A template family for gravitational waves from coalescing binary black holes

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in collaboration with

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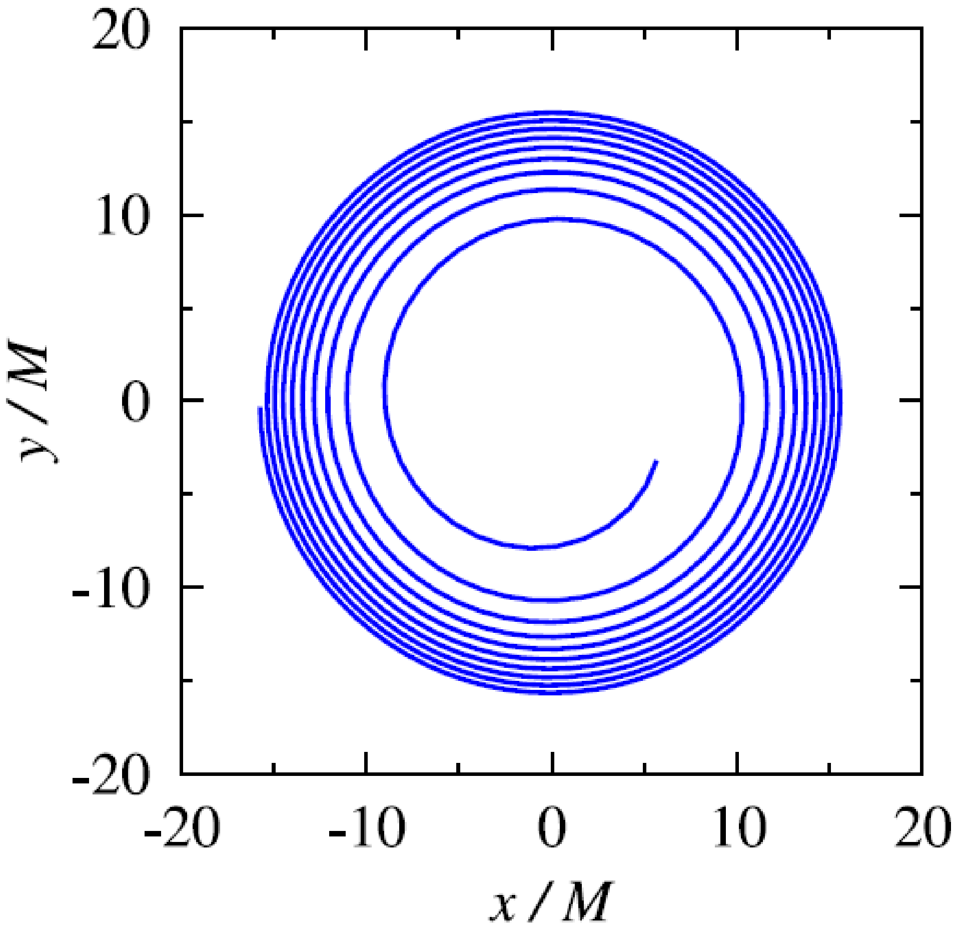
Outline

- **Introduction** to the modelling and data analysis from coalescing compact binaries.
- **Motivation & plans** Look for BH coalescences using a single template family.
- **Constructing coalescence waveforms** Matching PN and NR waveforms.
- **Phenomenological waveform family** parametrized by the physical parameters of the binary.
- **Template bank** Laying down templates.
- **Sensitivity** of the search.

Modelling of CCBs **INSPIRAL PHASE**

- Inspiral orbit is approximated as an adiabatic perturbation of many circular orbits.



Pic. BCV, 2002

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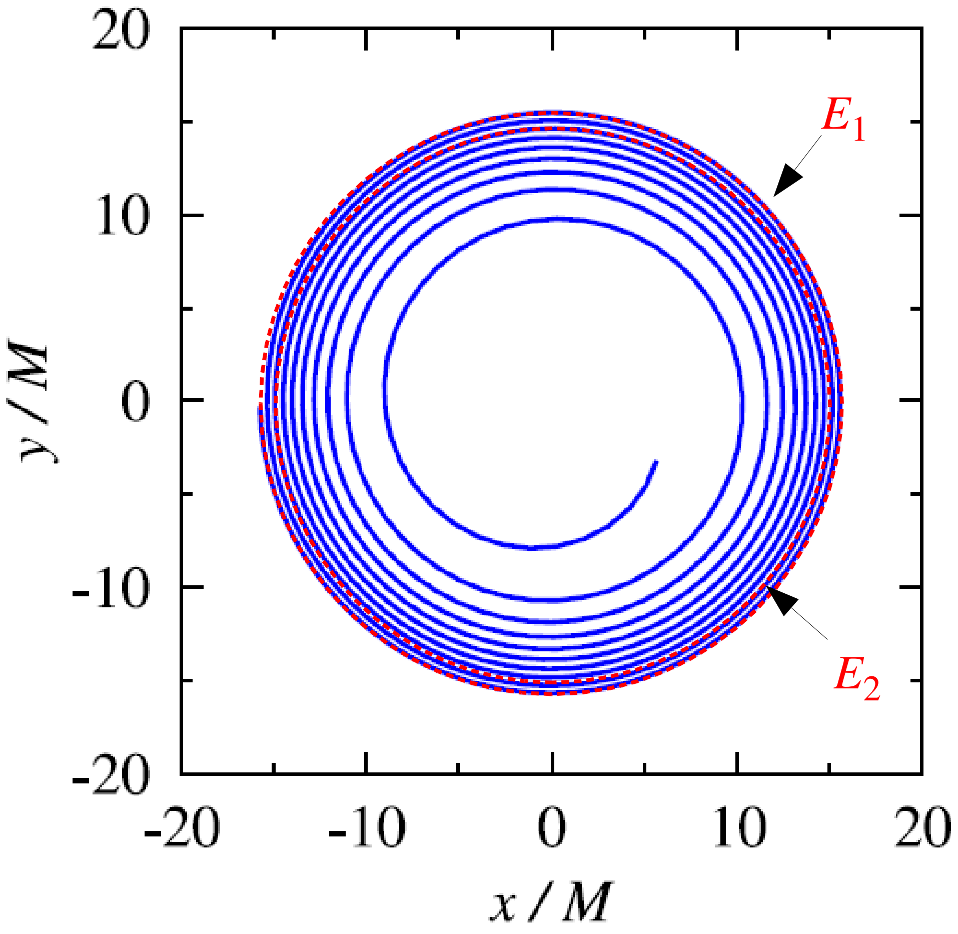

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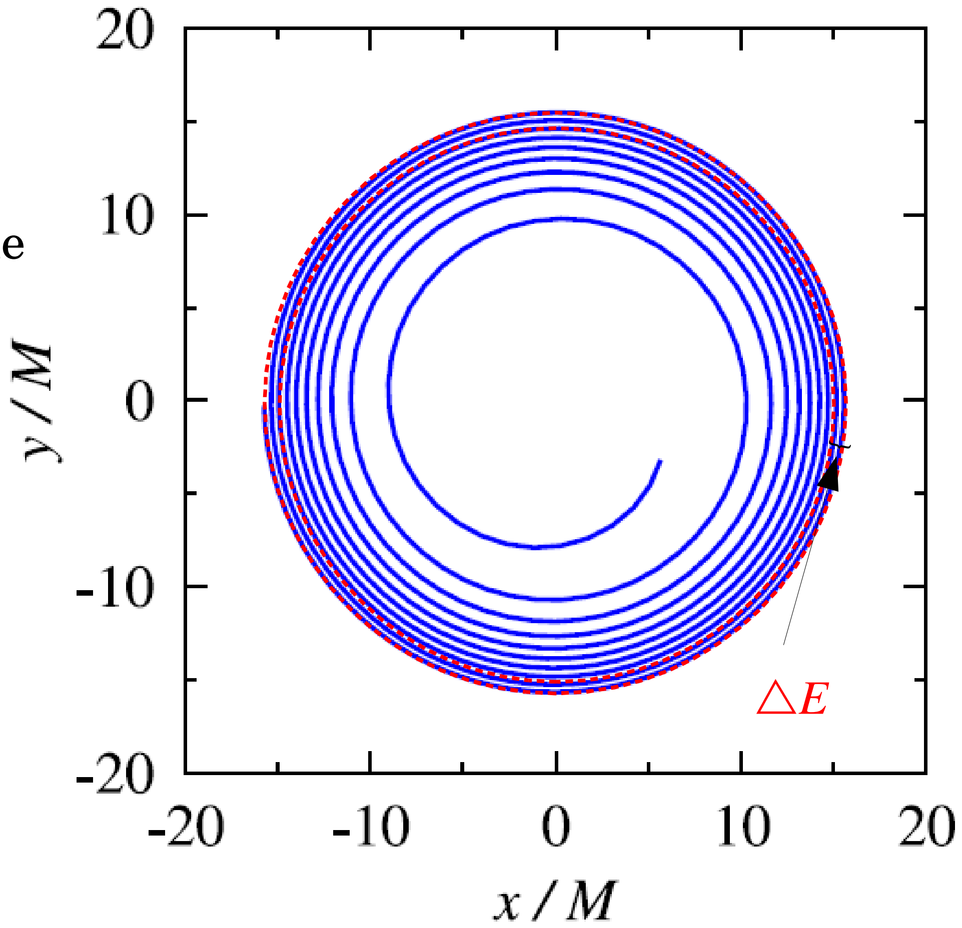
Modelling of CCBs **INSPIRAL PHASE**

- Inspiral orbit is approximated as an adiabatic perturbation of many circular orbits.

- GW phasing is computed from the 'energy-balance' argument.

$$\dot{E} = -\mathcal{F}$$



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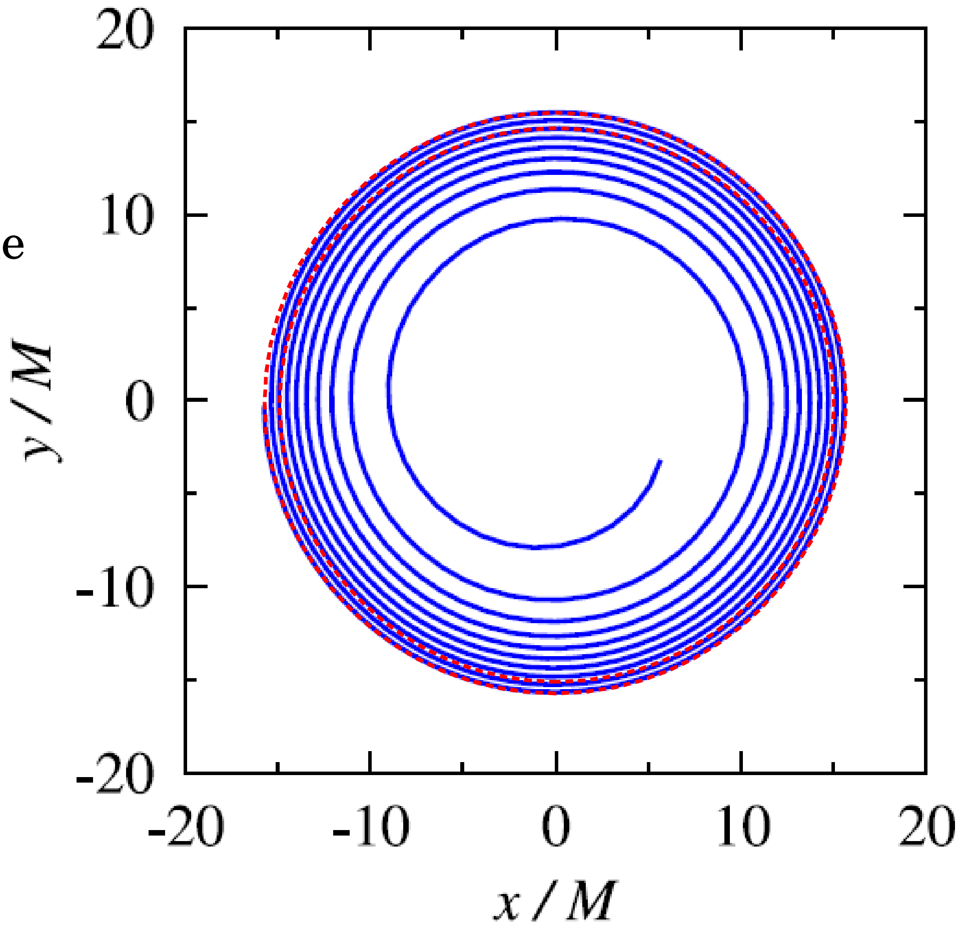
Modelling of CCBs **INSPIRAL PHASE**

- Inspiral orbit is approximated as an adiabatic perturbation of many circular orbits.
- GW phasing is computed from the 'energy-balance' argument.
- Binding energy and GW energy flux is computed as 'PN expansions' in terms of the velocity parameter v/c



Pic. BCV, 2002

$$E_{\rm 3PN}(x;\eta)=-\frac{1}{2}\eta x\Bigg\{1-\frac{1}{12}(9+\eta)x-\frac{1}{8}\left(27-19\eta+\frac{\eta^2}{3}\right)x^2+\left[\frac{-675}{64}+\left(\frac{209323}{4032}-\frac{205\pi^2}{96}-\frac{110\lambda}{9}\right)\eta-\frac{155}{96}\eta^2-\frac{35}{5184}\eta^3\right]x^3+\mathcal{O}(x^4)\Bigg\}, \tag{4.1}$$

where $\lambda=-1987/3080\simeq-0.6451$ [15–18]. The corresponding $E'(v,\eta)$ appearing in the phasing formula reads,

$$E'_{\rm 3PN}(v;\eta)=-\eta v\Bigg\{1-\frac{1}{6}(9+\eta)v^2-\frac{3}{8}\left(27-19\eta+\frac{\eta^2}{3}\right)v^4+4\left[\frac{-675}{64}+\left(\frac{209323}{4032}-\frac{205\pi^2}{96}-\frac{110\lambda}{9}\right)\eta-\frac{155}{96}\eta^2-\frac{35}{5184}\eta^3\right]v^6+\mathcal{O}(v^8)\Bigg\}. \tag{4.2}$$

We use this expression truncated at the necessary orders to construct the various approximate templates. To compute a fiducial exact waveform, we use the exact energy function in the test mass limit supplemented by the finite mass corrections up to 3PN in the spirit of the hybrid approximation [48]. In other words, the fiducial exact energy $E'(v;\eta)$ will look like

$$=-\eta v\Bigg\{\frac{-E'_{\rm exact}(v)}{\eta v}-\frac{\eta}{6}v^2-\frac{3}{8}\left(-19\eta+\frac{\eta^2}{3}\right)v^4$$

$$\frac{205\pi^2}{}-\frac{110\lambda}{9}\Bigg)\eta$$

are included only up to 2PN instead of 3PN, $v_{\rm Iso}\simeq$ 0.4113. It is worth pointing out that $v_{\rm Iso}^{\rm 2PN\text{-}Pade}\simeq 0.4456$ [41] and it is not unreasonable to expect that, with 3PN η-corrections the differences between various different ways of determining the Iso converge. (For the purposes of our analysis, we have checked that there is no drastic change in our conclusions due to these differences and hence we use uniformly the value $v_{\rm Iso}=0.4082$).

B. The flux function

The flux function in the case of comparable masses has been calculated up to 3.5PN accuracy [19–25], and is given by:

$$\mathcal{F}(v;\eta)=\frac{32}{5}\eta^2v^{10}\left[\sum_{k=0}^{7}A_k(\eta)v^k+B_6(\eta)v^6\ln v+\mathcal{O}(v^8)\right], \tag{4.4}$$

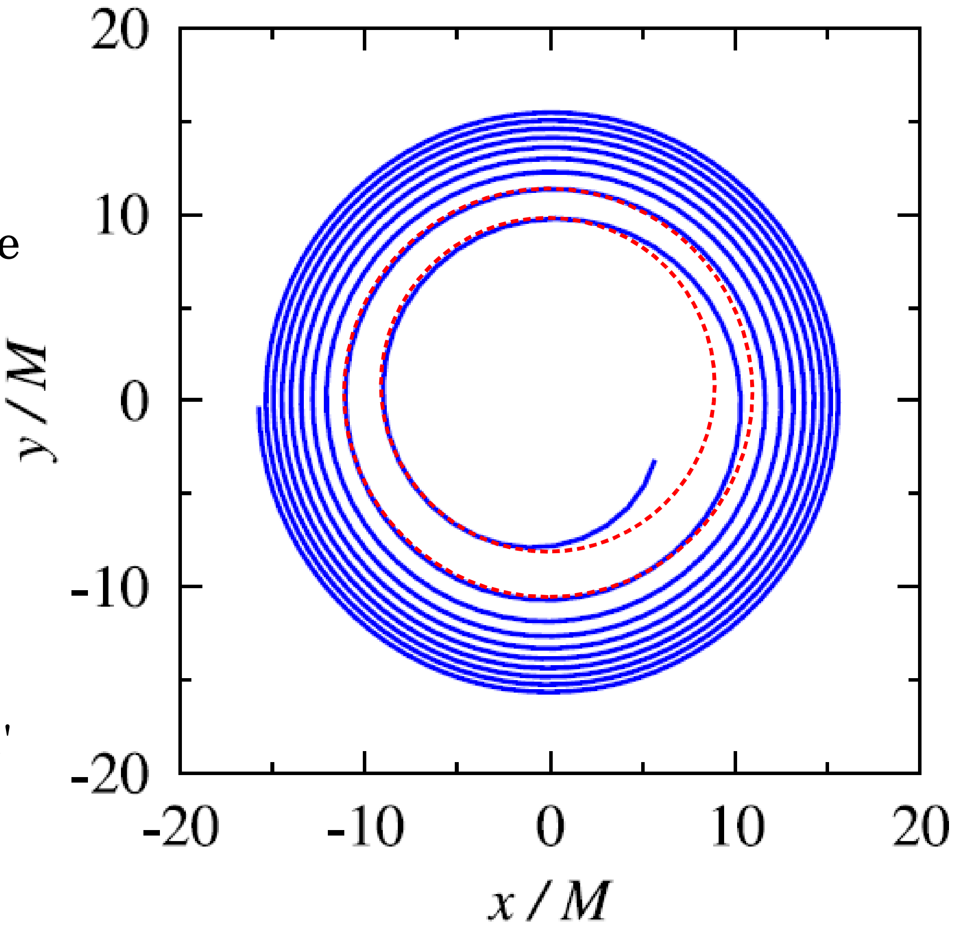
where

$$A_0(\eta)=1,\quad A_1(\eta)=0,\quad A_2(\eta)=-\frac{1247}{336}-\frac{35}{12}\eta,$$
$$A_3(\eta)=4\pi,\quad A_4(\eta)=-\frac{44711}{9072}+\frac{9271}{504}\eta+\frac{65}{18}\eta^2,$$
$$A_5(\eta)=-\left(\frac{8191}{672}+\frac{535}{24}\eta\right)\pi,$$
$$A_6(\eta)=\frac{6643739519}{69854400}+\frac{16\pi^2}{3}-\frac{1712}{105}\gamma+\left(-\frac{11497453}{272160}+\frac{41\pi^2}{48}+\frac{176\lambda}{9}-\frac{88\Theta}{3}\right)\eta-\frac{94403}{3024}\eta^2-\frac{775}{324}\eta^3-\frac{1712}{105}\ln 4, \tag{4.5}$$

$$\frac{16285}{}+\frac{176419}{1512}\eta+\frac{19897}{378}\eta^2\Bigg)\pi,$$

Modelling of CCBs **INSPIRAL PHASE**

- Inspiral orbit is approximated as an adiabatic perturbation of many circular orbits.
- GW phasing is computed from the 'energy-balance' argument.
- Binding energy and GW energy flux is computed as 'PN expansions' in terms of the velocity parameter v/c.
- But the '*adiabatic approximation*' breaks down as the binary approaches close to the merger stage.



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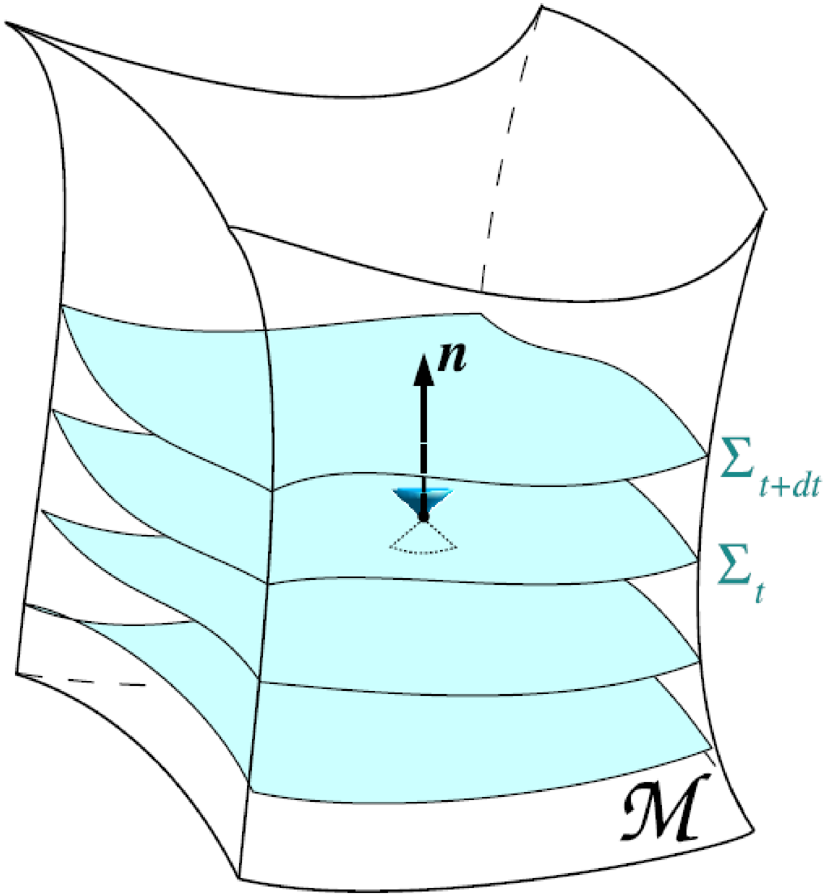
Modelling of CCBs **MERGER & RING-DOWN PHASES**

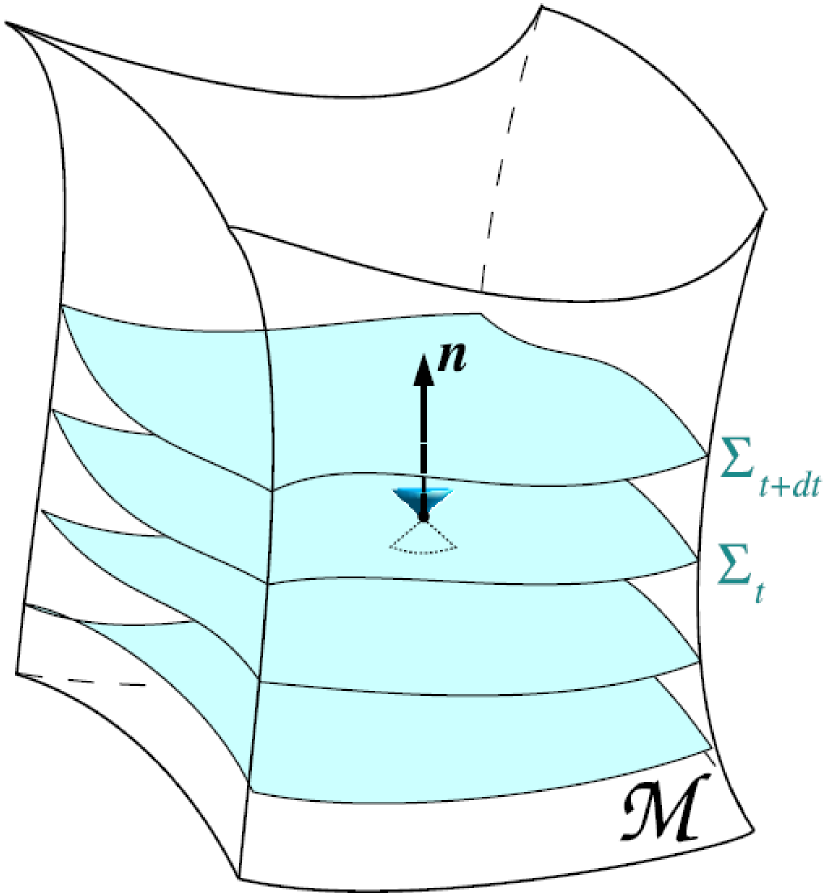
- In the “3+1” formulation of GR, the spacetime is foliated by a family of spacelike hypersurfaces Σ_t (level surfaces of the time parameter t).



Pic. Gourgoulhon, 2007

Modelling of CCBs **MERGER & RING-DOWN PHASES**

- In the "3+1" formulation of GR, the spacetime is foliated by a family of spacelike hypersurfaces Σ_t (level surfaces of the time parameter t).
- Prescribe suitable "initial data" for the first slice. Numerically evolve the slices along t – Time-varying tensor fields in the "ordinary" 3-space.



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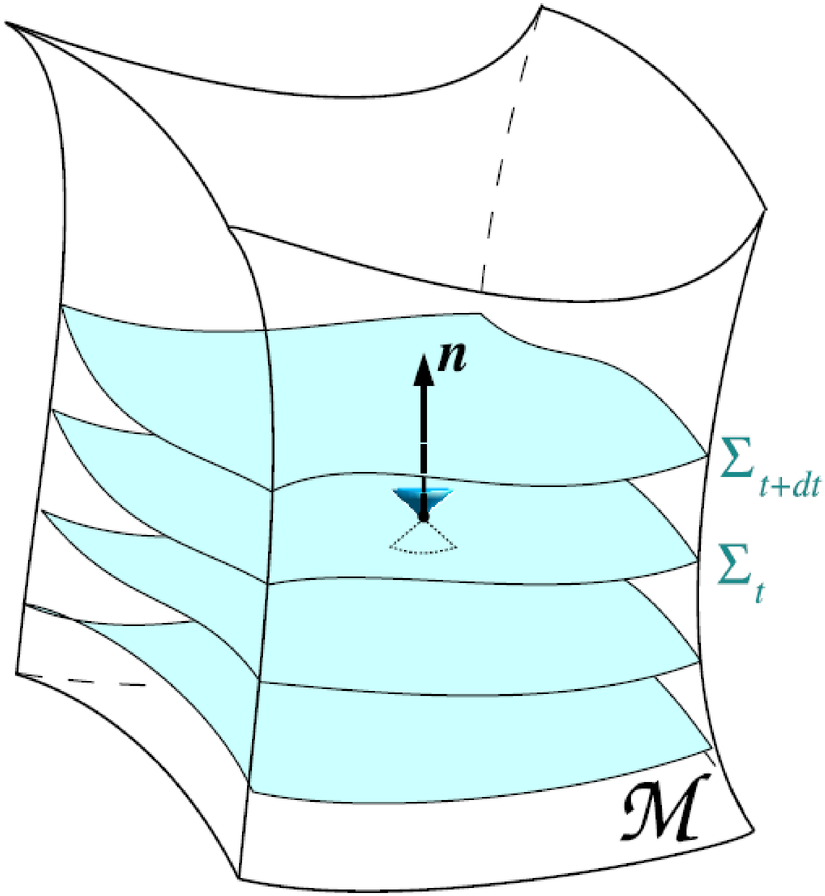
Modelling of CCBs **MERGER & RING-DOWN PHASES**

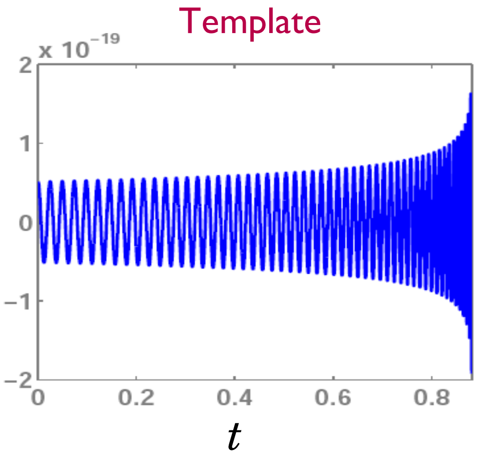
- In the "3+1" formulation of GR, the spacetime is foliated by a family of spacelike hypersurfaces Σ_t (level surfaces of the time parameter t).
- Prescribe suitable "initial data" for the first slice. Numerically evolve the slices along t – Time-varying tensor fields in the "ordinary" 3-space.
- GWs can be computed, for e.g., from the *Newman-Penrose Weyl scalar* $\Psi_4 = \ddot{h}_+ - \mathrm{i}\,\ddot{h}_\times$

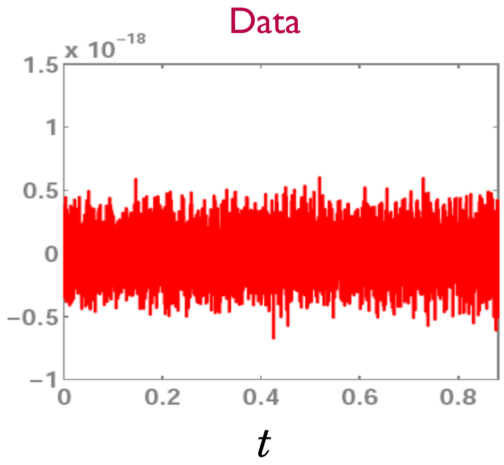


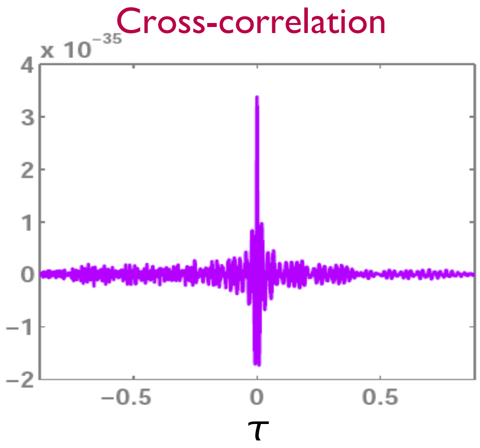
Pic. Gourgoulhon, 2007

Data analysis of CCBs **INSPIRAL PHASE**

- Physical (analytical) waveforms from PN theory. Also, detection template families (*e.g.* BCV) constructed to match a number of theoretical waveforms.
- Since waveforms are *known,* the optimal filter to search these signals in the noise is *matched filter,* which essentially involves cross-correlating the data and the signal template.

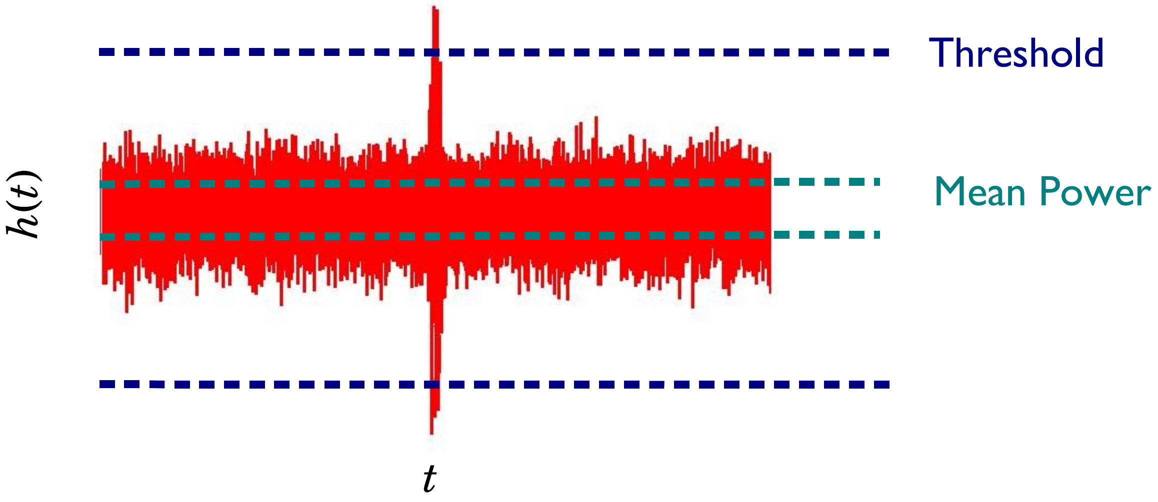

Template


Data


Cross-correlation

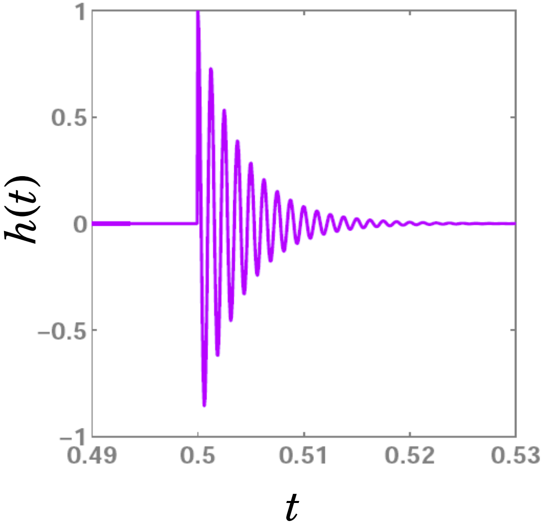
Data analysis of CCBs **MERGER PHASE**

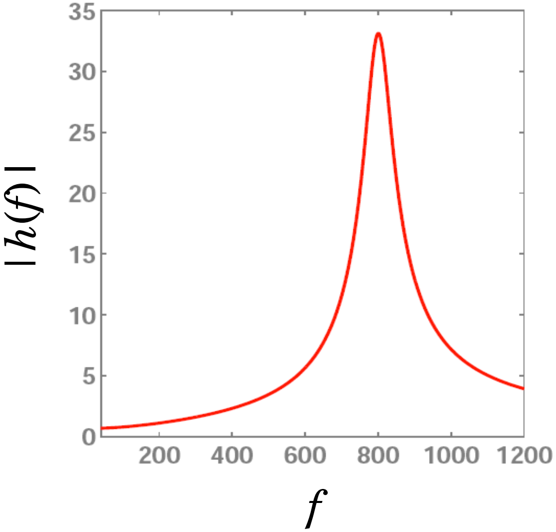
- Waveforms were not known until recently.
- Searches use *excess-power* techniques, which are tuned to detect large morphologies of waveforms. Look for short-lived excitations of power in the data, which are less-likely to be associated with the background noise distribution – a non-optimal filter.



Data analysis of CCBs **RING-DOWN PHASE**

- Waveforms are exponentially-damped sinusoids, parametrized by the mass and spin of the BH (known from BH perturbation theory).
- Since waveforms are known, matched filtering can be used to search for the signals.





Motivation of the work

- Recent progress in Numerical Relativity in solving the binary BH problem.
- Gravitational waveforms from the non-perturbative merger phase can also be computed.
- Finally allows us to coherently search for all three stages (inspiral, merger and ring-down) of the binary BH coalescence.

A single template bank for BH coalescence

- Coherently search for all three stages of the BH coalescence signals (non-spinning binaries) using a single template bank.
- But the high computational cost makes it infeasible to generate enough numerical waveforms to densely cover the entire parameter space to be searched over using matched filtering technique.

A single template bank for BH coalescence

- **Issue** How to construct a bank of templates?
  - Too expensive to compute a bank of NR waveforms dense enough in the (M, η) parameter space.
  - A phenomenological template bank (with parametrized waveforms) which has very good overlap with the 'target signals'.
- **Issue** How to construct the 'target' waveforms?
  - Need waveforms containing all three stages of the BH coalescence - too expensive to (numerically) evolve the binary from very large separations.
  - Match PN inspiral waveforms with NR (merger+ring-down) waveforms in a region where both calculations are valid.

Matching PN and NR waveforms

- Minimize the 'distance' between PN and NR waveforms over a matching region (a few cycles long), thus construct hybrid waveforms.

$$\delta = \min_{\boldsymbol{\mu},a} \left[\sum_{i=+,\times} \int_{t_1}^{t_2} \left[h_i^{\mathrm{PN}}(t,\boldsymbol{\mu}) - a\, h_i^{\mathrm{NR}}(t,\boldsymbol{\nu}) \right]^2 \mathrm{d}t \right]$$

- Minimisation is carried out over the parameters $\boldsymbol{\mu} = \{t_0, \phi_0\}$ and an amplitude scaling factor a.

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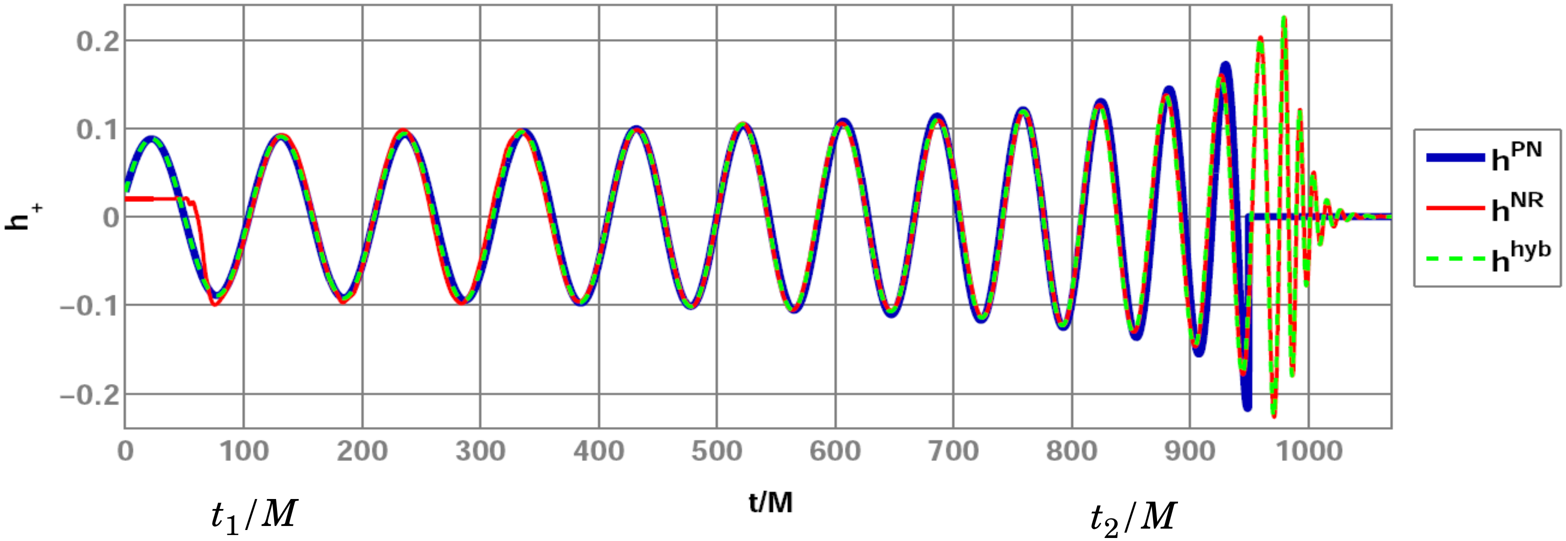
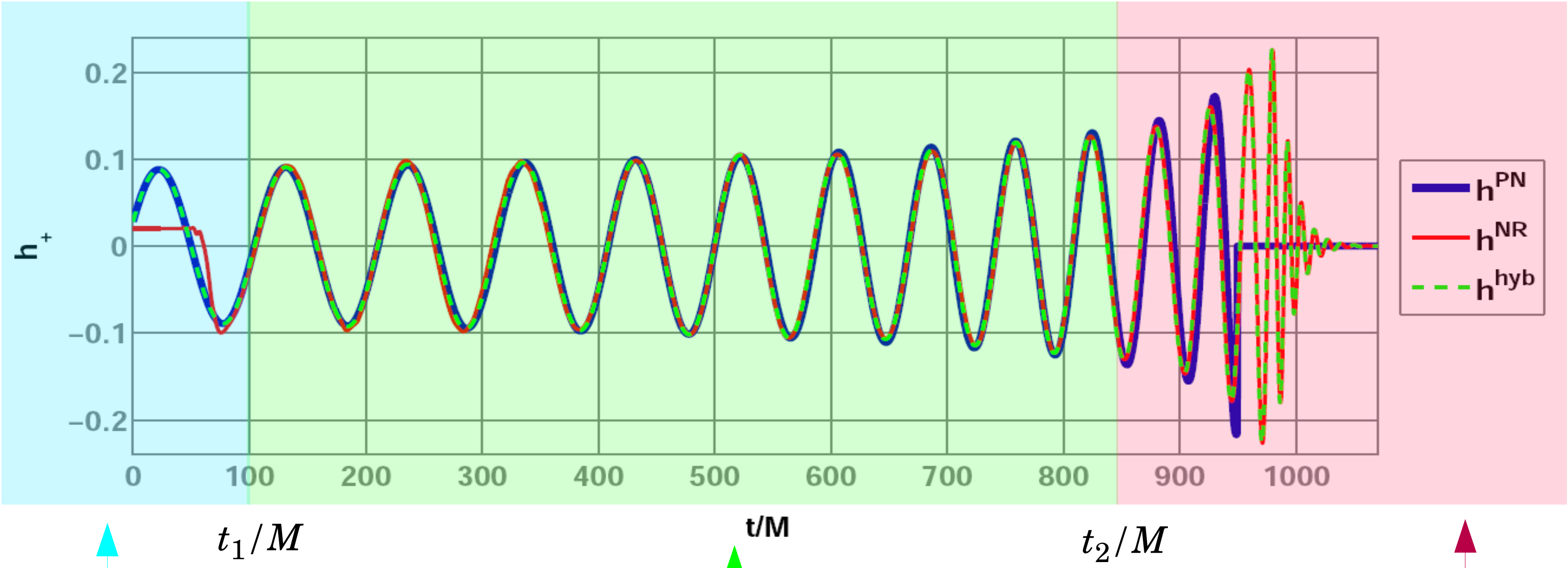
Constructing hybrid waveforms



PN 3.5PN TaylorT1 waveform
NR AEI equal-mass simulation

- Combine NR waveforms with the best-matched PN waveforms

Constructing hybrid waveforms

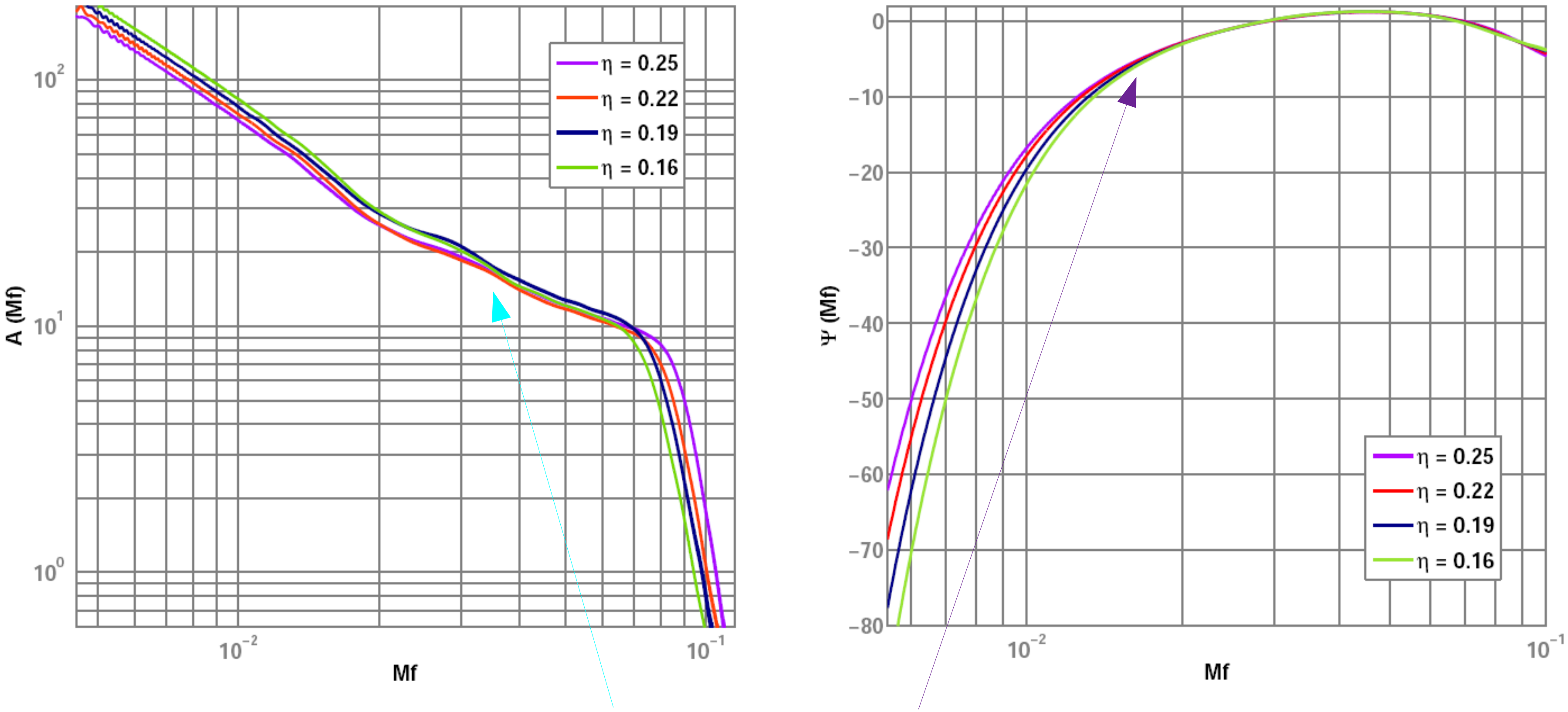


Hybrid waveform = PN waveform

Hybrid waveform = NR waveform

Hybrid waveform = Linear combination of PN & NR waveforms

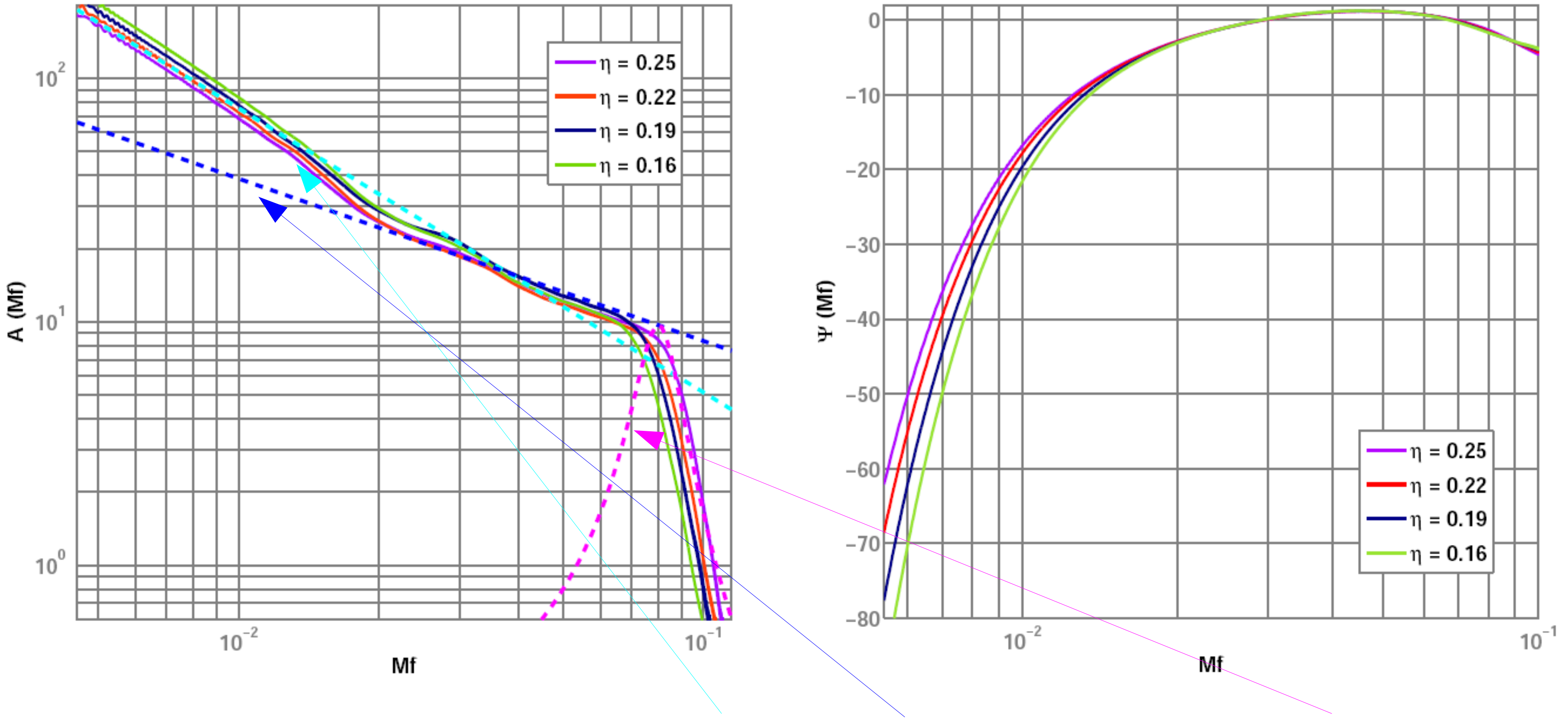
Hybrid waveforms in the Fourier domain



- Fourier domain magnitude and phase of hybrid waveforms.

PN 3.5PN TaylorT1 waveform
NR Jena unequal-mass simulations

A phenomenological parametrization



- **Magnitude** by two power-laws ($f^{-7/6}$ and $f^{-2/3}$) and a Lorentzian $L(f_{\mathrm{ring}}, \sigma)$.
- **Phase** Expansion in powers of f (motivation from the SPA).

Phenomenological waveforms

- The parametrised phenomenological waveform is written as

$$u(f) = \mathcal{A}_{\rm eff}(f)\, e^{i\Psi_{\rm eff}(f)}$$

where

$$\mathcal{A}_{\rm eff}(f) = \begin{cases} (f/f_{\rm merg})^{-7/6} & \text{if } f < f_{\rm merg} \\ (f/f_{\rm merg})^{-2/3} & \text{if } f_{\rm merg} \le f < f_{\rm ring} \\ w\,\mathcal{L}(f, f_{\rm ring}, \sigma) & \text{if } f_{\rm ring} \le f < f_{\rm cut} \end{cases}$$

$$\Psi_{\rm eff}(f) = 2\pi f t_0 + \phi_0 + \sum_{k=0}^{9} \psi_k\, f^{(k-5)/3}$$

Phenomenological waveforms

- The parametrised phenomenological waveform is written as

$$u(f) = \mathcal{A}_{\rm eff}(f)\, e^{i\Psi_{\rm eff}(f)}$$

where

Transition frequencies

$$\mathcal{A}_{\rm eff}(f) = \begin{cases} (f/f_{\rm merg})^{-7/6} & \text{if } f < f_{\rm merg} \\ (f/f_{\rm merg})^{-2/3} & \text{if } f_{\rm merg} \le f < f_{\rm ring} \\ w\,\mathcal{L}(f, f_{\rm ring}, \sigma) & \text{if } f_{\rm ring} \le f < f_{\rm cut} \end{cases}$$

'Spread' of the Lorentzian

Cutoff frequency

$$\Psi_{\rm eff}(f) = 2\pi f t_0 + \phi_0 + \sum_{k=0}^{9} \psi_k\, f^{(k-5)/3}$$

Phasing coefficients

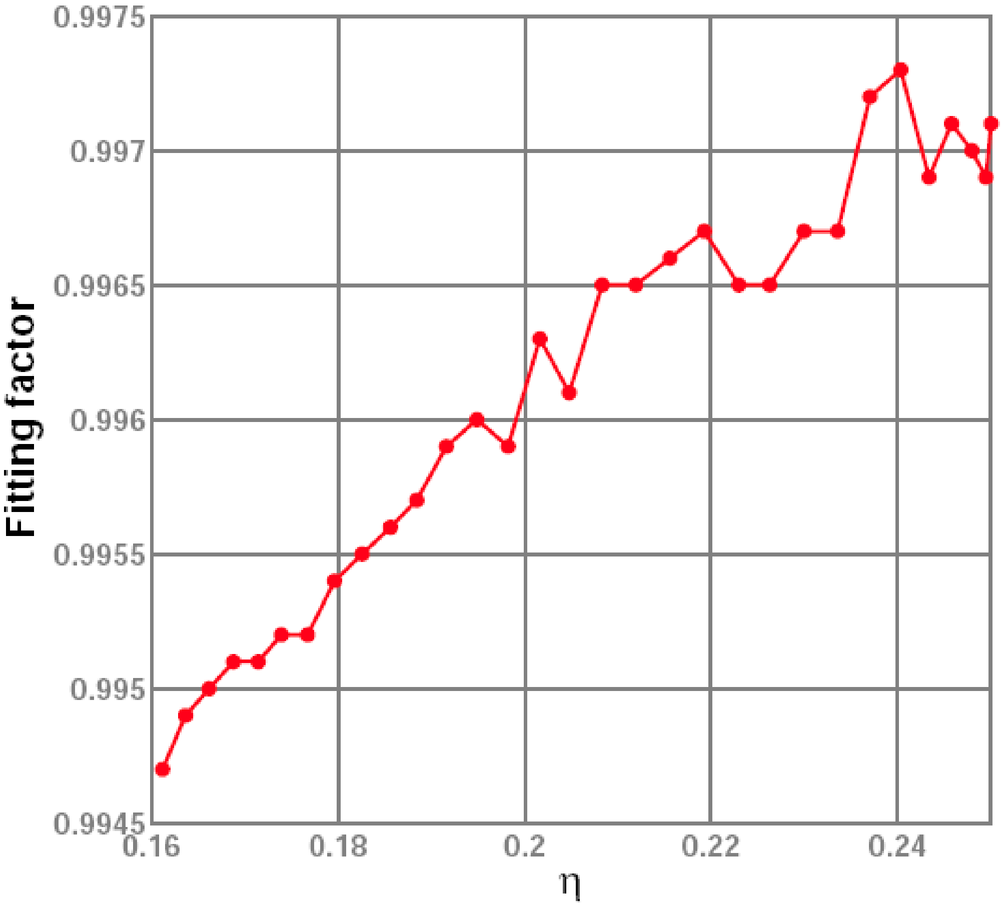
Characterising the quality of the phenomenological waveforms

- **Fitting factor** Ratio of the SNR that can be achieved with suboptimal template to the SNR obtained with an optimal template -- overlap between the (normalized) template and signal, maximized over the parameters of the template.

$$\mathrm{FF}\left(h(f,\boldsymbol{\mu}),u(f,\boldsymbol{\alpha})\right)=\frac{\max_{\boldsymbol{\alpha}}\langle h(f,\boldsymbol{\mu}),u(f,\boldsymbol{\alpha})\rangle}{\sqrt{\langle h(f,\boldsymbol{\mu}),h(f,\boldsymbol{\mu})\rangle}\sqrt{\langle u(f,\boldsymbol{\alpha}),u(f,\boldsymbol{\alpha})\rangle}},$$

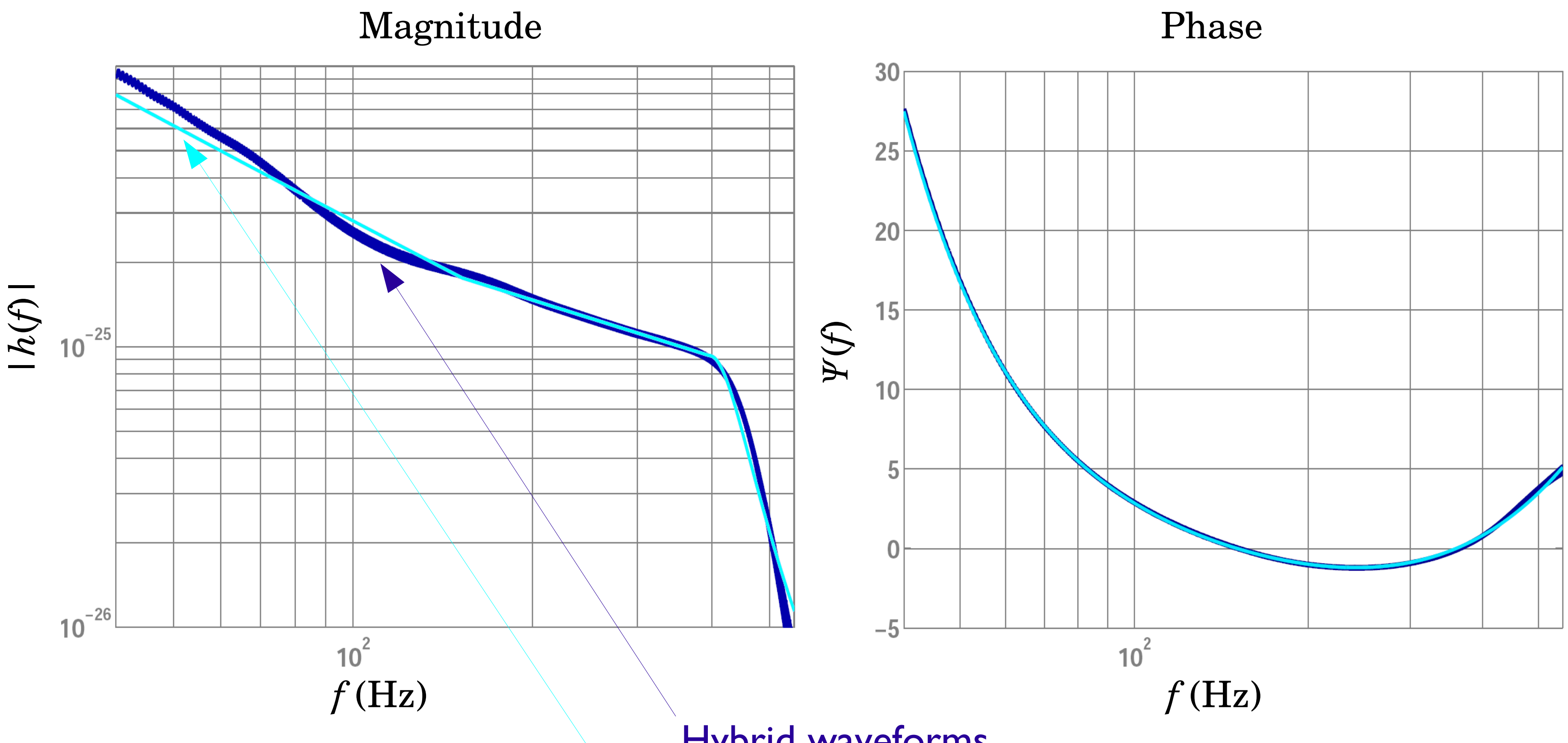
Fitting factors with the hybrid waveforms

- **Hybrid waveforms**
  Jena UM simulations $(0.16 \leqslant \eta \leqslant 0.25)$ + 3.5PN inspiral.
- **Noise spectrum**
  White noise.



Hybrid waveforms and the 'best-matched' templates

- $M = 40\,M_\odot,\ \eta = 0.25$ system, Initial LIGO noise spectrum.



Magnitude

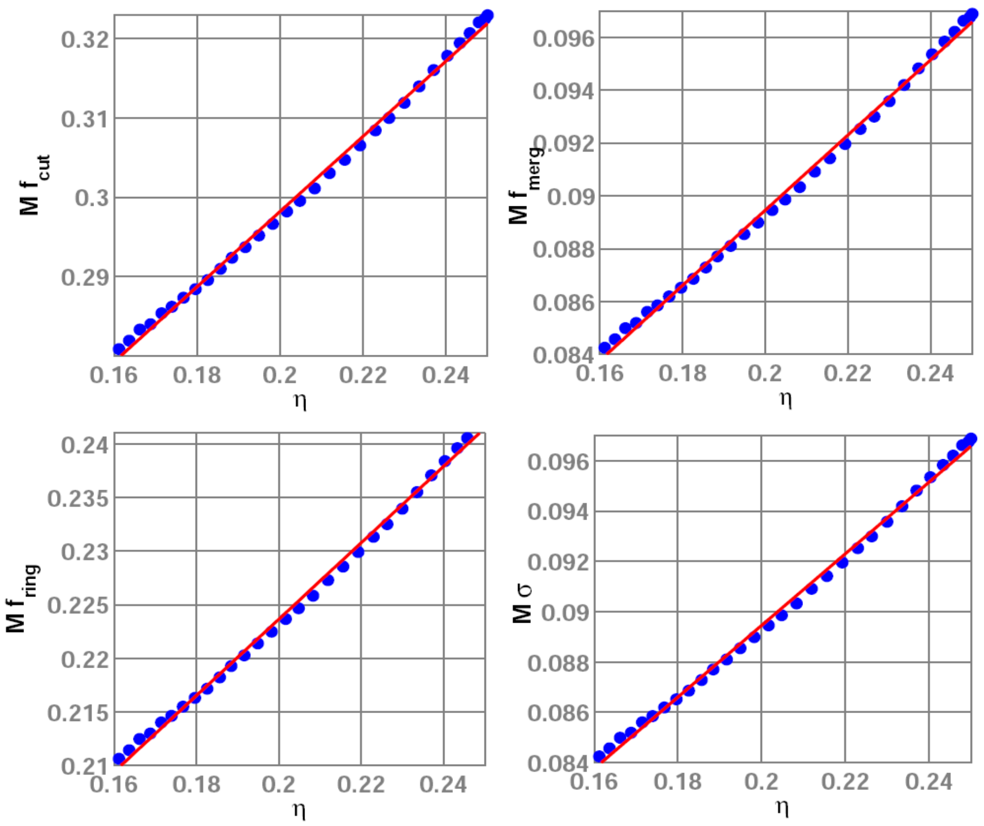
Phase

Hybrid waveforms

Phenomenological waveforms

From phenomenological parameters to physical parameters

- Possible to reparametrize the 'best-matched' phenomenological waveforms in terms of the physical parameters of the binary.

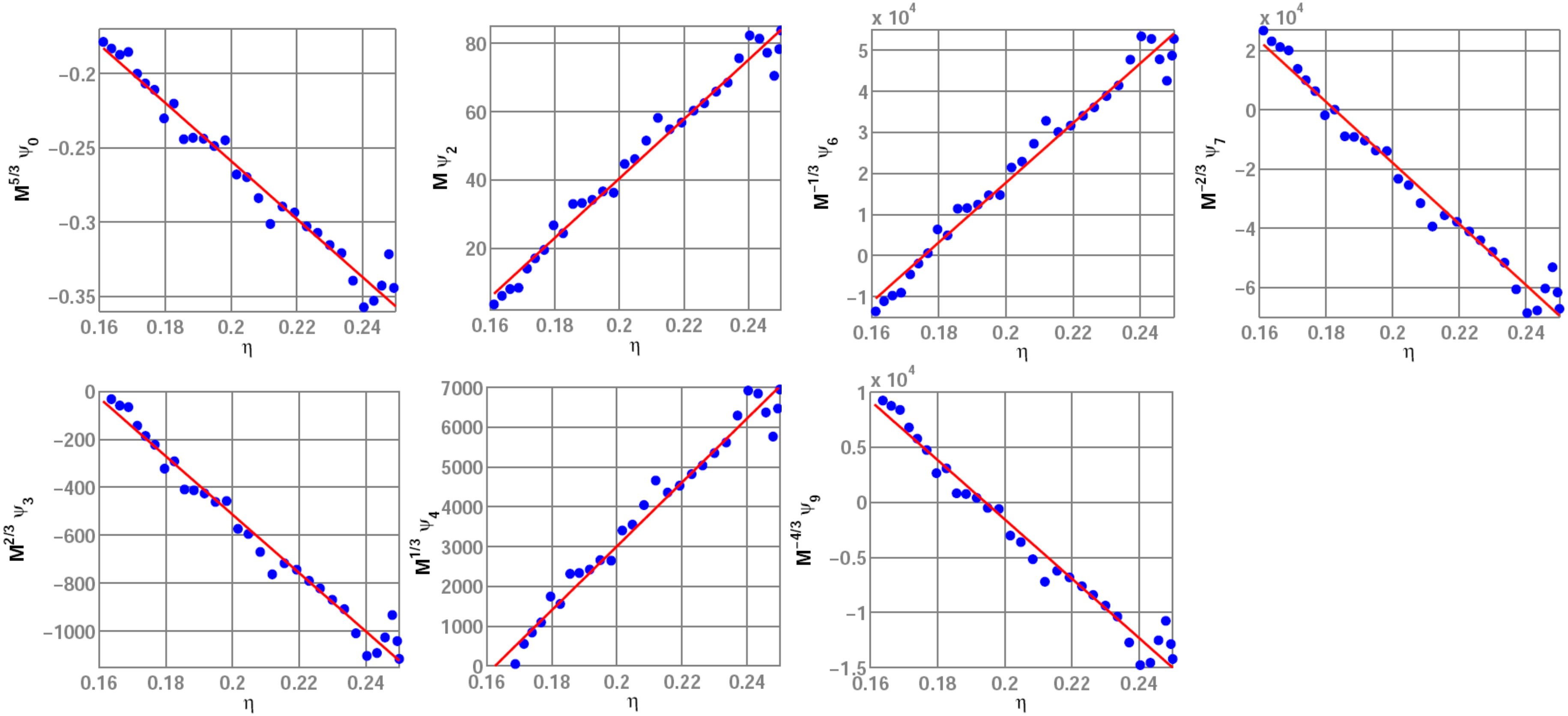


Best-matched phenomenological parameters

Physical parameters of the binary

From phenomenological parameters to physical parameters

Best-matched phenomenological parameters



Physical parameters of the binary

Reparametrization

- “Best-matched” phenomenological parameters can be written in terms of the physical parameters of the binary as

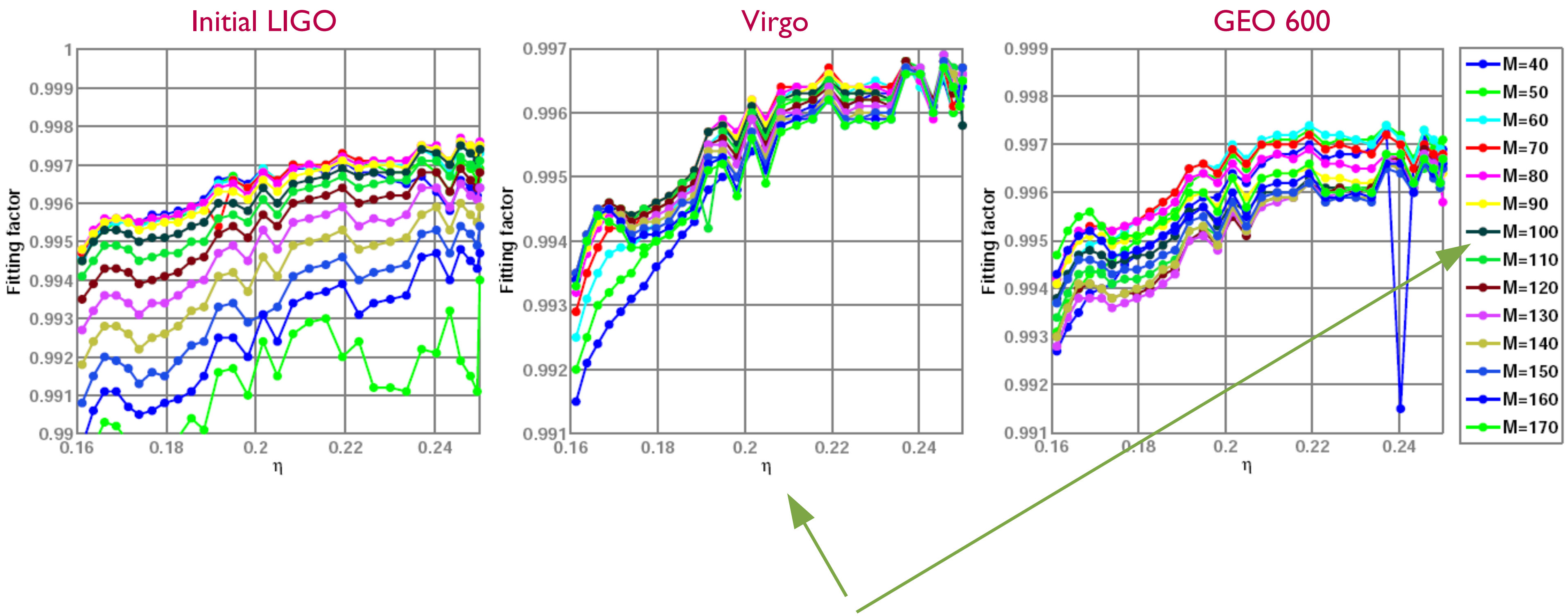
$$\begin{aligned} \pi M \boldsymbol{\alpha} &= \mathbf{a}\,\eta + \mathbf{b}\,, \quad \boldsymbol{\alpha} = \{f_{\rm merg}, f_{\rm ring}, f_{\rm cut}, \sigma\} \\ (\pi M)^{(5-k)/3}\,\boldsymbol{\beta} &= \mathbf{c}\,\eta + \mathbf{d}\,, \quad \boldsymbol{\beta} = \{\psi_0,\, \psi_2,\, \psi_3,\, \psi_4,\, \psi_6,\, \psi_7,\, \psi_9\} \end{aligned}$$

Polynomial fits

Polynomial coefficients

$$\begin{aligned} \mathbf{a} &= \{a_0, a_1, a_2, a_3\} \\ \mathbf{b} &= \{b_0, b_1, b_2, b_3\} \\ \mathbf{c} &= \{c_0, c_2, c_3, c_4, c_6, c_7, c_9\} \\ \mathbf{d} &= \{d_0, d_2, d_3, d_4, d_6, d_7, d_9\} \end{aligned}$$

“Reparametrized” templates



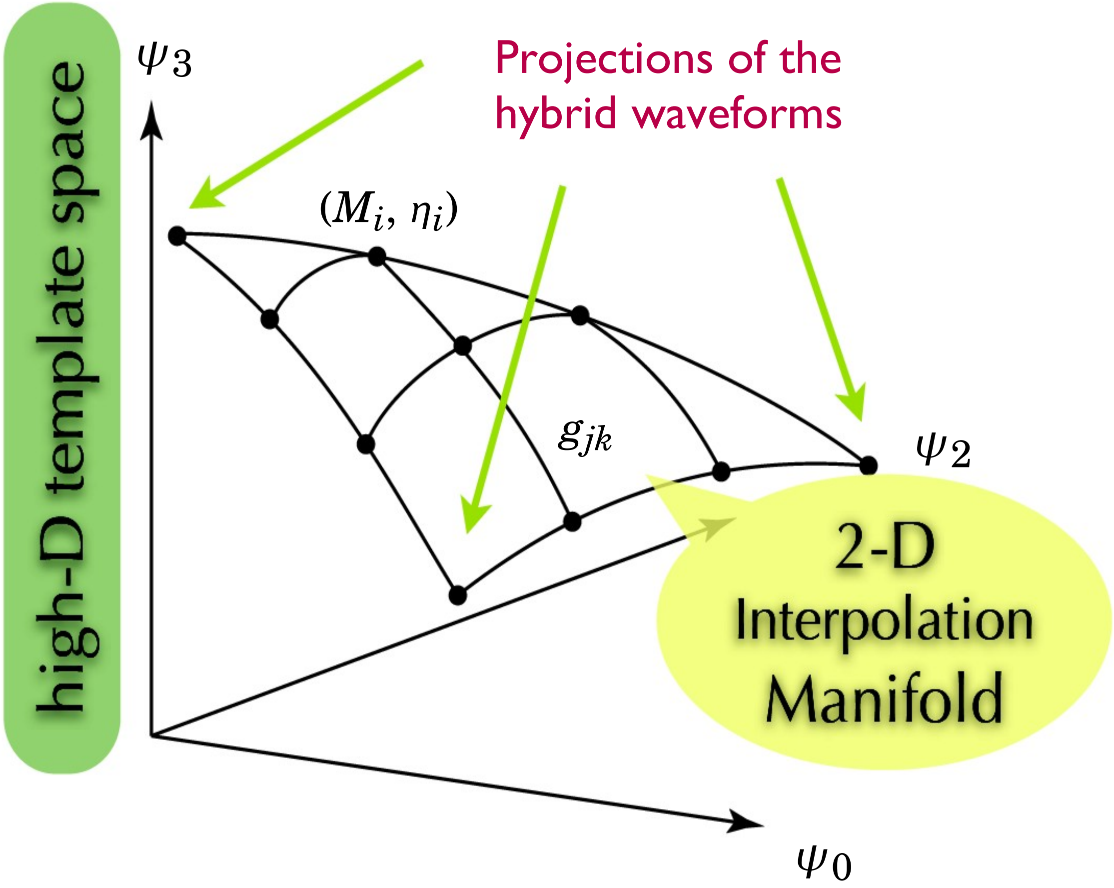
Physical parameters of the binary

- Two-parameter templates with ≥ 0.99 overlaps with the hybrid waveforms!

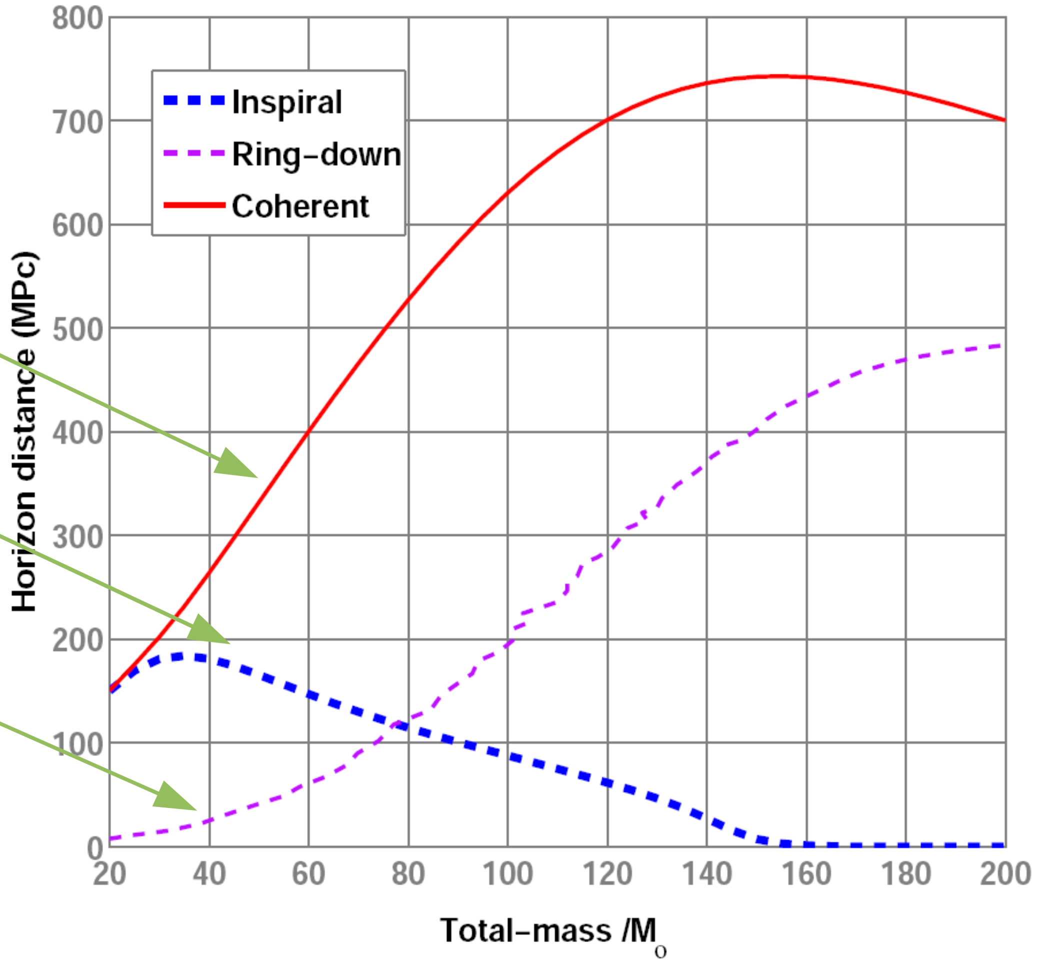
Template bank

- The re-parametrized template family is a two dimensional manifold embedded in a higher dimensional space.

- A metric can be defined on this manifold, which gives the notion of distance. This can be used to lay down templates allowing a given missmatch between neighboring templates.



Sensitivity of the search



Using inspiral+merger+ring-down templates

Using standard PN templates (truncated at ISCO)

Using ring-down templates

- Effective distance to optimally-oriented systems which can produce an optimal SNR of 8 at Initial LIGO.

Summary

- Recent progress in Numerical Relativity in modelling the non-perturbative merger phase of the binary BH coalescence problem.

- Constructed a set of hybrid waveforms by matching PN and NR predictions – waveforms containing the inspiral, merger, and ring-down stages.

- Proposed a phenomenological waveform family which has very good overlaps with the hybrid waveforms. These waveforms can be parametrized in terms of the physical parameters of the system – a two-dimensional template bank (for non-spinning binaries).

- This template bank might enable us to extend the present inspiral searches to higher mass binary black hole systems – increased reach of the current generation of ground based detectors.