A MODE-SUM APPROACH TO THE SELF-FORCE IN CURVED SPACETIME

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1. Gravitational Waves and Radiation Reaction

- A possible source of gravitational waves for LISA detection: a ***binary inspiral*** (**EMRI**).

- Describing the theoretical waveform requires an accurate description of the orbital evolution of the small black hole: including the effects of ***radiation reaction*** and the ***self-force***.

What is Self-force?

- The self-force is the ***interaction*** of a particle with its ***own field*** (scalar, electromagnetic, gravitational).

- Self-force includes ***radiation reaction***, and also includes ***conservative effects***.

- It ***changes*** the ***world-line*** of the particle ***away from*** a ***geodesic***.

- **e.g.** Self-force of a scalar field:

$$F^a_{\text{self}} = q\nabla^a\psi^{\text{R}},$$

q: charge of a particle,
ψ^{R}: field that results from the particle itself but does ***not*** include the ***singular*** part.

2. General Formal Schemes for Calculating Self-force

- **Dirac** (1938): an ***electron*** moving in ***flat*** spacetime,
  Dewitt and Brehme (1960): an ***electron*** moving in ***curved*** spacetime (extended Dirac's problem),
  Mino, Sasaki and Tanaka (1997), **Quinn and Wald** (1997): ***Gravitational*** self-force of particles in ***curved*** spacetime,
  Quinn (2000): a ***scalar*** point particle in ***curved*** spacetime

- **e.g.** Quinn's self-force of a ***scalar*** field:

$$\begin{aligned} m\dot{v}^a &= q\nabla^a\psi_{\rm in} + \frac{1}{3}q^2(\ddot{v}^a - \dot{v}^2 v^a) + \frac{1}{6}q^2(R^a{}_b v^b + v^a R_{bc} v^b v^c) \\ &\quad -\frac{1}{12}q^2 R v^a + \lim_{\varepsilon\to 0} q^2 \int_{-\infty}^{\tau-\varepsilon} \nabla^a G^{\rm ret}\left(x^a(\tau), z^a(\tau')\right) d\tau'. \end{aligned}$$

$\Rightarrow G^{\rm ret}\left(x^a(\tau), z^a(\tau')\right)$: bi-scalar retarded Green's function for the wave eqn.

$$\nabla^2\psi^{\rm ret} = -4\pi J_0 \qquad \text{with} \qquad J_0 = q\int_\Gamma (-g)^{-1/2}\delta^{(4)}\left(x^a - z^a(\tau)\right) d\tau.$$

Note the ***tail*** integral term.

Scalar self-force exists; ***radiation reaction*** occurs even for a particle in free fall.

3. Evaluation of Scalar Self-force in Schwarzschild spacetime by *Mode-sum Regularization* (Barack and Ori (2002); Mino, Nakano, and Sasaki (2002); Detweiler, Messaritaki, and Whiting (2003))

- Self-force is obtained by ***subtraction***:

$$\mathcal{F}_a^{\mathrm{self}} = \lim_{p\to p'} \left[\mathcal{F}_a^{\mathrm{ret}}(p) - \mathcal{F}_a^{\mathrm{S}}(p')\right] = \lim_{p\to p'} \mathcal{F}_a^{\mathrm{R}}(p').$$

(**S** : **S**ingular **S**ource, **R** : **R**egular **R**emainder)

- Use ***spherical harmonic decompositions***:

$$\psi^{\mathrm{ret/S}} = \sum_{\ell m} \psi_{\ell m}^{\mathrm{ret/S}}(r,t) Y_{\ell m}(\theta,\phi).$$

Determine $\psi_{\ell m}^{\mathrm{ret}}(r,t)$ $(\to \mathcal{F}_{\ell\, a}^{\mathrm{ret}})$ ***numerically*** and $\psi_{\ell m}^{\mathrm{S}}(r,t)$ $\left(\to \mathcal{F}_{\ell\, a}^{\mathrm{S}}\right)$ ***analytically***.

- Self-force is finally computed by ***mode-sum regularization***:

$\psi^{\mathrm{ret/S}}$ is ***infinite*** but $\psi_{\ell m}^{\mathrm{ret/S}}$ is ***finite*** at the location of the particle, .

$$\begin{aligned}
\mathcal{F}_a^{\mathrm{self}} &= \lim_{p\to p'} \nabla_a \left(\psi^{\mathrm{ret}} - \psi^{\mathrm{S}}\right) \\
&= \lim_{p\to p'} \nabla_a \left\{\sum_{\ell m} \left[\psi_{\ell m}^{\mathrm{ret}}(r,t) - \psi_{\ell m}^{\mathrm{S}}(r,t)\right] Y_{\ell m}(\theta,\phi)\right\} \\
&= \sum_{\ell=0}^{\infty} \lim_{p\to p'} \left(\mathcal{F}_{\ell\, a}^{\mathrm{ret}} - \mathcal{F}_{\ell\, a}^{\mathrm{S}}\right) \\
&= \sum_{\ell=0}^{\infty} \left\{\lim_{p\to p'} \mathcal{F}_{\ell\, a}^{\mathrm{ret}} - \left[\left(\ell + \frac{1}{2}\right) A_a + B_a + \frac{C_a}{\ell + \frac{1}{2}} - \frac{2\sqrt{2} D_a}{(2\ell-1)(2\ell+3)} + O(\ell^{-4})\right]\right\},
\end{aligned}$$

where A_a, B_a, C_a, D_a are termed ***regularization parameters***.

3.A. Numerical Computation of $\mathcal{F}^{\rm ret}_{\ell\, a}$ (Detweiler, Messaritaki, and Whiting (2003))

- ***Solve numerically*** the wave eqn. for a charge q moving along $\Gamma(\tau)$, described by $z^a(\tau)$:

$$\nabla^2\psi^{\rm ret} = -4\pi j_0 \qquad \text{with} \qquad j_0 = q\int_\Gamma (-g)^{-1/2}\delta^{(4)}\left(x^a - z^a(\tau)\right)d\tau.$$

- **e.g.** Charge moving along a ***circular*** orbit at radius $r_{\rm o}$ with angular frequency Ω about a Schwarzschild BH:

$$z^i(t) \Leftrightarrow (r = r_{\rm o}, \theta = \pi/2, \phi = \Omega t).$$

  Then

$$j_0 = \sum_{\ell m}\frac{q_{\ell m}}{4\pi r_{\rm o}}\delta(r - r_{\rm o})e^{i\omega_m t}Y_{\ell m}(\theta,\phi)$$

  with

$$\omega_m = -m\Omega, \qquad q_{\ell m} = \frac{4\pi q}{r_{\rm o}}\frac{Y^*_{\ell m}(\pi/2,0)}{dt/d\tau}, \qquad \frac{dt}{d\tau} = \frac{1}{\sqrt{1-3M/r_{\rm o}}}.$$

  Decompose

$$\psi^{\rm ret} = \sum_{\ell m}\psi^{\rm ret}_{\ell m}(r)e^{i\omega_m t}Y_{\ell m}(\theta,\phi)$$

  and solve the wave eqn.

$$\frac{d^2\psi^{\rm ret}_{\ell m}}{dr^2} + \frac{2(r-M)}{r(r-2M)}\frac{d\psi^{\rm ret}_{\ell m}}{dr} + \left[\frac{\omega^2 r^2}{(r-2M)^2} - \frac{\ell(\ell+1)}{r(r-2M)}\right]\psi^{\rm ret}_{\ell m} = -\frac{q_{\ell m}}{r_{\rm o}-2M}\delta(r - r_{\rm o})$$

  with B.C. requiring only ingoing waves at the event horizon ($r \to 2M$) and outgoing waves at infinity ($r \to \infty$).
  Finally,

$$\mathcal{F}^{\rm ret}_{\ell\, r} = \sum_{m=-\ell}^{\ell}\left.\frac{d\psi^{\rm ret}_{\ell m}}{dr}\right|_{r_{\rm o}}.$$

3.B. Analytical Calculation of $\mathcal{F}^{\rm S}_{\ell\,a}$ (Barack and Ori (2002); Mino, Nakano, and Sasaki (2002); Detweiler, Messaritaki, and Whiting (2003); Kim(2004))

- Determine ***regularization parameters*** analytically

$$\mathcal{F}^{\rm S}_{\ell\,a} = \left(\ell + \frac{1}{2}\right) A_a + B_a + \frac{C_a}{\ell + \frac{1}{2}} - \frac{2\sqrt{2}D_a}{(2\ell - 1)(2\ell + 3)} + O(\ell^{-4}).$$

- **e.g.** Charge moving along a ***general*** orbit about a Schwarzschild BH (Kim(2004)):

$$\mathcal{F}^{\rm S}_{\ell\,a} = \ell \quad \text{mode} \quad \text{decomp.} \quad \text{of} \quad \partial_a \psi^{\rm S}, \qquad \text{where} \qquad \psi^{\rm S} = \frac{q}{\sqrt{\mathcal{X}^2 + \mathcal{Y}^2 + \mathcal{Z}^2}}.$$

$$\Leftarrow \qquad \mathcal{X}^A = \Lambda^A{}_B \left[M^B{}_b (x^b - x^b_{\rm o}) + \frac{1}{2} M^B{}_b \, \Gamma^b_{cd}\Big|_{\rm o} (x^c - x^c_{\rm o})(x^d - x^d_{\rm o}) \right] + O[(x - x_{\rm o})^3]$$

(***Thorne-Hartle-Zhang normal coords.***) with

$$\begin{aligned} M^A{}_a &= \text{diag} \ \left[f^{1/2}, f^{-1/2}, r_{\rm o}, -r_{\rm o} \right] \qquad \left(f \equiv 1 - \frac{2M}{r_{\rm o}} \right), \\ \Lambda^A{}_B &= \text{Lorentz} \ \ \text{boost} \ \ \text{by} \ \ \text{four-velocity} \ \ \text{of} \ \ \text{particle.} \end{aligned}$$

Then

$$A_t = \text{sgn}(\Delta) \frac{q^2}{r_{\rm o}^2} \frac{\dot{r}}{\lambda}, \quad A_r = -\text{sgn}(\Delta) \frac{q^2}{r_{\rm o}^2} \frac{E}{f\lambda}, \quad A_\phi = 0, \quad A_\theta = 0, \quad B_t = \frac{q^2}{r_{\rm o}^2} E \dot{r} \left[\frac{F_{3/2}}{\lambda^{3/2}} - \frac{3F_{5/2}}{2\lambda^{5/2}} \right],$$

$$B_r = \frac{q^2}{r_{\rm o}^2} \left[-\frac{F_{1/2}}{\lambda^{1/2}} + \frac{(f - 2\dot{r}^2)\, F_{3/2}}{2f\lambda^{3/2}} + \frac{3\dot{r}^2 F_{5/2}}{2f\lambda^{5/2}} \right], \quad B_\phi = \frac{q^2}{J} \dot{r} \left[\frac{F_{1/2} - F_{3/2}}{\lambda^{1/2}} + \frac{3(F_{5/2} - F_{3/2})}{2\lambda^{3/2}} \right], \quad B_\theta = 0,$$

$$C_t = C_r = C_\phi = C_\theta = 0, \qquad \cdots, \qquad \text{where} \quad \Delta \equiv r - r_{\rm o} \quad \text{and} \quad \lambda \equiv (1 + J^2/r_{\rm o}^2).$$

3.C. An Example: Self-force on ***circular*** orbits about Schwarzschild BH

$$\begin{aligned}\mathcal{F}_r^{\rm self} &= \sum_\ell \lim_{r\to r_{\rm o}} \left[\mathcal{F}^{\rm ret}_{\ell\, r}(r) - \mathcal{F}^{\rm S}_{\ell\, r}(r)\right] = \sum_\ell \mathcal{F}^{\rm R}_{\ell\, r}(r_{\rm o}) \\ &= \sum_\ell \left\{ \lim_{r\to r_{\rm o}} \mathcal{F}^{\rm ret}_{\ell\, r}(r) - \left[\left(\ell+\frac{1}{2}\right)A_r + B_r - \frac{2\sqrt{2}D_r}{(2\ell-1)(2\ell+3)} + O(\ell^{-4})\right]\right\}\end{aligned}$$

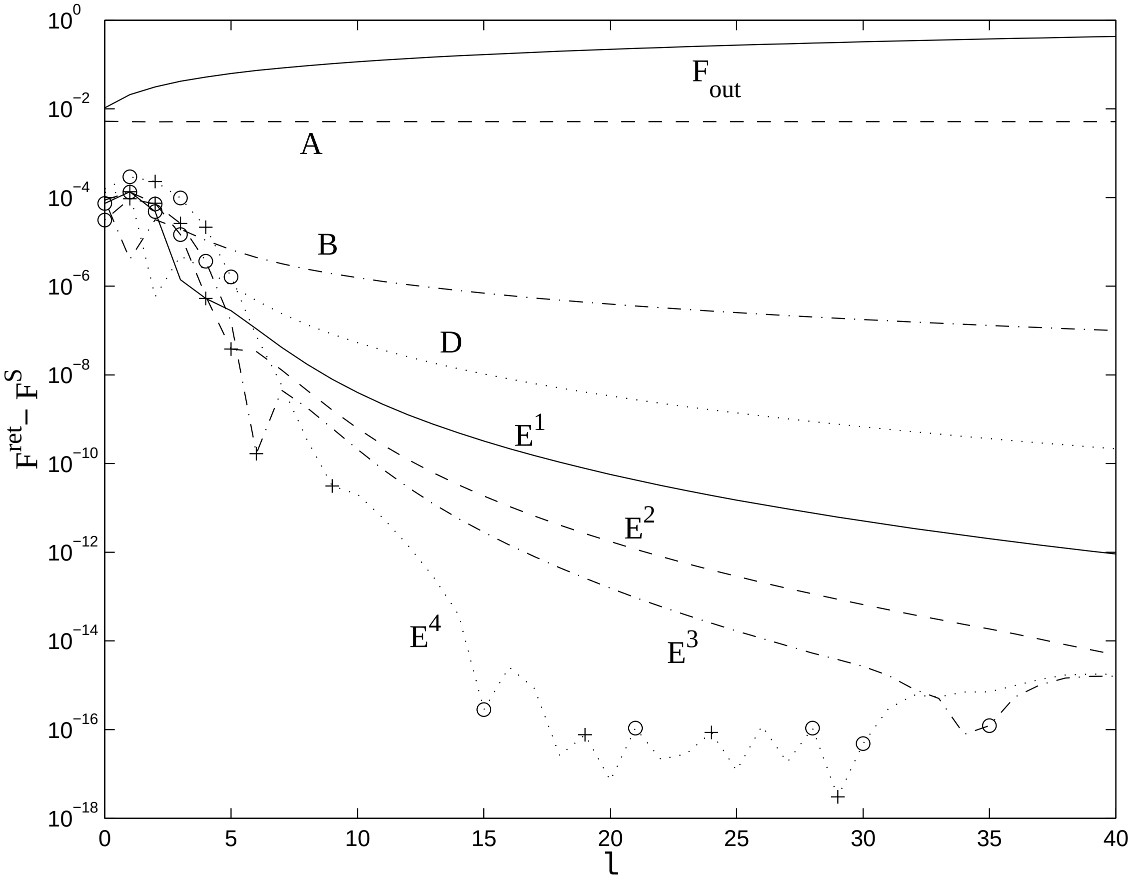

Figure 1: $\mathcal{F}^{\rm self}_{\ell\, r}$ vs. ℓ (by courtesy of Detweiler et al, 2003): for $r_{\rm o} = 10M$ to $\ell = 40$, $\mathcal{F}_r^{\rm self} = 1.37844828(2) \times 10^{-5}$

4. Gravitational Self-force in Schwarzschild Spacetime by Mode-sum Regularization (in progress)

- Problems with ***Mino-Sasaki-Tanaka-Quinn-Wald Gravitational Self-force***: ***not gauge invariant*** and ***depending upon the gauge condition***.
- Determine the ***effects of self-force***: ***changes*** in ***gauge invariant quantities*** due to the ***self-force effects***.

e.g. For a ***circular*** orbit, a ***direct observable*** Ω (orbital angular frequency) or a combination of observables $E - \Omega J$
$\Rightarrow$ The particle moves on a geodesic of $g_{ab} + h^{\mathrm{R}}_{ab}$ and

$$\Omega^2 = (d\phi/dt)^2 = \left(u^\phi/u^t\right)^2 = \frac{M}{r^3} - \frac{r-3M}{2r^2} u^a u^b \partial_r h^{\mathrm{R}}_{ab},$$

$$(E - \Omega J)^2 = \left(1 - \frac{3M}{r}\right)\left(1 - u^a u^b h^{\mathrm{R}}_{ab} + \frac{1}{2} r u^a u^b \partial_r h^{\mathrm{R}}_{ab}\right),$$

where $u_a = (u_t,\, u_r,\, u_\theta,\, u_\phi) = (-E,\, 0,\, 0,\, J)$ (four-velocity of particle).
Mode-sum regularization can be similarly employed using ***tensor spherical harmonic decomposition***

$$h^{\mathrm{R}}_{ab} = h^{\mathrm{ret}}_{ab} - h^{\mathrm{S}}_{ab} = \sum_{\ell m}\left[h^{\mathrm{ret}\,\ell m}_{ab} - h^{\mathrm{S}\,\ell m}_{ab}\right] = \sum_{\ell}\left[h^{\mathrm{ret}\,\ell}_{ab} - h^{\mathrm{S}\,\ell}_{ab}\right],$$

$$\partial_c h^{\mathrm{R}}_{ab} = \partial_c h^{\mathrm{ret}}_{ab} - \partial_c h^{\mathrm{S}}_{ab} = \sum_{\ell m}\left[\partial_c h^{\mathrm{ret}\,\ell m}_{ab} - \partial_c h^{\mathrm{S}\,\ell m}_{ab}\right] = \sum_{\ell}\left[\partial_c h^{\mathrm{ret}\,\ell}_{ab} - \partial_c h^{\mathrm{S}\,\ell}_{ab}\right].$$

(**S** : **S**ingular **S**ource, **R** : **R**egular **R**emainder)

5. Self-forces in Kerr Spacetime by Mode-sum Regularization (in progress)

- ***Extremely difficult*** to handle the problem in ***strong field regime***
- Study of a ***simple*** case of ***scalar*** self-force: in ***weak field***, ***weak spin*** limit ($M/r \ll 1$, $a/r \ll 1$ ($a \equiv J/M$)) with ***circular***, ***equatorial*** orbits
  $\Rightarrow$ ***Simplify the Kerr geometry*** and ***modify the coordinates*** such that to leading order in M/r and a/r

$$ds^2 \cong -\left(1 - \frac{2M}{r}\right) d\tilde{t}^2 + \left(1 + \frac{2M}{r}\right) dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\tilde{\phi}^2\right),$$

where

$$\tilde{t} \equiv t - (a \sin^2\theta)\phi \qquad \text{and} \qquad \tilde{\phi} \equiv \phi - \left(\frac{a}{r^2}\right) t.$$

Then, $\tilde{\nabla}^2$ is ***spherically symmetric in the new geometry*** $\tilde{x}^a = \left(\tilde{t}, r, \theta, \tilde{\phi}\right)$, and we solve

$$\tilde{\nabla}^2 \psi^{\text{ret}} = -4\pi j^0(\tilde{x}),$$

where

$$j^0(\tilde{x}) = q \int (-g)^{-1/2} \delta^4\left(\tilde{x}^a - \tilde{z}^a(\tau)\right) d\tau = q(-g)^{-1/2}\left(d\tilde{t}/d\tau\right)^{-1} \delta^3\left(\tilde{x}^i - \tilde{z}^i\left(\tilde{t}\right)\right)$$

and

$$\tilde{z}^i\left(\tilde{t}\right) = \left(r_{\text{o}}, \pi/2, \tilde{\Omega}\tilde{t}\right) \quad \text{with} \quad \tilde{\Omega} \equiv \Omega - a\left(r_{\text{o}}^{-2} + \Omega^2\right) \qquad \Leftrightarrow \qquad z^i(t) = \left(r_{\text{o}}, \pi/2, \Omega t\right),$$

to calculate

$$F_a^{\text{self}} = q^2 \lim_{p \to p'} \partial_a \psi^{\text{R}} = q^2 \lim_{p \to p'} \partial_a \left(\psi^{\text{ret}} - \psi^{\text{S}}\right) = \sum_{\ell=0}^{\infty} \lim_{p \to p'} \left(\mathcal{F}_{\ell a}^{\text{ret}} - \mathcal{F}_{\ell a}^{\text{S}}\right).$$

- For a scalar charge q moving along a ***general*** orbit in the plane with $\theta_{\rm o}$ about a Kerr BH,

$$\mathcal{F}^{\rm S}_{\ell\, a} = \left(\ell + \frac{1}{2}\right) A_a + B_a + \frac{C_a}{\ell + \frac{1}{2}} + O(\ell^{-2})$$

with regularization parameters

$$\begin{aligned}
A_t &= \mathrm{sgn}(r - r_{\rm o}) \frac{q^2 \Sigma^2 \sin^2\theta_{\rm o} \dot{r}}{\mathcal{A}\sin^2\theta_{\rm o}\,(\Sigma + u_\theta^2) + \Sigma^2 L_z^2}, \\
A_r &= -\mathrm{sgn}(r - r_{\rm o}) \frac{q^2 \Sigma \sin\theta_{\rm o} \left[\mathcal{A}\sin^2\theta_{\rm o}\,(\Sigma + (\Sigma^2/\Delta)\dot{r}^2 + u_\theta^2) + \Sigma^2 L_z^2\right]^{1/2}}{\Delta^{1/2}\left[\mathcal{A}\sin^2\theta_{\rm o}\,(\Sigma + u_\theta^2) + \Sigma^2 L_z^2\right]},\ A_\theta = 0,\ A_\phi = 0, \\
B_a &= \sqrt{2} q^2 \sin\theta_{\rm o} \Big[-R_{\rm o}^{-3/2}\left(P_{ab}\,\Gamma^b_{cd}\big|_{\rm o} + 2P_{bc}\,\Gamma^b_{ad}\big|_{\rm o}\right)\left\langle(\sin\Phi)^{\mathcal{M}}(\cos\Phi)^{2-\mathcal{M}}\chi^{-3/2}\right\rangle \\
&\quad +6R_{\rm o}^{-5/2} P_{ac}P_{bd}\,\Gamma^b_{ef}\big|_{\rm o}\left\langle(\sin\Phi)^{\mathcal{N}}(\cos\Phi)^{4-\mathcal{N}}\chi^{-5/2}\right\rangle\Big] \qquad (\mathcal{M} = \delta^c{}_\theta + \delta^d{}_\theta,\ \ \mathcal{N} = \delta^c{}_\theta + \delta^d{}_\theta + \delta^e{}_\theta + \delta^f{}_\theta), \\
C_a &= 0 \qquad \cdots \quad ,
\end{aligned}$$

where

$$\Delta \equiv r_{\rm o}^2 + a^2 - 2Mr_{\rm o}, \quad \Sigma \equiv r_{\rm o}^2 + a^2\cos^2\theta_{\rm o}, \quad \mathcal{A} \equiv \left(r_{\rm o}^2 + a^2\right)^2 - a^2\Delta\sin^2\theta_{\rm o},$$

$$P_{ab} \equiv u_{{\rm o}a}u_{{\rm o}b} + g_{ab}\big|_{\rm o}, \quad Q_{ab} \equiv 2u_{{\rm o}a}u_{{\rm o}b} + g_{ab}\big|_{\rm o},$$

$$R_{\rm o} \equiv \left[\left(-\Sigma - u_\theta^2 + L_z^2 + \frac{\mathcal{A}\sin^2\theta_{\rm o}}{\Sigma}\right)^2 + 4u_\theta^2 L_z^2\right]^{1/2} + \Sigma + u_\theta^2 + L_z^2 + \frac{\mathcal{A}\sin^2\theta_{\rm o}}{\Sigma}, \quad \chi \equiv 1 - \alpha\sin^2(\Phi + \beta),$$

$$\alpha \equiv 2R_{\rm o}^{-1}\left[\left(-\Sigma - u_\theta^2 + L_z^2 + \mathcal{A}\sin^2\theta_{\rm o}/\Sigma\right)^2 + 4u_\theta^2 L_z^2\right]^{1/2},\ \beta = \tan^{-1}\left[2u_\theta L_z/\left(-\Sigma - u_\theta^2 + L_z^2 + \mathcal{A}\sin^2\theta_{\rm o}/\Sigma\right)\right]$$

- ***Gravitational*** self-force in Kerr:
  Similarly, we may also try to solve

$$E_{ab} = -8\pi T_{ab}\left(\tilde{x}\right),$$

where

$$E_{ab} = \tilde{\nabla}^2 h^{\mathrm{ret}}_{ab} + \tilde{\nabla}_a \tilde{\nabla}_b h^{\mathrm{ret}} - 2\tilde{\nabla}_{(a}\tilde{\nabla}^c h^{\mathrm{ret}}_{b)c} + 2R_a{}^c{}_b{}^d h^{\mathrm{ret}}_{cd} + g_{ab}\left(\tilde{\nabla}^c\tilde{\nabla}^d h^{\mathrm{ret}}_{cd} - \tilde{\nabla}^2 h^{\mathrm{ret}}\right)$$

and

$$T_{ab}\left(\tilde{x}\right) = \mu \int u^a u^b (-g)^{-1/2} \delta^4\left(\tilde{x}^a - \tilde{z}^a(\tau)\right) d\tau,$$

to calculate

$$\frac{\mathrm{d}E}{\mathrm{d}\tau} = -\frac{1}{2}u^a u^b \frac{\partial h^{\mathrm{R}}_{ab}}{\partial t} = -\frac{1}{2}u^a u^b \frac{\partial\left(h^{\mathrm{ret}}_{ab} - h^{\mathrm{S}}_{ab}\right)}{\partial t} \quad \text{and} \quad \frac{\mathrm{d}J}{\mathrm{d}\tau} = \frac{1}{2}u^a u^b \frac{\partial h^{\mathrm{R}}_{ab}}{\partial \phi} = \frac{1}{2}u^a u^b \frac{\partial\left(h^{\mathrm{ret}}_{ab} - h^{\mathrm{S}}_{ab}\right)}{\partial \phi}.$$

DISCUSSIONS & QUESTIONS

- How to include the ***self-force effects*** in the ***orbital evolution***.
- ***Dephasing effects*** due to ***conservative*** self-force / ***Adiabatic approach***.
- Developing techniques for the self-force in ***Kerr*** in ***strong field regime***.