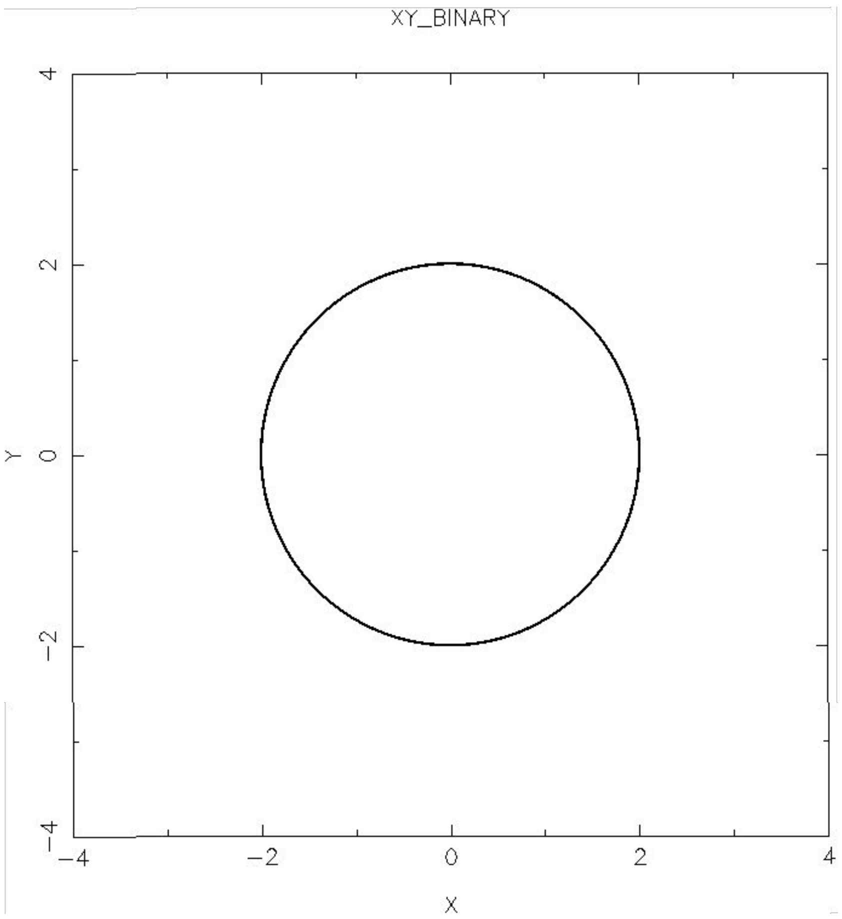
$x_{1,0}=1.0, \; y_1=z_1=0.0$
$v_{y1,0}=0.5, \; v_{x1,0}=v_{z1,0}=0.0$

$x_{2,0}=-1.0, y_2=z_2=0.0$
$v_{y2,0}=-0.5, \; v_{x2,0}=v_{z12,0}=0.0$



Circular orbits in 1PN dynamics

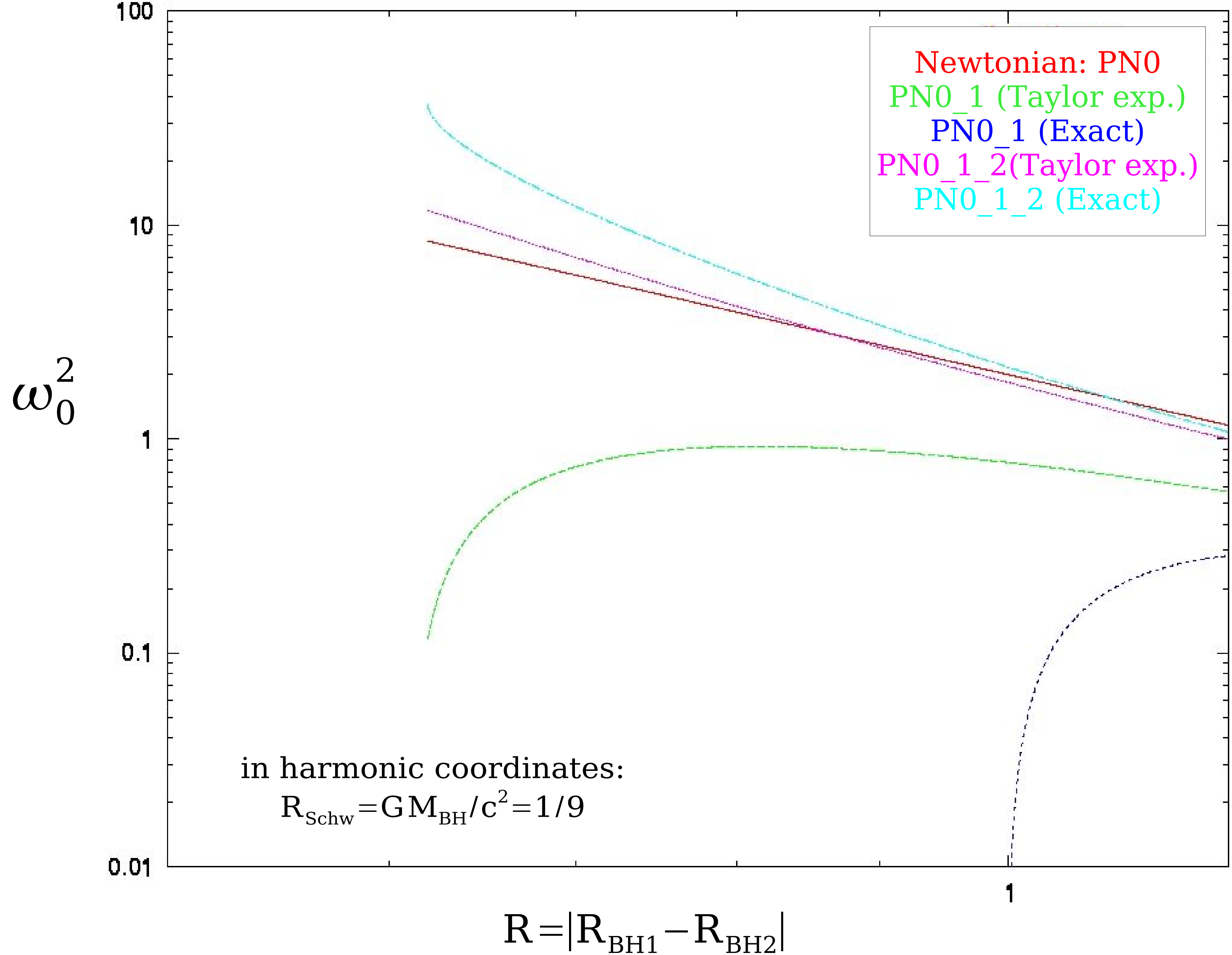




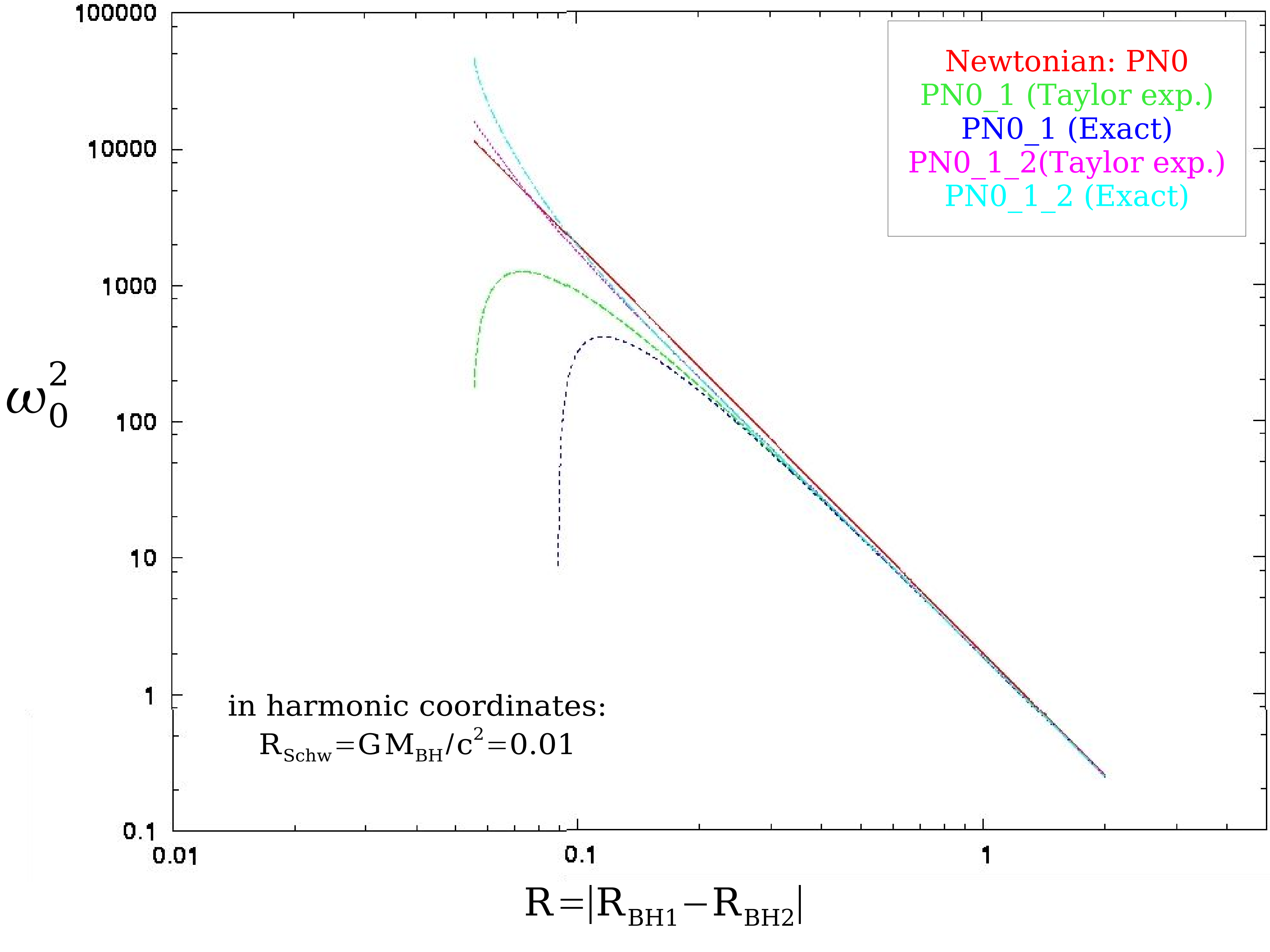
$$\omega_0^2 = \frac{GM}{r_0^2}\left[1 + (\nu - 3) GM / r_0 c^2\right] + O(1/c^4)$$

$$\omega_0^2 = \frac{GM}{r_0^2} \frac{1 - (4 + 2\nu) GM / r_0 c^2}{1 - (1 + 3\nu) GM / r_0 c^2}$$

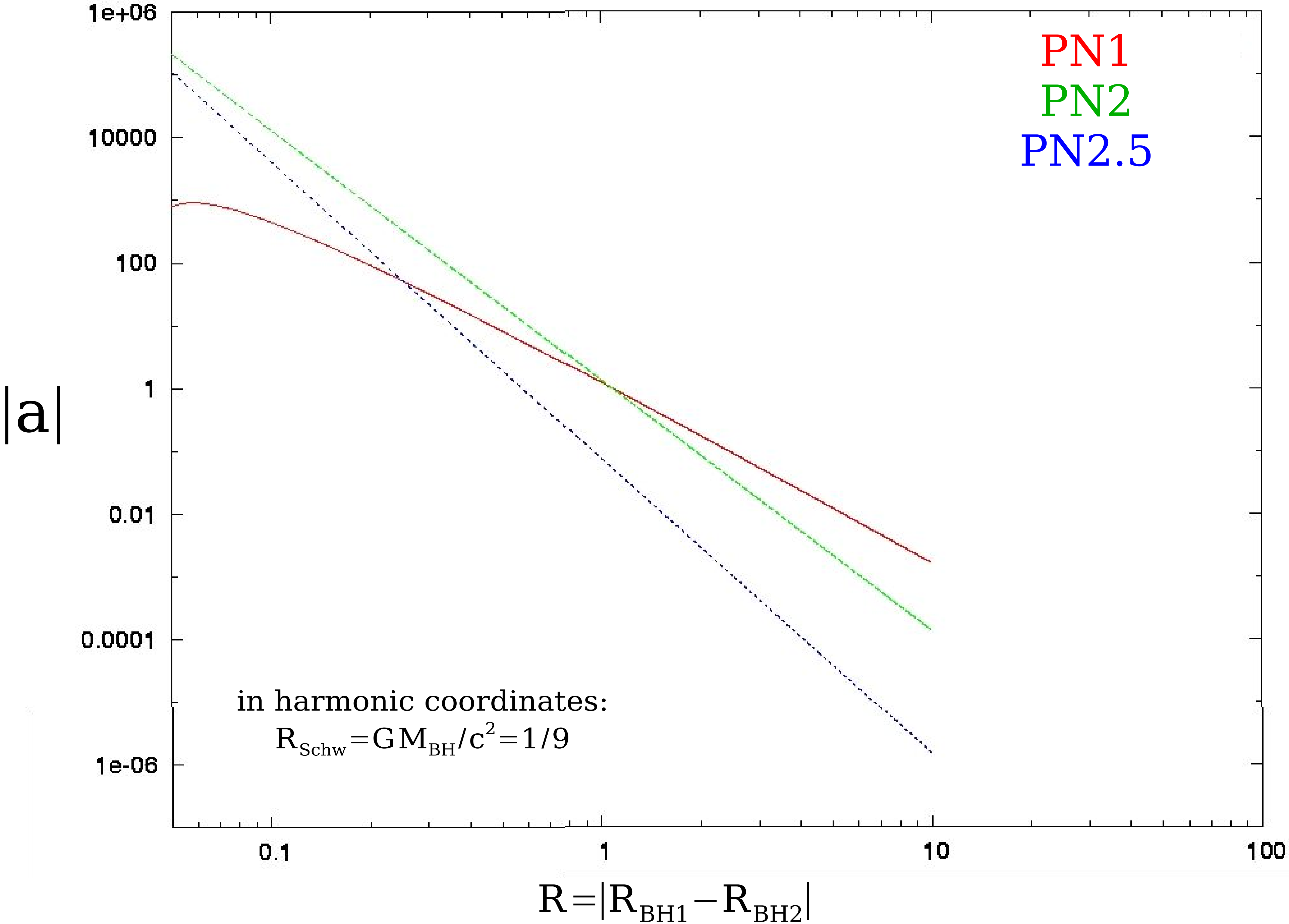
Circular orbit frequency (c=3)



Circular orbit freq. (c=10)



Magnitude of PN aceleration terms (c=3)



Linear Stability Analysis

The PN equations of motion in polar coordinates are:

$$\dot{r} = u$$
$$\dot{u} = -\frac{GM}{r^2}[1+A+Bu]+r\omega^2$$
$$\dot{\omega} = -\omega\left[\frac{GM}{r^2}B+\frac{2u}{r}\right]$$

Consider small perturbations around the circular orbit:

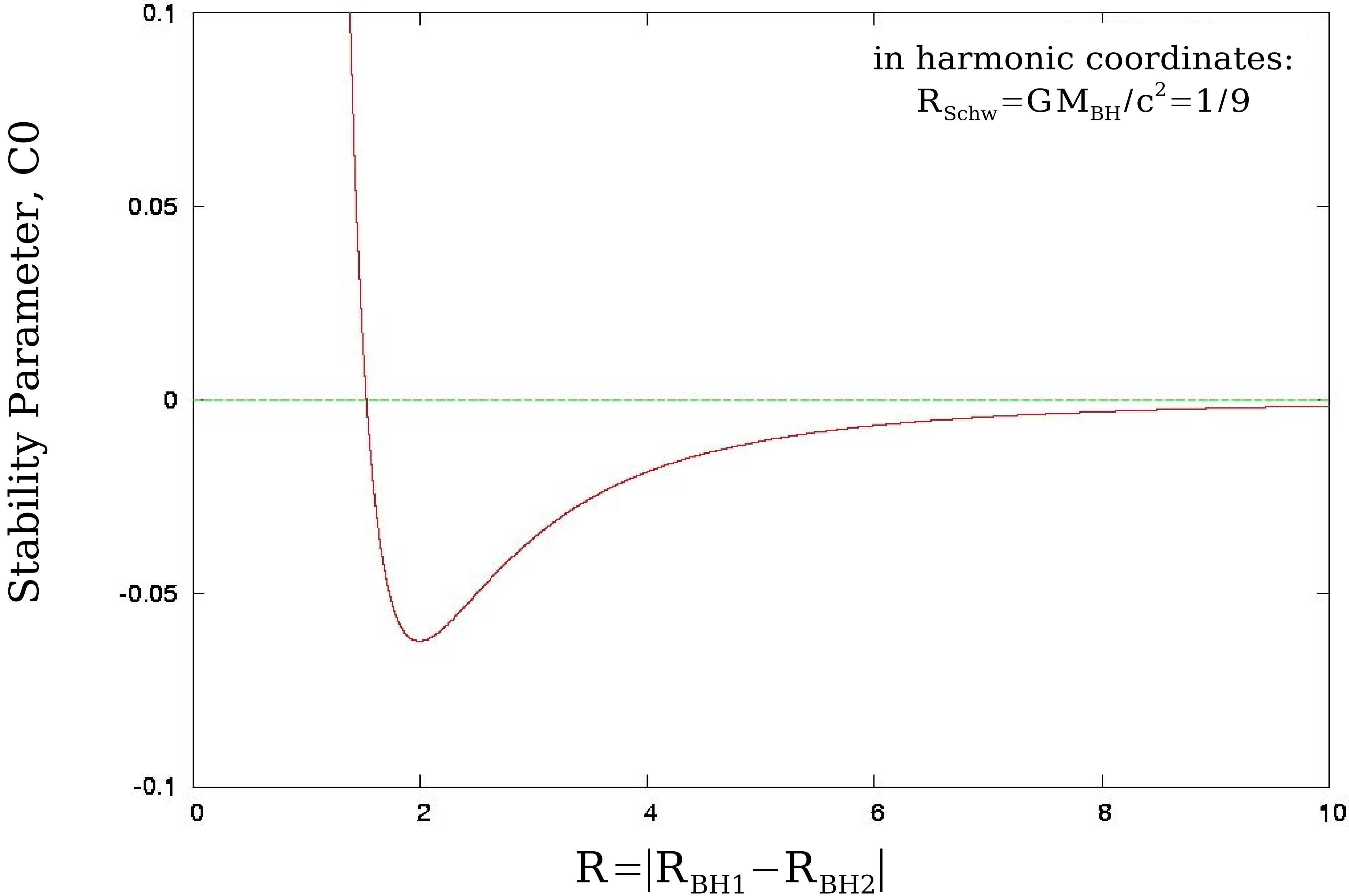
$$r=r_0+\epsilon_0 \qquad \dot{\epsilon}_r=\dot{r}=\epsilon_u$$
$$u=\epsilon_u \quad \longrightarrow \quad \dot{\epsilon}_u=\dot{u}=\alpha_0(r_{0,}\omega_0)\ \epsilon_r+\beta_0(r_{0,}\omega_0)\ \epsilon_\omega$$
$$\omega=\omega_0+\epsilon_\omega \qquad \dot{\epsilon}_\omega=\dot{\omega}=\gamma_0(r_{0,}\omega_0)\ \epsilon_u$$

Solve for the eigenvalues λ to find: $\lambda=\sqrt{-(\alpha_0+\beta_0\gamma_0)}$

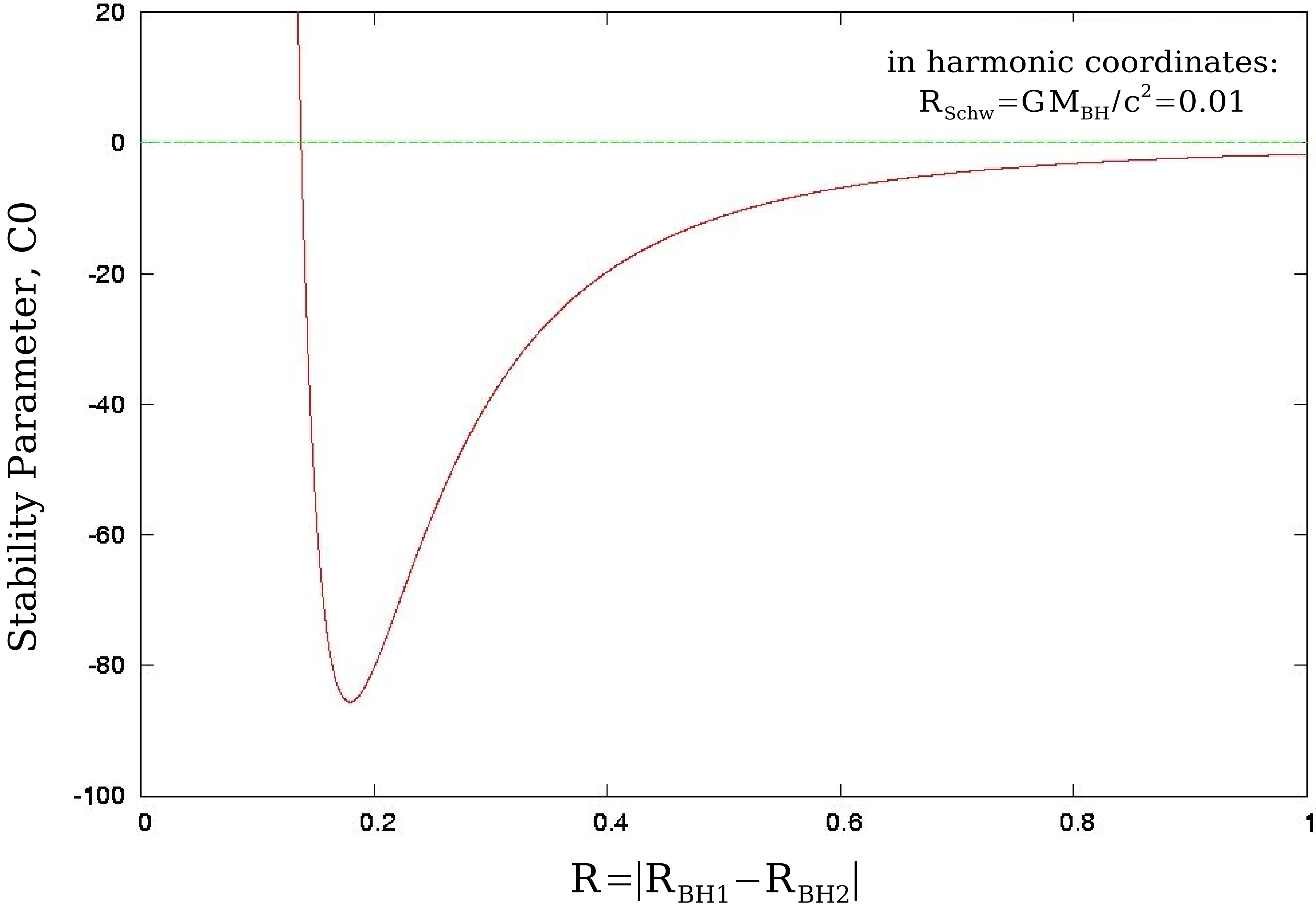
Thus: criteria for stability is $$\boxed{C_0=\alpha_0+\beta_0\gamma_0<0}$$

For solutions of the type: $\epsilon_r=E_r e^{i\lambda t},\ \epsilon_u=E_u e^{i\lambda t}, \epsilon_\omega=E_\omega e^{i\lambda t}$

Linear stability analysis for circular orbits, PN1+PN2 (c=3)



Linear stability analysis for circular orbits, PN1+PN2 (c=10)



Summary

1) **Massive BH binary problem**: already under detailed study in Heidelberg and Potsdam (at least!), PN corrections implemented and tested, still space for improvements in accuracy and efficiency of numerical integrator for the BH and nearby field stars interactions depending on astrophysical scenario;

2) **Resonant Relaxation**: very high N (10 M) required for realism, for many orbits, great accuracy and very expensive in CPU time, (1-4) M calculations are feasible, 10M if combined collisional and tree code (not there yet!);

3) **EMRIs**: very big challenge from the numerical point of view, highly eccentric orbits for many periods (with PN dynamics), very large mass ratio (difficult for chain integrator), algorithmic regularizations, new symplectic or time-reversible methods

4) Compact binaries and stellar BH populations in globular clusters: direct N-body codes ready.

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