# A New Class of Post-Newtonian Approximants to the Dynamics of Inspralling Compact Binaries: Test-Mass in the Schwarzschild Spacetime

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# *Overview*

- Background and motivation of the present work.
- *Standard* approximation to the phasing formula.
- *Standard* and *complete non-adiabatic* approximants.
- *Complete adiabatic* approximation to the phasing formula.
- Tools for the study: *overlaps, effectualness, faithfulness.*
- Construction of waveforms.
- Results and summary.

# *Stages of binary coalescence*

- **Inspiral** : well modelled by the PN approximation methods.
- **Plunge** : extended approximation  methods like EOB approximation.
- **Merger** : fully general relativistic solution of Einstein's eqns. - Numerical Relativity.
- **Ring-down** : BH perturbation theory.

This study is restricted to the inspiral phase

# *Adiabatic Inspiral*

- **Adiabatic approximation:** change in orbital frequency is small compared to the orbital frequency itself.

$$\dot{\nu}/\nu^2 \ll 1 \quad \rightarrow \quad \dot{E} = -\mathcal{F}$$

Orbital Frequency

Binding Energy

Energy flux of GWs

Thus the inspiral orbit can be thought of as an adiabatic perturbation over a number of circular orbits.

# *Phasing Formula*

- The adiabatic approximation along with the definition of the GW frequency $2\pi F = d\phi/dt$ gives the time evolution of the GW phase

$$\frac{d\varphi}{dt} = \frac{2v^3}{m}, \qquad \frac{dv}{dt} = -\frac{\mathcal{F}(v)}{mE'(v)},$$

where the binding energy and flux functions are given as PN expansions. Given the phasing, the waveform can be constructed as

$$h(t) = \frac{4A\eta m}{D} v^2(t) \cos[\varphi(t) + \varphi_C],$$

# *A recap of the notation in PN theory*

- The crucial inputs of the phasing formula are the binding energy and GW flux, which are given as PN expansions of $v/c$.
- The energy function corresponds to the conserved energy of the binary. In the conservative dynamics of the binary, wherein there is no dissipation, the energy is expressed as a PN expansion in $(v/c)^2$, with the dominant order termed Newtonian or 0PN and a correction at order $(v/c)^{2n}$ called $n$PN, with the dynamics involving only even powers of $v/c$.
- When dissipation is added to the dynamics, then the equation of motion will have terms of both odd and even powers of v/c. Thus, a correction of order $(v/c)^m$ is termed as $m/2$ PN.

# *Standard adiabatic approximation to the phasing formula*

- The *standard approximation* to the phasing formula uses the energy and flux functions both to the same *relative* accuracy.
- Including the radiation reaction at dominant order, however, is not a first order correction to the dynamics of the system, rather it is a correction that arises at 2.5PN.
- Thus, the phasing of the waves when translated to the relative motion of the bodies implies that the dynamics is described by the dominant Newtonian force and a correction at an order $(v/c)^5$, but neglecting conservative force terms that occur at orders $(v/c)^2$ and $(v/c)^4$.

- In order to make this point clearer, let's consider some phasing models constructed directly from the dynamics: *non-adiabatic approximants*.
- The *Lagrangian* templates studied by Buonanno, Chen & Vallisnery (BCV) can be thought of as examples of *standard* non-adiabatic approximants; since following 'standard' choices (of acceleration terms), they result in gaps in the acceleration.

# *Lagrangian approximants (BCV)*

- The Lagrangian models studied by BCV are specified by the acceleration experienced by the binary system.

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}, \qquad \frac{d\mathbf{v}}{dt} = \mathbf{a}$$

acceleration (given as a PN expansion)

$$\frac{d\phi}{dt} = \omega, \quad v^2 = \omega^2 r^2.$$

$$\varphi = 2\phi.$$

GW phase

Obital phase

# *Standard non-adiabatic approximants:*

➲ Choses the acceleration terms consistent with the 'standard' treatment of the phasing formula, resulting in gaps in the acceleration.

- 0 PN : $\mathbf{a} = a_N + a_{2.5RR}$
- 1 PN : $\mathbf{a} = a_N + a_{1PN} + a_{2.5RR} + a_{3.5RR}$

# *Complete non-adiabatic approximation: An alternative*

- Consistently includes all acceleration terms up to the respective PN order.

- 0 PN : $\mathbf{a} = a_N + a_{1PN} + a_{2PN} + a_{2.5RR}$
- 1 PN : $\mathbf{a} = a_N + a_{1PN} + a_{2PN} + a_{2.5RR} + a_{3PN} + a_{3.5RR}$

# *Natural adiabatic extensions of the complete non-adiabatic approximants:*

- In the *adiabatic* approximation, energy/flux functions can be thought of as carrying the information of the conservative/radiative dynamics of the binary.
- We propose a new and simple *complete adiabatic approximant* constructed from the energy and flux functions.
- Given the 0PN flux (leading to an acceleration at 2.5PN), we choose the energy function at 2PN (equivalent to 2PN conservative dynamics) instead of the standard choice 0PN(equivalent to 0PN conservative dynamics). In general, given the flux at $n$ PN-order, a corresponding complete adiabatic approximant is constructed by choosing the energy function at order $[n+2.5]$PN, where $[p]$ denotes the integer part of $p$.
- The complete adiabatic approximation, in spirit, corresponds to the dynamics where there are no missing post-Newtonian terms in the acceleration.

- We compare the overlaps of the standard and complete adiabatic approximant templates with the exact waveforms (in the adiabatic approximation) for a test-particle orbiting a Schwarzschild black hole.
- The 'performance' of these two family of templates is measured by their *effectualness* and *faithfulness*.

# *Effectualness and faithfulness*

- Overlap of an approximate waveform *A(t)* with the exact waveform *X(t)* is defined as:

$$\Omega = \frac{\langle A, X \rangle}{\sqrt{\langle A, A \rangle \langle X, X \rangle}}$$

where

$$\langle A, B \rangle = 2 \int_0^\infty \frac{df}{S_h(f)} \tilde{A}(f) \tilde{B}^*(f) + \mathrm{C.C.}$$

One-sided PSD of the detector noise

Fourier transform of *A(t)*

- *Faithfulness* (measure of smaller bias in the estimation of parameters):

$$F = \text{max\_extrinsic\_params}\,(\Omega)$$

- *Effectualness* (measure of larger overlaps with the exact waveform):

$$E = \text{max\_all\_params}\,(\Omega)$$

# *Exact and approximate waveforms*

- **Exact energy function** for a test-particle $m_2$ moving in circular orbit in the background of a Schwarzschild BH of mass $m_1$ is given in terms of $x = v^2$ as

$$E_{\text{exact}}(x) = \eta \, \frac{1-2x}{\sqrt{1-3x}} \, ,$$

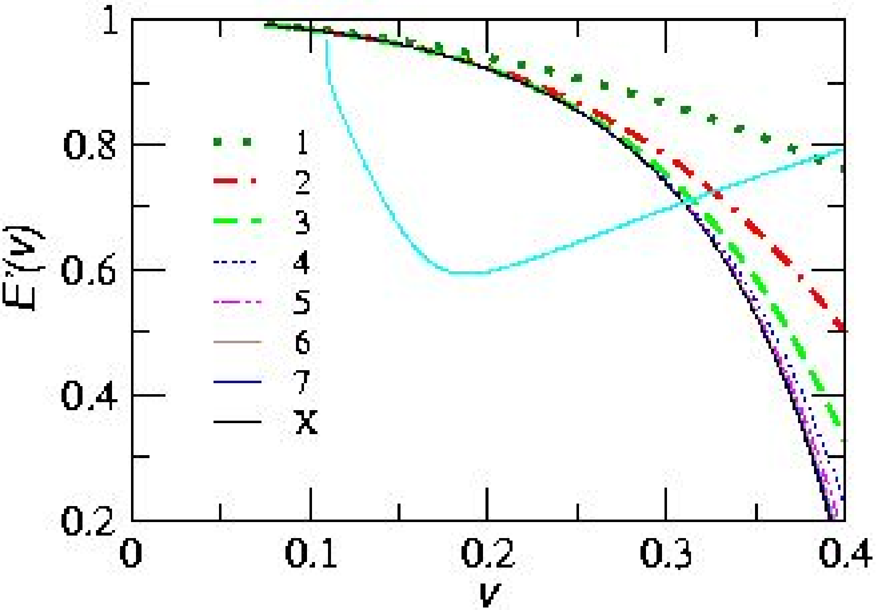
where

$\eta = m_1 m_2 / (m_1 + m_2)^2$

is the

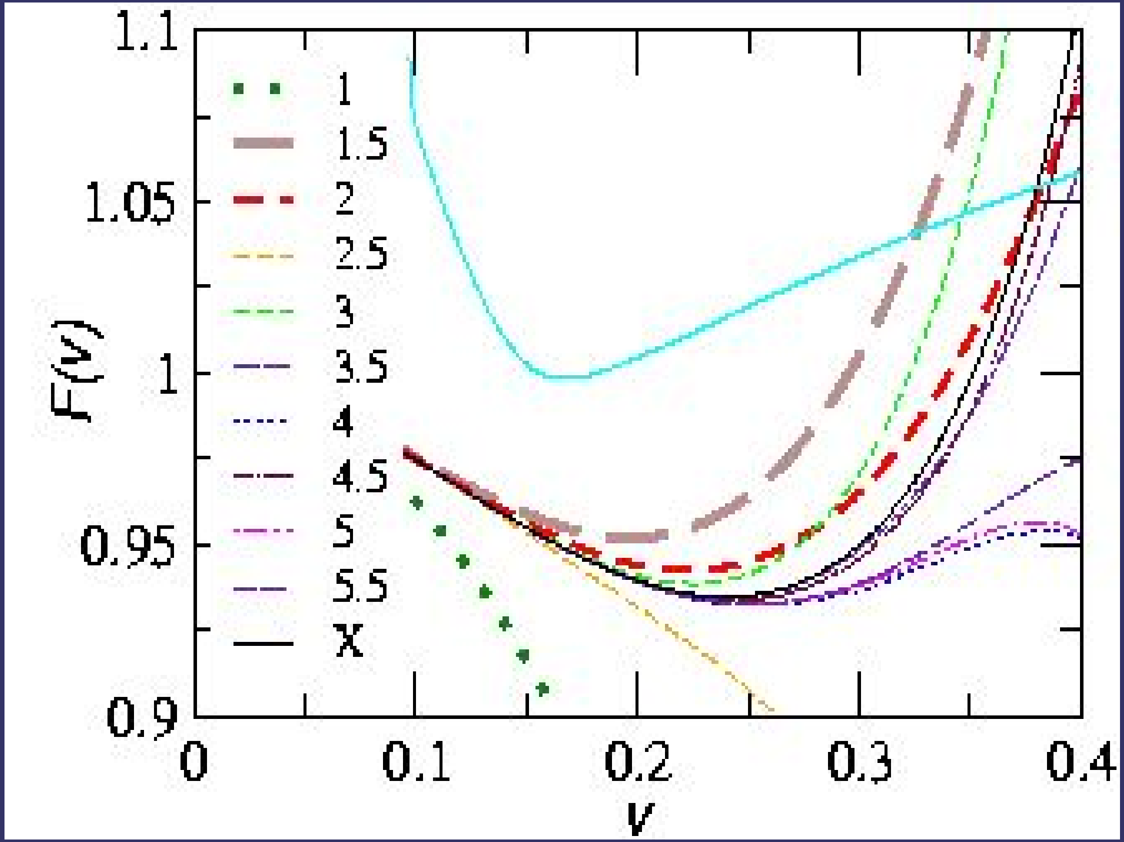
*symmetric-mass ratio*

of the binary



# *Exact and approximate waveforms*

- **Exact flux function** has been numerically calculated by E. Poisson (1995). Also analytical calculation of the flux is available up to 5.5PN (Tagoshi et. al.)



# *Calculating the overlaps*

- Using the exact/approximate energy and flux functions, the exact/approximate waveforms are constructed using the phasing formula, and the overlaps are calculated.
- Binaries studied : $(1M_\odot, 10M_\odot)$, $(1M_\odot, 50M_\odot)$ and $(1M_\odot, 100M_\odot)$ binaries.
- Overlaps are first calculated assuming a flat power spectral density to the detector noise (*white-noise*): viewed as a general mathematical question concerning the nature of PN templates.
- As a next step, overlaps are calculated for the initial LIGO noise spectrum.

# *Standard Vs. complete adiabatic approximants*

- **White-noise study:**
- The complete adiabatic approximants lead to a remarkable improvement in the *effectualness* (i.e. larger overlaps with the exact signal) at lower PN (< 3PN) orders. However, standard adiabatic approximants of order >= 3PN are nearly as good as the complete adiabatic approximants for the construction of *effectual* templates.
- In general, *faithfulness* (i.e. smaller biases in the estimation of parameters) of complete approximants is also better than that of standard approximants. But there are some cases of anomalous behavior where the *complete* performs worse than the *standard*.

# *Standard Vs. complete adiabatic approximants*

- **Initial LIGO study:**
- The complete adiabatic approximants lead to a remarkable improvement in the *effectualness* at lower PN (< 3PN) orders. However, standard adiabatic approximants of order >= 3PN are nearly as good as the complete adiabatic approximants for the construction of effectual templates.
- *Faithfulness* of complete approximants is almost *always* better than that of standard approximants.

# *Summary and conclusion*

- The standard adiabatic approximation to the phasing of GWs from ICBs is based on the PN expansions of the binding energy and GW flux both truncated at the same *relative* PN order.
- Viewed as a problem concerning the dynamics of the binary, standard approximation is equivalent to neglecting certain conservative terms in the PN expansion of the acceleration.
- A new PN *complete adiabatic approximant* based on energy and flux functions is proposed.
- We have evaluated the performance of the standard adiabatic vis-a-vis complete adiabatic approximants, in terms of their *effectualness* and *faithfulness*

# *Summary and conclusion*

- The complete adiabatic approximants lead to a remarkable improvement in the *effectualness* at lower PN (< 3PN) orders. However, standard adiabatic approximants of order >= 3PN are nearly as good as the complete adiabatic approximants for the construction of *effectual* templates.
- *Faithfulness* of complete approximants is also, in general, better than that of standard approximants.

# Acknowledgments

- Luc Blanchet, Alessandra Buonanno
- K. G. Arun
- Iyan Taylor, Roger Philip

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