Mapping spacetime with LISA, Ryan's theorem and its generalization

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Sep 21, 2006

EMRI meeting, AEI, Potsdam

Question

From the observation of EMRI waveforms, how can we extract

- **the spacetime information of the central body**

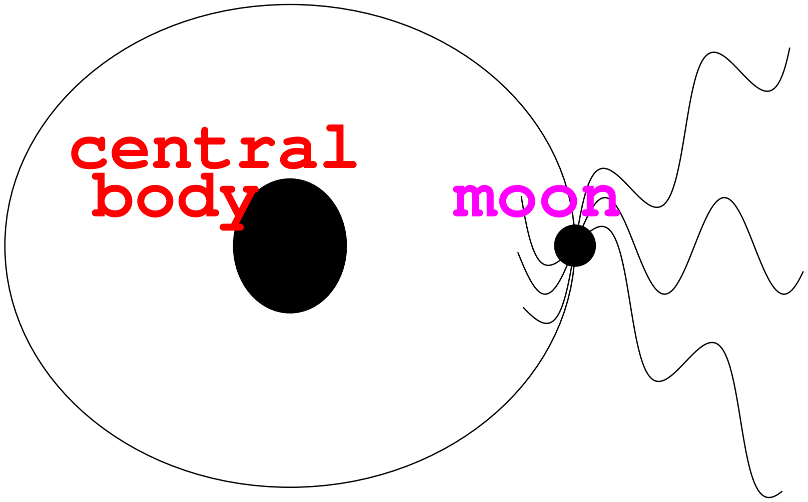
  spacetime multipole moments, mass and current which are defined by STF tensors

- **the orbital parameters of the small object**

  orbital semi-latus rectum p, eccentricity e, and inclination angle ι

- **tidal coupling information**

  energy and angular momentum flux into the central body



Assumptions

- **The central body has a spacetime metric which is stationary, axisymmetric, reflection symmetric across the equatorial plane and asymptotically flat. (SARSAF spacetime)**

  Much more general than Kerr metric.

- **Adiabactical approximation.**

  The inspiraling moon is sufficiently compact and its mass is sufficiently small such that the radiation reaction timescale is much longer than its orbital period.

  Radiation reaction in general SARSAF spacetime is difficult.

  Approximate by snapshot waveforms, which are emitted by moon in fixed geodesics.

Ryan's theorem (PRD, 52 5707)

- **Ryan: 1. Assume the geodesics are nearly circular and nearly equatorial. So that there are three fundamental frequencies, orbital frequency Ω_ϕ , oscillating frequencies Ω_ρ PN parameter $v = (M\Omega_\phi)^{1/3}$.**

  2. Neglect tidal coupling between central object and the moon.

$$\frac{\Omega_\rho}{\Omega_\phi} = 3v^2 - 4\frac{S_1}{M^2}v^3 + \left(\frac{9}{2} - \frac{3}{2}\frac{M_2}{M^3}\right)v^4 - 10\frac{S_1}{M^2}v^5 + \left(\frac{27}{2} - 2\frac{S_1^2}{M^4} - \frac{21}{2}\frac{M_2}{M^3}\right)v^6 + \cdots$$

$$\frac{\Omega_z}{\Omega_\phi} = 2\frac{S_1}{M^2}v^3 + \frac{3}{2}\frac{M_2}{M^3}v^4 + \left(7\frac{S_1^2}{M^4} + 3\frac{M_2}{M^3}\right)v^6 + \left(11\frac{S_1 M_2}{M^5} - 6\frac{S_3}{M^4}\right)v^7 + \cdots$$

$$-\left.\frac{dE}{dt}\right|_{orbit} = \frac{32}{5}\left(\frac{\mu}{M}\right)^2 v^{10}\left[1 - \frac{1247}{336}v^2 + 4\pi|v|^3 - \frac{44711}{9072}v^4 - \frac{11}{4}\frac{S_1}{M^2}v^3 + \frac{1}{16}\frac{S_1^2}{M^4}v^4 + \cdots\right]$$

$$\frac{\Delta E}{\mu} = \frac{1}{3}v^2 - \frac{1}{2}v^4 + \frac{20}{9}\frac{S_1}{M^2}v^5 \cdots$$

My generalization

- **Generic geodesics and waveforms in SARSAF spacetime**

- **Extract tidal coupling information (Preliminary results)**

- **Some preparations**

  Specify gauge: Weyl coordinate (ρ, z, ϕ, t).

  metric: $-F(dt - \omega d\phi)^2 + \frac{1}{F}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2]$.

  In SARSAF spacetime

  Spacetime metric $\Leftrightarrow$ **Mass and current moments** M_i, S_i.

Geodesics in SARSAF spacetime

- **Three known constants of motion, (E, L, norm).**
- **If there exists a fourth integral of motion, (Carter constant in Kerr metric), geodesic equation is completely separable. Geodesics have multi-periodicities.**

$$\rho(\tau) = \sum_{mn} \rho_{mn} e^{i(m\Lambda_\rho + n\Lambda_z)\tau}, \quad \phi(\tau) = \Lambda_\phi \tau + \sum_{mn} \phi_{mn} e^{i(m\Lambda_\rho + n\Lambda_z)\tau}$$

$$z(\tau) = \sum_{mn} z_{mn} e^{i(m\Lambda_\rho + n\Lambda_z)\tau}, \quad t(\tau) = \Lambda_t \tau + \sum_{mn} t_{mn} e^{i(m\Lambda_\rho + n\Lambda_z)\tau}$$

This can be proved using action-angle variable theory. See (Goldstein, Classical Mechanics Chapter

- **Even without fourth integral of motion, KAM theorem still predicts multi-periodicities.**

KAM theorem in Newtonian gravity

(See Ilya's talk on geodesics in SARSAF spacetime.)

- **Geodesic equation in general SARSAF spacetime is complicated. Here I do a simpler example, a point particle moves in an axisymmetric, stationary Newtonian potential.**

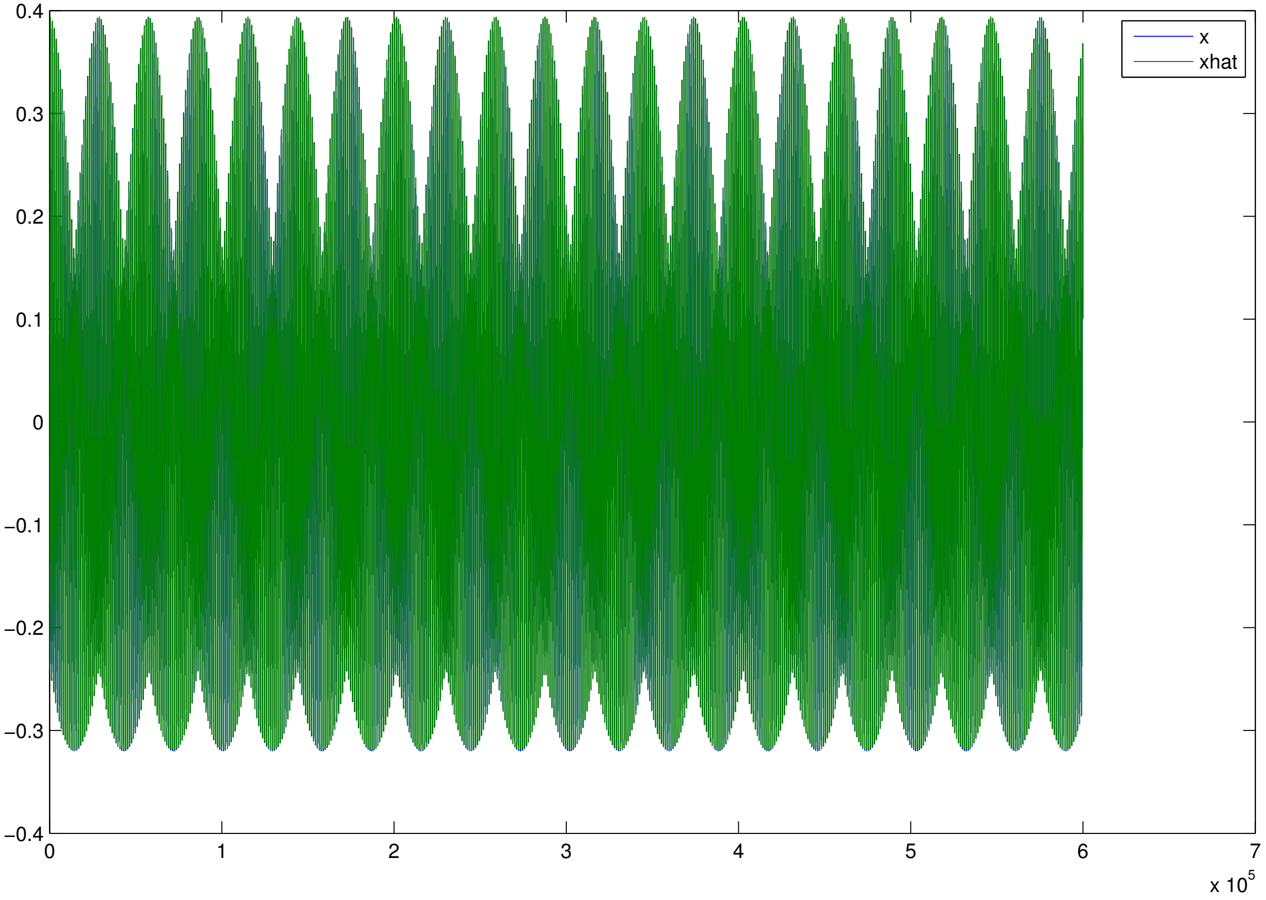
$$V(r,\theta) = -\frac{M_0}{r} + \frac{M_2}{r^3}P_2(\cos\theta) + \frac{M_4}{r^5}P_4(\cos\theta)$$

I use standard fourth-order Runge-Kutta method to do integration.

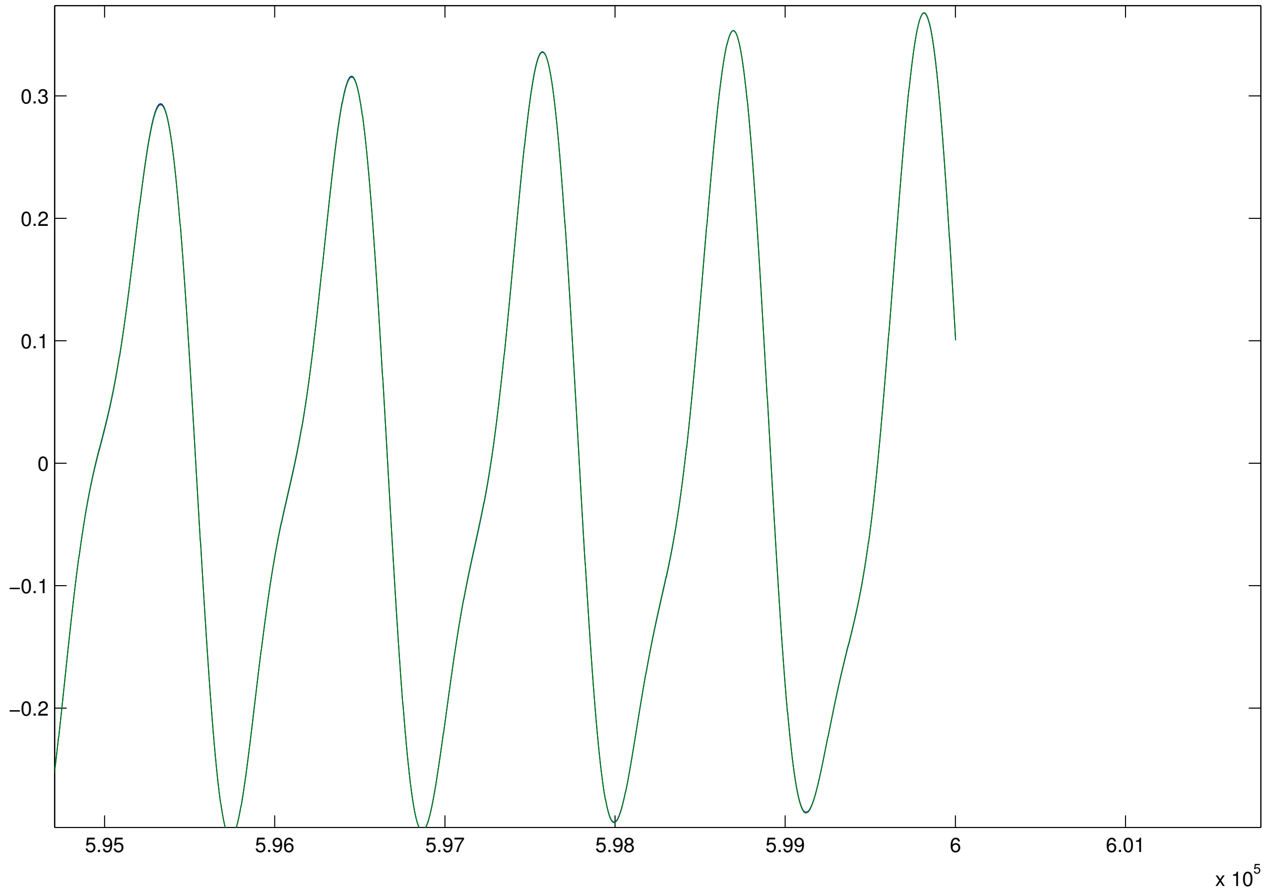
Initial conditions: $r_0 = 9.5M_0, \theta_0 = \pi/2, v_r = 0, v_\theta = 0.2.$

Integration time: from t=0 to t=$10^4 M_0$

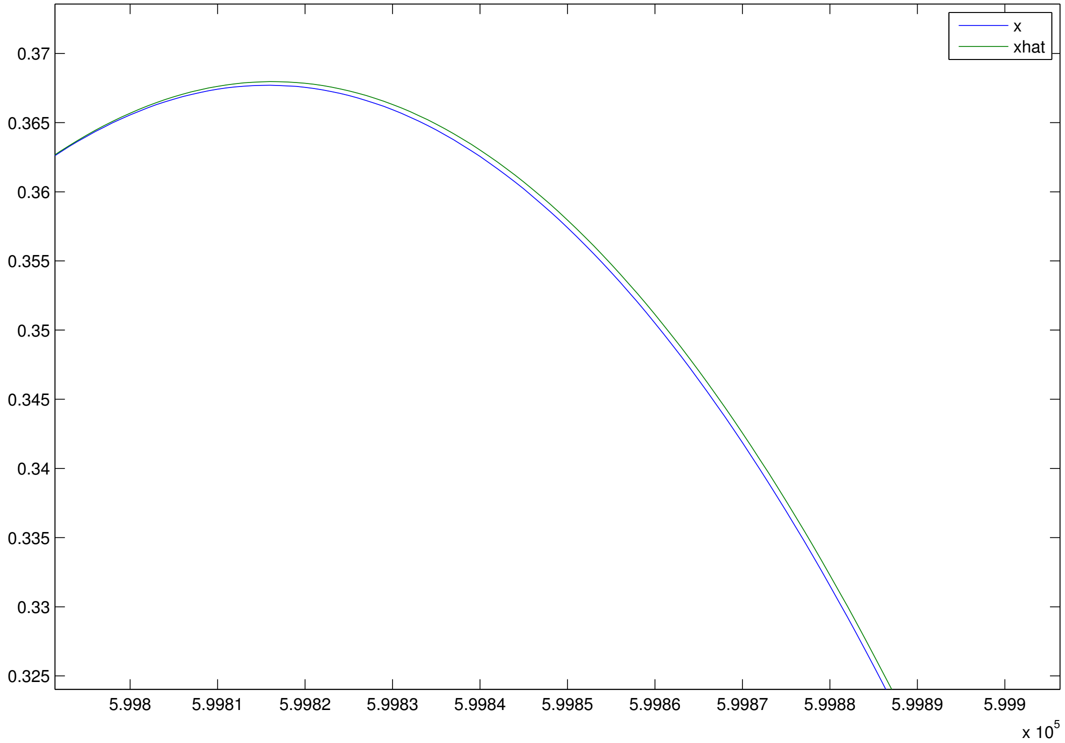
Numerics



Zoom in



Zoom in again

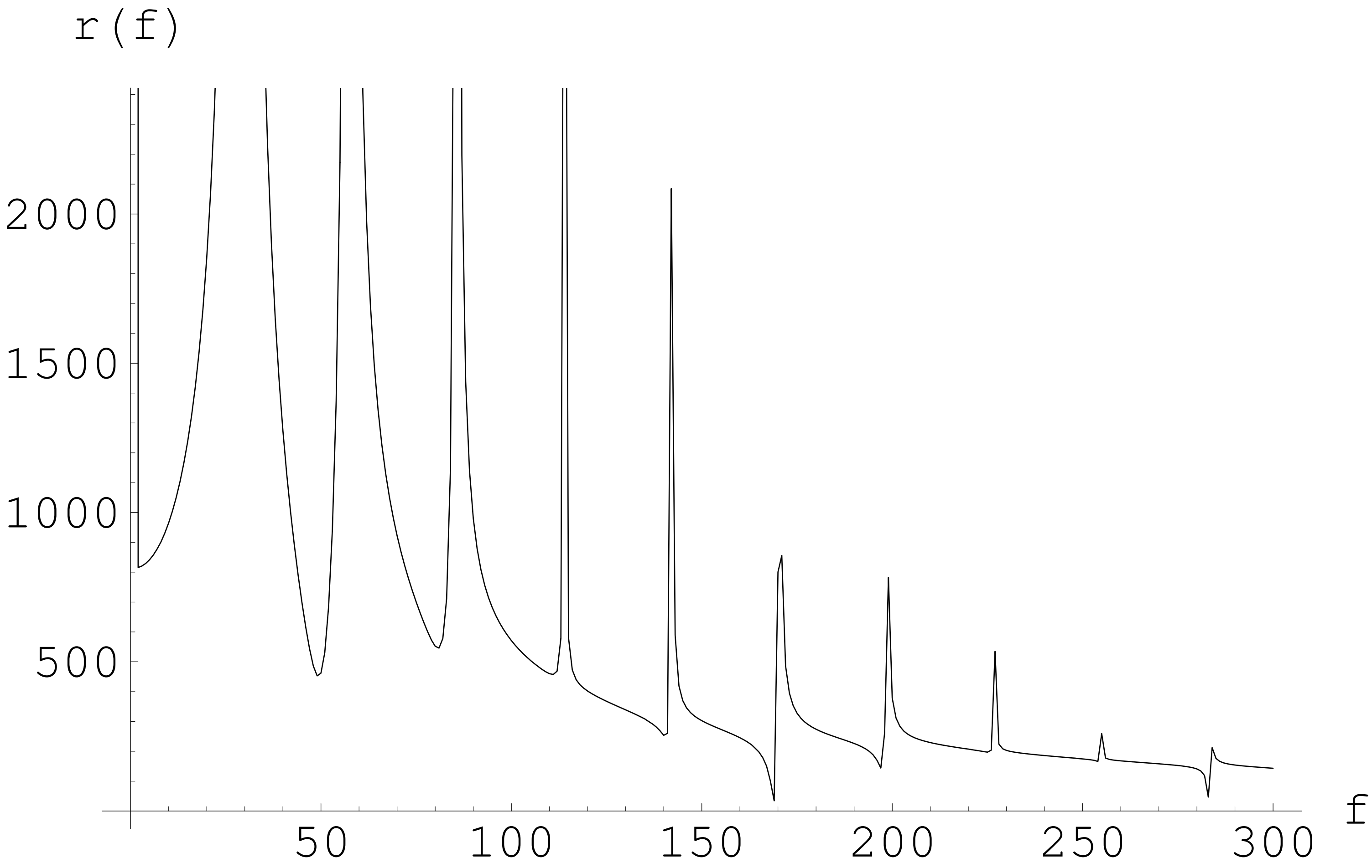


Blue curve: numerical solution of $r(t)$

Green curve: fitting function, $\hat{r}(t) = \sum_{i=1}^{9} A_i \cos(\omega_i + \phi_i)$**.**

all ω_i **can be written as** $m\omega_1 + n\omega_2$**, (m,n are integers)**

Fast Fourier Transform



KAM theorem and its application

- **Let H_0 to be an integrable Hamiltonian function with a torus T_0 and set of frequencies ω, having an incommensurate frequency vector ω^*. Let H_0 be perturbed by some periodic function H_1, KAM theorem states that, if H_1 is small enough, then for almost every ω^*, there exists an invariant torus $T(\omega^*)$ of the perturbed system such that $T(\omega^*)$ is "close to" $T_0(\omega^*)$. Moreover, the tori $T(\omega^*)$ form a set of positive measures whose complement has a measure which tends to zero as $H_1 \to 0$.**

Geodesics in SARSAF spacetime

- **To conclude, it's reasonable to expect that the geodesics in SARSAF spacetimes are quasi multi-periodic. In the following, I will discuss the gravitational wave snapshots by the moon when it moves in these quasi multi-periodic geodesics.**

- **However, there do exist chaotic orbits in SARSAF spacetime. But all the chaotic orbits we found are close to the central object. (More in Ilya's talk)**

Gravitational Waves in SARSAF spacetime

- **Green's function in curved background**

Symmetry in this problem: **stationary and axisymmetric**

$$G = \int d\omega \sum_m G_m(\rho_0, \rho', z_0, z') e^{im(\phi_0 - \phi')} e^{i\omega(t_0 - t')}$$

- **Source terms**

Leading order stress energy tensor

$$T^{\alpha\beta} = \int d\tau \mu \frac{dx'^{\alpha}}{d\tau} \frac{dx'^{\beta}}{d\tau} \delta^{(4)}(x' - x(\tau))$$

- **Gravitational waves**

$$h^{\alpha\beta}(t) = \int GT = \sum h^{\alpha\beta}_{mnl} e^{-i(m\Omega_\phi + n\Omega_\rho + l\Omega_z)t}$$

$$\Omega_\phi = \Lambda_\phi / \Lambda_t, \qquad \Omega_z = \Lambda_z / \Lambda_t, \qquad \Omega_\rho = \Lambda_\rho / \Lambda_t,$$

Nice frequency properties!

What can we observe?

- **Short time, (snapshot, static) Observables:** $h^{\alpha\beta}_{mnl}, \quad \Omega_\phi, \Omega_\rho, \Omega_z$

  Perihelion precession? $\Delta\phi = 2\pi \times \left(\frac{\Omega_\rho}{\Omega_\phi} - 1\right)$

- **Long time, (secular, dynamical) Observables:** $h^{\alpha\beta}_{mnl}(t), \quad \Omega_\phi(t), \Omega_\rho(t), \Omega_z(t)$

What can we do from observation?

- **Only use frequency information: counting degrees of freedom.**

  Assume detection of N snapshots, so we have 3N data. $(\Omega^i_\rho, \Omega^i_\phi, \Omega^i_z)$

  Each snapshot has its own orbital parameters, so we have 3N unknowns (p_i, e_i, ι_i)

  Furthermore, we have ∞ unknown spacetime multipole moments.

  No easy method!

- **Why Ryan succeeded?**

Adding constraints to orbits!

- **Ryan assume the geodesics to be nearly equatorial and nearly circular.**

  Assume detection of N snapshots, so we have 3N data. $(\Omega_\rho^i, \Omega_\phi^i, \Omega_z^i)$

  Each snapshot has its own orbital parameters, so we have N unknowns (p_i)**.**

  We do have 3N-N=2N extra freedom!

Generalization to eccentric, equatorial orbits.

- I assumed the geodesics to be nearly equatorial or nearly circular

  Assume detection of N snapshots, so we have 3N data. $(\Omega_\rho^i, \Omega_\phi^i, \Omega_z^i)$

  Each snapshot has its own orbital parameters, so we have 2N unknowns (p_i, e_i).

  We still have 3N-2N=N extra freedom!

- Analytical expressions for $\Omega_\rho, \Omega_z, \Omega_\phi$ are worked out. (Quite complicated!)

  $\Omega_A = \Omega_A(p, e, M, S, Q, \cdots)$, here $A = \rho, \phi, z$.

- Start from the first snapshot, I do iteration.

  Step i, use i snapshots, assume only the first i multipoles nonvanishing

  Step i+1, use i+1 snapshots, we solve for the first i+1 multipoles.

  Collect the results for mass, they should converge to some stable limit.

  Similar for the other multipoles.

- Further generalization? No constraint on orbits?

Wave generation in SARSAF spacetime

- **This time, we need to use the amplitude information $h^{\mu\nu}_{mnl}$, or the time-evolution of three frequencies $\Omega_\phi(t), \Omega_\rho(t), \Omega_z(t)$. Both require to treat wave generation in SARSAF spacet**
- **No problem! Numerical relativity can do it.**
- **But what about inner boundary condition?** This changes tidal coupling.

Tidal coupling (Preliminary results)

- **Stationary Phase Approximation waveform**

  $\tilde{h}(f) = \mathcal{A} f^{-7/6} e^{i\psi(f)}$

  $\psi(f)$ **depends on** dE/df **(obtained from snapshots) and** dE/dt

- **Luminosity**

$$\underbrace{\frac{dE}{dt}}_{\text{observable}} = \underbrace{\left.\frac{dE}{dt}\right|_{\infty}}_{\text{universal}} + \underbrace{\left.\frac{dE}{dt}\right|_{H}}_{\text{non-universal}}$$

universal: weak dependence on the nature of the central object.

non-universal: strong dependence on the nature of the central object.

Conclusion

- **Two assumptions: SARSAF spacetime, adiabatic approximation.**
- **What we know:**
  1. **Waveform:** $h^{\mu\nu} = \sum_{mnl} h^{\mu\nu}_{mnl} e^{-i(m\Omega_\rho + n\Omega_\phi + l\Omega_z)t}$
  2. **We can extract metric and orbital information from $(\Omega_\rho, \Omega_z, \Omega_\phi)$, if the orbits are equatorial, eccentric (or circular, inclined).**
  3. **We can also extract tidal coupling information. (need check)**
- **What we don't know:**

  Generic geodesics. This requires good understanding of radiation reaction in general SARSAF spacetime.
- **Difficulties:**
  1. **Geodesics in non-integrable spacetime, (three integrals, four coordinates.)**
  2. **Inner boundary conditions.**